Ivan Chajda; Jan Kühr

A note on interval MV-algebras

Mathematica Slovaca, Vol. 56 (2006), No. 1, 47--52

Persistent URL: http://dml.cz/dmlcz/130876

Math. Slovaca, **56** (2006), No. 1, 47–52

Mathematica Slovaca

Dedicated to Professor Tibor Katriňák

A NOTE ON INTERVAL MV-ALGEBRAS

IVAN CHAJDA — JAN KÜHR

(*Communicated by Sylvia Pulmannová*)

ABSTRACT. We show that every interval of an MV-algebra M is an MV-algebra again; the operations on the interval are defined by means of certain polynomial functions over M.

It is well known that every complemented modular lattice $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is relative complemented, i.e., if $y \in L$ is a complement of $x \in [a, b]$ in $\mathbf{L}$, then $z = (y \wedge b) \vee a$ is its relative complement in the interval $[a, b]$. Since any MV-algebra is a bounded distributive lattice with respect to its natural order, the question arises whether every interval can be endowed with an MV-structure.

Recall from [1] that an *MV-algebra* is an algebra $\mathbf{M} = (M, \oplus, \neg, 0)$ of type $\langle 2, 1, 0\rangle$ satisfying the identities

(MV1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,
(MV2) $x \oplus y = y \oplus x$,
(MV3) $x \oplus 0 = x$,
(MV4) $\neg\neg x = x$,
(MV5) $x \oplus \neg 0 = \neg 0$,
(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

MV-algebras were introduced by C. C. Chang in [2] as an algebraic counterpart of the Łukasiewicz many valued propositional calculus. Due to D. Mundici's result [5], MV-algebras are intervals in abelian lattice-ordered groups:

2000 Mathematics Subject Classification: Primary 06D35.
Keywords: MV-algebra, interval MV-algebra.

Given an abelian lattice-ordered group $\mathbf{G} = (G, +, -, 0, \vee, \wedge)$ and $0 \leq u \in G$, the structure $\Gamma(\mathbf{G}, u) = ([0,u], \oplus, \neg, 0)$ is an MV-algebra, where

$$x \oplus y := (x+y) \wedge u \quad \text{and} \quad \neg x := u - x \qquad \text{for all} \quad x, y \in [0,u],$$

and also conversely, every MV-algebra is isomorphic to $\Gamma(\mathbf{G}, u)$ for some abelian lattice-ordered group $\mathbf{G}$ and $0 \leq u \in G$.

For basic properties of MV-algebras we refer to [1].

Let $\mathbf{M}$ be an MV-algebra. If we put

$$x \leq y \qquad \text{if and only if} \qquad \neg x \oplus y = 1,$$

then $\leq$ is a partial order on M, the *natural order* of $\mathbf{M}$, and $\mathbf{L}(\mathbf{M}) = (M, \leq)$ is a bounded distributive lattice in which

$$x \vee y = \neg(\neg x \oplus y) \oplus y \qquad \text{and} \qquad x \wedge y = \neg(\neg x \vee \neg y).$$

An element $x \in M$ is called *boolean* if there is the complement x' of x in the lattice $\mathbf{L}(\mathbf{M})$; we use $B(\mathbf{M})$ to denote the set of all boolean elements of $\mathbf{M}$. As a matter of fact, $\mathbf{B}(\mathbf{M}) = (B(\mathbf{M}), \vee, \wedge, 0, 1)$ is a boolean sublattice of $\mathbf{L}(\mathbf{M})$ in which $x' = \neg x$. Moreover, $x \in \mathbf{B}(\mathbf{M})$ if and only if $x \oplus x = x$. It is worth adding that if $x \in B(\mathbf{M})$, then $x \oplus y = x \vee y$ for each $y \in M$.

We proved in [3] that MV-algebras are polynomially equivalent with the class of algebras $\mathbf{A} = (A, \circ, 0, 1)$ of type $\langle 2, 0, 0 \rangle$ satisfying the identities

(A1) $x \circ 1 = 1$, $1 \circ x = x$,
(A2) $(x \circ y) \circ y = (y \circ x) \circ x$,
(A3) $x \circ (y \circ z) = y \circ (x \circ z)$.

If $\mathbf{M}$ is an MV-algebra, then upon defining $x \circ y := \neg x \oplus y$, the algebra $\mathbf{A}(\mathbf{M}) = (M, \circ, 0, 1)$ fulfils (A1), (A2) and (A3). Conversely, let $\mathbf{A}$ be an algebra satisfying the axioms (A1), (A2) and (A3), and define $\mathbf{M}(\mathbf{A}) = (A, \oplus, \neg, 0)$ by $x \oplus y := (x \circ 0) \circ y$ and $\neg x := x \circ 0$; then $\mathbf{M}(\mathbf{A})$ is an MV-algebra. More generally, letting $x \leq y$ if and only if $x \circ y = 1$ we define the *induced order* of $\mathbf{A}$ which makes every interval $[a, 1]$ into a lattice with $x \vee y = (x \circ y) \circ y$ and $x \wedge y = ((x \circ y) \circ (x \circ a)) \circ a$. Hence for any $a \in A$, $\mathbf{M}(a,1) = ([a,1], \oplus_a, \neg_a, a)$ is an MV-algebra, where $x \oplus_a y := (x \circ a) \circ y$ and $\neg_a x := x \circ a$. As an immediate consequence we obtain:

PROPOSITION 1. ([3]) *Let* $\mathbf{M}$ *be an* MV*-algebra,* $a \in M$*. Define*

$$x \oplus_a y := \neg(\neg x \oplus a) \oplus y, \qquad \neg_a x := \neg x \oplus a.$$

Then $\mathbf{M}(a,1) = ([a,1], \oplus_a, \neg_a, a)$ *is an* MV*-algebra.*

MV-algebras in the sense of the above definition are *right-*MV*-algebras* since they are representable in the positive cones (the "right-hand side") of abelian lattice-ordered groups. But we can define a new binary operation $\odot$ via

$$x \odot y := \neg(\neg x \oplus \neg y),$$

which leads to the concept of *left-MV-algebras* that are obtained from the negative cones (the "left-hand side") of abelian lattice-ordered groups:

A *left-MV-algebra* is an algebra $\mathbf{M}' = (M, \odot, \neg, 1)$ of type $\langle 2,1,0\rangle$ satisfying the axioms

(MV1′) $(x \odot y) \odot z = x \odot (y \odot z)$,
(MV2′) $x \odot y = y \odot x$,
(MV3′) $x \odot 1 = x$,
(MV4′) $\neg\neg x = x$,
(MV5′) $x \odot \neg 1 = \neg 1$,
(MV6′) $\neg(\neg x \odot y) \odot y = \neg(\neg y \odot x) \odot x$.

Let $\mathbf{M}'$ be a left-MV-algebra and define $x * y := x \odot \neg y$. It is easily seen that this binary operation has the following properties:

(A1′) $x * 0 = x$, $0 * x = 0$,
(A2′) $x * (x * y) = y * (y * x)$,
(A3′) $(x * y) * z = (x * z) * y$.

A tedious but straightforward calculation yields the analogue of Proposition 1:

PROPOSITION 2. *Let $\mathbf{A} = (A, *, 0)$ be an algebra of type $\langle 2,0\rangle$ satisfying the identities (A1′), (A2′) and (A3′). Let $a \in A$. Define*

$$x \odot^a y := x * (a * y) \quad \text{and} \quad \neg^a x := a * x \qquad \text{for} \quad x, y \in [0, a].$$ [1]

Then $\mathbf{M}'(0,a) = \big([0,a], \odot^a, \neg^a, a\big)$ is a left-MV-algebra.

Proof. First observe that

$$x * (a * y) = \big(a * (a * x)\big) * (a * y) = \big(a * (a * y)\big) * (a * x) = y * (a * x),$$

which verifies (MV2′).

(MV1′):

$$\begin{aligned}(x \odot^a y) \odot^a z &= \big(x * (a * y)\big) * (a * z)\\ &= \big(y * (a * x)\big) * (a * z)\\ &= \big(y * (a * z)\big) * (a * x)\\ &= x * \Big(a * \big(y * (a * z)\big)\Big)\\ &= x \odot^a (y \odot^a z).\end{aligned}$$

(MV3′): By (A1′) and (A2′) we have $a * a = 0$, and hence

$$x \odot^a a = x * (a * a) = x * 0 = x.$$

[1] The *induced order* of $\mathbf{A}$ is defined by $x \le y$ if and only if $x * y = 0$.

(MV4′):
$$\neg^a\neg^a x = a * (a * x) = x\,.$$

(MV5′):
$$\neg^a a \odot^a x = (a * a) * (a * x) = 0 = \neg^a a\,.$$

(MV6′): We have
$$\neg^a(\neg^a x \odot^a y) \odot^a y = y * \Big(a * \Big(a * \big(y * \big(a * (a * x)\big)\big)\Big)\Big) = y * (y * x)$$
and similarly $\neg^a(\neg^a y \odot^a x) \odot^a x = x * (x * y)$. □

COROLLARY 3. *If* $\mathbf{M}' = (M, \odot, \neg, 1)$ *is a left-*MV*-algebra and* $a \in M$, *the upon defining* $x \odot^a y := x \odot \neg(a \odot \neg y)$ *and* $\neg^a x := a \odot \neg x$, *the structure* $\mathbf{M}'(0,a) = \big([0,a], \odot^a, \neg^a, a\big)$ *is a left-*MV*-algebra.*

P r o o f. If $\mathbf{M}'$ is a left-MV-algebra, then the algebra $\mathbf{A}(\mathbf{M}') = (M, *, 0)$ fulfils the above axioms, and so, by the proposition, $\mathbf{M}'(0,a)$ is a left-MV-algebra, where $x \odot^a y = x * (a * y) = x \odot \neg(a \odot \neg y)$ and $\neg^a x = a * x = a \odot \neg x$. □

COROLLARY 4. *Let* $\mathbf{M}$ *be an* MV*-algebra,* $a \in M$. *Define* $\mathbf{M}(0,a) = \big([0,a], \oplus^a, \neg^a, 0\big)$ *via* $x \oplus^a y := (x \oplus y) \wedge a$ *and* $\neg^a x := \neg(x \oplus \neg a)$. *Then* $\mathbf{M}(0,a)$ *is an* MV*-algebra.*

P r o o f. For any $x, y \in [0,a]$ we have
$$\begin{aligned} x \oplus^a y &= \neg^a(\neg^a x \odot^a \neg^a y) \\ &= a \odot \neg\big(a \odot \neg x \odot \neg\big(a \odot \neg(a \odot \neg y)\big)\big) \\ &= a \wedge \neg\big(\neg x \odot \neg(a \wedge y)\big) \\ &= a \wedge \neg(\neg x \odot \neg y) \\ &= a \wedge (x \oplus y) \end{aligned}$$
and $\neg^a x = a \odot \neg x = \neg(\neg a \oplus x)$. □

Combining Proposition 1 and Corollary 4, we get the promised description of interval MV-algebras:

THEOREM 5. *Let* $\mathbf{M}$ *be an* MV*-algebra and let* $a, b \in M$ *with* $a \leq b$, *where* $\leq$ *is the natural order of* $\mathbf{M}$. *Define* $\mathbf{M}(a,b) = \big([a,b], \oplus_a^b, \neg_a^b, a\big)$ *by*
$$x \oplus_a^b y := \big(\neg(\neg x \oplus a) \oplus y\big) \wedge b \qquad \textit{and} \qquad \neg_a^b x := \neg(x \oplus \neg b) \oplus a\,.$$
Then $\mathbf{M}(a,b)$ *is an* MV*-algebra.*

Proof. Let $x, y \in [a, b]$. Then

$$x \oplus_a^b y = (x \oplus_a y) \wedge b = (\neg(\neg x \oplus a) \oplus y) \wedge b$$

and

$$\begin{aligned}\neg_a^b x &= \neg_a(x \oplus_a \neg_a b) \\ &= \neg\big(\neg(\neg x \oplus a) \oplus \neg b \oplus a\big) \oplus a \\ &= \neg\big((x \vee a) \oplus \neg b\big) \oplus a \\ &= \neg(x \oplus \neg b) \oplus a\,.\end{aligned}$$

□

COROLLARY 6. *Let* $\mathbf{M}$ *be an* MV*-algebra and let* $a, b \in M$ *such that* $a \leq b$*, where* $\leq$ *is the natural order of* $\mathbf{M}$*. If* $a, b \in B(\mathbf{M})$*, then in the* MV*-algebra* $\mathbf{M}(a, b)$ *we have* $x \oplus_a^b y = x \oplus y$ *and* $\neg_a^b x = (\neg x \wedge b) \vee a$.

Proof. For any $x, y \in [a, b]$,

$$\begin{aligned}x \oplus_a^b y &= \big(\neg(\neg x \oplus a) \oplus y\big) \wedge b \\ &= \big(\neg(\neg x \vee a) \oplus y\big) \wedge b \\ &= \big((x \wedge \neg a) \oplus y\big) \wedge b \\ &= (x \oplus y) \wedge (\neg a \oplus y) \wedge b \\ &= x \oplus y\end{aligned}$$

since $\neg a \oplus y \geq \neg a \oplus a = 1$ and $x \oplus y \leq b \oplus b = b$. □

COROLLARY 7. *Let* $\mathbf{A} = (A, \circ, 1)$ *be an algebra satisfying* (A1), (A2) *and* (A3)*. Let* $a, b \in A$ *such that* $a \leq b$*, where* $\leq$ *is the induced order of* $\mathbf{A}$*. Define*

$$x \oplus_a^b y := \Big(\big(b \circ \big((x \circ a) \circ y\big)\big) \circ (b \circ a)\Big) \circ a \qquad \text{and} \qquad \neg_a^b x := (b \circ x) \circ a\,.$$

Then $\mathbf{M}(a, b) = \big([a, b], \oplus_a^b, \neg_a^b, a\big)$ *is an* MV*-algebra.*

Proof. The structure $\big([a, 1], \oplus_a, \neg_a, a\big)$ is an MV-algebra, where $x \oplus_a y = (x \circ a) \circ y$ and $\neg_a x = x \circ a$, and therefore, $\mathbf{M}(a, b)$ is an MV-algebra, where

$$\begin{aligned}x \oplus_a^b y &= (x \oplus_a y) \wedge b \\ &= \big((x \circ a) \circ y\big) \wedge b \\ &= \Big(\big(b \circ \big((x \circ a) \circ y\big)\big) \circ (b \circ a)\Big) \circ a\end{aligned}$$

as $[a, 1]$ is a lattice with $x \wedge y = \big((y \circ x) \circ (y \circ a)\big) \circ a$ for any $x, y \in [a, 1]$, and

$$\begin{aligned}\neg_a^b x &= \neg_a(x \oplus_a \neg_a b) \\ &= \big((x \circ a) \circ (b \circ a)\big) \circ a \\ &= \big(b \circ \big((x \circ a) \circ a\big)\big) \circ a \\ &= (b \circ x) \circ a\,.\end{aligned}$$

□

REFERENCES

[1] CIGNOLI, R. L. O.—D'OTTAVIANO, I. M. L.—MUNDICI, D.: *Algebraic Foundations of Many-valued Reasoning*, Kluwer Acad. Publ., Dordrecht-Boston-London, 2000.

[2] CHANG, C. C.: *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc. **88** (1958), 467–490.

[3] CHAJDA, I.—HALAŠ, R.—KÜHR, J.: *Distributive lattices with sectionally antitone involutions*, Acta Sci. Math. (Szeged) **71** (2005), 19–33.

[4] CHAJDA, I.—HALAŠ, R.—KÜHR, J.: *Implication in MV-algebras*, Algebra Universalis **52** (2004), 377-382.

[5] MUNDICI, D.: *Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus*, J. Funct. Anal. **65** (1986), 15–63.

Received May 24, 2004

Department of Algebra and Geometry
Faculty of Science
Palacký University
Tomkova 40
CZ-779 00 Olomouc
CZECH REPUBLIC

E-mail: chajda@inf.upol.cz
kuhr@inf.upol.cz