Jaroslav Nešetřil

Infinite precise objects

Mathematica Slovaca, Vol. 28 (1978), No. 3, 253--260

Persistent URL: http://dml.cz/dmlcz/130924

Math. Slovaca 28, 1978, No. 3, 253—260

INFINITE PRECISE OBJECTS

JAROSLAV NEŠETŘIL

Introduction

This paper was motivated by the following problem (see [2], [3]): Is it possible to construct for each integer $h \geqq 2$ two non-isomorphic graphs G and G' with the following properties:

1. G and G' have the same infinite number c of vertices;
2. for each pair (a, b) of different vertices of G (of G', respectively) there exists exactly one path of the legth $\leqq k$ joind a and b;
3. G and G' are not trees.

Such graphs are called infinite *tied* graphs of type (c, k) (or infinite Moore graphs). Tied graphs with finite number of vertices are just few and they have been almost completely characterized (see [1], [5], [11]). Using a simple construction we shall show that infinite tied graphs are as many as possible. This solves the above problem. Moreover we prove a stronger result.

As usually every ordinal number will be considered as the set of all smaller ordinals (ordinal numbers will be denoted by $\iota, \lambda, \ldots$). A cardinal number (usually denoted by c) will be identified with the smallest ordinal number of the same cardinality.

Theorem A: *For each integer $k \geqq 2$ and infinite cardinal number c there exists a family G_ι, $\iota < 2^c$ of tied graphs with parameters (c, k) such that there is no homomorphism $G_\iota \to G_\lambda$ for any $\iota \neq \lambda$.*

(This is much stronger than the fact that the graphs G_ι are non-isomorphic. A *homomorphism* $f: G = (V, E) \to G' = (V', R')$ is a mapping $f: V \to V'$ which satisfies $[f(x), f(y)] \in E'$ for every $[x, y] \in E$.)

This is proved in §2. The method of the proof is general enough to obtain similar results for tactical cinfigurations and friendship graphs. This is contained in §3.

During the final preparations of this paper I learned that a similar technique was used independently by V. Chvátal and A. Kotzig in a special case; their paper will be published in Canad. Math. Bull. (they constructed 2^c non-isomorphic friendship graphs with c vertices).

Finally, let us remark that the results of this paper may be interpreted negatively: infinite precise objects are not rare and there is such a "jump" between finite and infinite precise objects that one perhaps should not use the term "precise" for the latter.

§2. Proof of Theorem A

A *graph* $G=(V, E)$ means an undirected loopless graph; V, E may be infinite sets. A *digraph* G is a couple (V, R) where $R \subseteq V \times V$; both V and R may be infinite. The class of all digraphs will be denoted by **DiGra**.

Let $t \geqslant 3$ be a fixed natural number. Denote by $\mathbf{Cyc}(t)$ the class of all graphs without cycles of the lenght $\leqslant t$.

The chromatic number of a graph is denoted by $\chi(G)$. It is known by [7] that for every natural number n there exists a graph $G \in \mathbf{Cyc}(t)$ such that $\chi(G)=n$.

Let $k \geqslant 2$ be fixed. Put $t=2k$. Obviously every subgraph of a tied graph with parameters (c, k) belongs to the class $\mathbf{Cyc}(t)$. Denote by $\mathbf{Tied}(k)$ the class of all tied graphs with parameters (c, k) for some c. Therefore $\mathbf{Tied}(k) \subseteq \mathbf{Cyc}(t)$.

Now we consider the opposite procedure: We start with the class $\mathbf{Cyc}(t)$ and we modify (in a canonical way) each graph from $\mathbf{Cyc}(t)$ into a graph from $\mathbf{Tied}(k)$. The construction is as follows:

Let $G=(V, E) \in \mathbf{Cyc}(t)$. Put $G_0=(V_0, E_0)=G$.

Let $G_n=(V_n, E_n) \in \mathbf{Cyc}(t)$ be defined.

For each pair x, y of vertices of G_n with $d_{G_n}(x, y)>k (d_{G_n}(x, y)$ is the distance of x and y in the graph $G_n)$ adjoin a path $P(x, y)$ from x to y of the length k in such a way that, with a possible exclusion of endvertices, these paths are mutually disjoint and have no vertices common with V_n.

Do this simultaneously for all pairs $\{x, y\} \subseteq V_n$ with $d_{G_n}(x, y)>k$. Call the resulting graph G_{n+1}. Clearly $G_n \subseteq G_{n+1}$. Put $\lim G=\cup(G_n ; n \in \mathbf{N})$. Let $\iota_G : G \rightarrow \lim G$ be the naturally defined inclusion.

The construction of $\lim G$ has first appeard in [3] and has been studied in [2].

The following two propositions give a basic information on the homomorphism properties of the class $\mathbf{Tied}(k)$:

Proposition 1: *Let $G, G' \in \mathbf{Tied}(k)$. Then every homomorphism f: $G \rightarrow G'$ is a distance preserving monomorphism (i.e. f is $1-1$ and $d_{G'}(f(x), f(y))=d_G(x, y)$ for all x, y).*

Let G, G' be graphs. a homomorphism f: $G \rightarrow G'$ is called *k-distance preserving if $d_G(x, y) = d_{G'}(f(x), f(y))$* whenever $d_G(x, y) \leqslant k$ and $d_{G'}(f(x), f(y))>k$ whenever $d_G(x, y)>k$.

Propostition 2: *Let $2k=t$, $k \geqslant 2$. Let $G \in \mathbf{Cyc}(t)$, $G' \in \mathbf{Cyc}(t)$. Then for every k-distance preserving homomorphism f: $G \rightarrow G'$ there exists exactly one homo-*

morphism $f'\colon \lim G \to \lim G'$ *such that* $f' \circ \iota_G = \iota_{G'} \circ f$. *The homomorphism* f' *will be denoted by* $\lim f$.

$$\begin{tikzcd} G & G' \arrow[l, "f"'] \\ \lim G \arrow[u, leftarrow, "\iota_G"] \arrow[r, "f'"] & \lim G' \arrow[u, leftarrow, "\iota_{G'}"'] \end{tikzcd}$$

Proof of Proposition 1: Let G, G', $f\colon G \to G'$ be given. Assume that $d_{G'}(f(x), f(y)) \neq d_G(x, y)$, consequently $d_{G'}(f(x), f(y)) < d_G(x, y)$. But then the unique path $P(x, y)$ of the length $\leqslant k$ which joins x and y in G has to be mapped into the unique path $P(f(x), f(y))$ of the length $\leqslant k$ which joins $f(x)$ and $f(y)$ in G'. As f is not distance preserving this implies that f is not $1-1$. Assume that $f(x) = f(y)$ for the vertices $x \neq y$. It can be easily seen that x and y are contained in a cycle C of the length $2k+1$. As the homomorphic image of an odd cycle contains an odd cycle and as $G' \in \mathbf{Cyc}(2k)$ we get a contradiction.

Proof of Proposition 2: Let $f\colon G \to G'$ be a k-distance preserving monomorphism. Define $\lim f = f'$ by: $f'|_{V_0} = f$; let $f'|_{V_1 - V_0}$ be defined by the condition $f'(P_{G_1}(x, y)) = P_{G_1'}(f(x), f(y))$ for every pair x, y satisfying $d_{G_0}(x, y) > k$ (as f is k-distance preserving it is also $d_{G_0'}(f(x), f(y)) > k$), where $P_{G_1}(x, y)$ the unique path of the length k in G_1 which connects x and y and analogously $P_{G_1}'(f(x), f(y))$. It can be verified that $f'|_{V_1}\colon G_1 \to G_1'$ is again k-distance preserving and consequently this procedure may be carried on by induction. This gives then definition of $f'\colon \lim G \to \lim G'$. It is easy to verify that f' is uniquely determined by f.

In the language of the category theory Propositions 1 and 2 mean that the category $\mathbf{Tied}(k)$ (with all homomorphisms as its morphisms) is a reflexive subcategory of the category $\mathbf{Cyc}(k)$ (considered with all k-distance preserving homomorphisms as its morphisms).

It is very important that the statement of Proposition 2 can be often reversed. In many cases for every homomorphism g: $\lim G \to \lim G'$, G, $G' \in \mathbf{Cyc}(2k)$ there exists a homomorphism $f\colon G \to G'$ such that $g = \lim f$. (Of course this is not always the case. It suffices to consider $G = G' =$ a discrete graph with at least 5 vertices.)

This reversed procedure may be established as follows:

1. Let $H = (W, F) \in \mathbf{Cyc}(t)$ be a finite graph with the following properties:
(i) $\chi(H) \geqslant 4$;
(ii) H is a vertex critical graph;
(iii) there are $a, b \in W$ such that $d_H(a, b) \geqslant k$.
By [7] such a graph exists.

2. Put $|W| = n$. We prove that for each infinite cardinal number $\mathbf{c}$ there exists a family H_ι, $\iota < 2^{\mathbf{c}}$, of digraphs with the following properties:

(i) $|H_\iota| \doteq \mathbf{c}$, $\iota < 2^{\mathbf{c}}$;

(ii) there exists no homomorphism $H_\iota \to H_\lambda$ for all $\iota \neq \lambda$, $\iota, \lambda < 2\mathbf{c}$;

(iii) the undirected graphs which arise from H_ι, $\iota < 2^{\mathbf{c}}$ (by destroying orientations of edges) belong to **Cyc**(n).

This follows from the calculus of rigid graphs: By [13] there exists a directed graph (X, R), $R \subseteq X \times X$, which satisfies $|X| = \mathbf{c}$ and for which there exists no non-identical homomorphism of (X, R) into itself — such graphs are called rigid. By [8], [9] there exists an undirected graph $\bar{H}$ with $\mathbf{c}$ vertices which is rigid and it may be further assumed that $\bar{H} \in$ **Cyc**(n). We may put $\bar{H} = (X, R)^*\tilde{H}$, where * is the šíp-construction (see step 3 bellow) and $\tilde{H} \in$ **Cyc**(n) is the rigid graph depicted in Fig. 1 (see [8], [10], [12]).

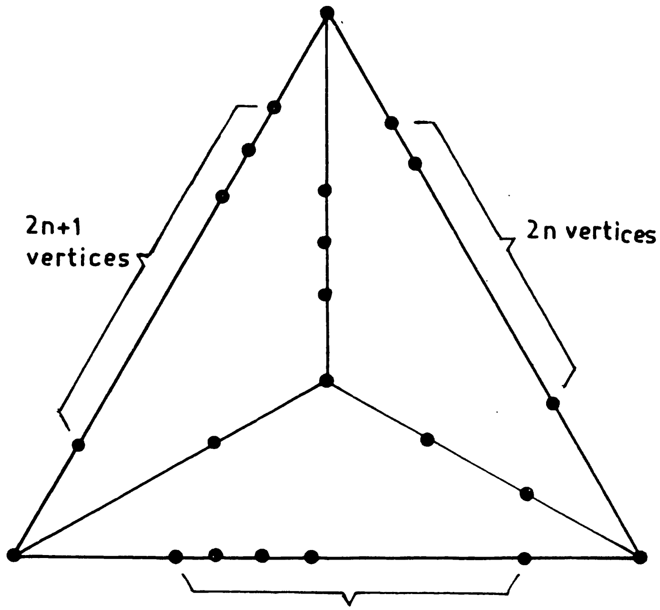

Fig. 1

Now it suffices to consider all orientations H_ι, $\iota < 2^{\mathbf{c}}$, of the graph $\bar{H}$. It is easy to see that this family of digraphs satisfies the above conditions.

3. Define the following construction (a concrete modification of the "šíp product" [8]): For a directed loopless graph $G' = (V', E')$ and an undirected graph $H = (W, F)$ with two fixed vertices a, b denote by $G^*(H, a, b)$ the following graph:

Put $\tilde{V}=E'\times W$ and let $\sim$ be equivalence on $\tilde{V}$ defined by the condition: $((x, y), z)\sim((x', y'), z')$ if and only if one of the following conditions holds:

i. $(x, y)=(x', y')$, $z=z'$;
ii. $y=x'$, $z=b$, $z'=a$;
iii. $x=y'$, $z=a$, $z'=b$;
iv. $x=x'$, $z=z'=a$;
v. $y=y'$, $z=z'=b$.

Denote by V the set of all equivalence classes of $\tilde{V}$ and define E by $E=\{[[x], [y]];\ x=(e, x'),\ y=(e, y'),\ [x, y]\in F\}$, where $[x]$ is the equivalence class of $\sim$ containing x.

(V, E) is the graph $G^*(H, a, b)$.

As is explained in [8], [9], the graph $G^*(H, a, b)$ is the graph G with each arrow replaced by a copy of H.

4. Put $G_\iota=H_\iota{}^*(H, a, b)$ for each $\iota<2^{\mathfrak{c}}$, where H_ι, $\iota<2^{\mathfrak{c}}$, is given in step 2 and (H, a, b) is given in step 1.

Claim: For each $\iota, \lambda<2^{\mathfrak{c}}$, $\iota\neq\lambda$, there exists no $1-1$ homomorphism $G_\iota\to G_\lambda$.

Proof: Let $f\colon G_\iota\to G_\lambda$. as $G_\iota=H_\iota{}^*(H, a, b)$ and H_ι is an orientation of the graph $\bar{H}\in\mathbf{Cyc}(n)$, $n=|V(H)|=|W|$ it follows that each cycle of the length $\leqslant n$ is contained in a "copy" the graph H in G_ι. It follows that f has to map copies of H in G_ι into copies of H in G_λ. Consequently f induces a monomorphism $f'\colon H_\iota\to H_\lambda$, which is a contradiction with step 2.

5. Now consider the family $\lim G_\iota$, $\iota<2^{\mathfrak{c}}$. Obviously $\lim G_\iota$ has $\mathfrak{c}$ vertices for each $\iota<2^{\mathfrak{c}}$. In order to finish the proof of Theorem A it suffices to prove:

Claim: Let $\iota\neq\lambda$, $\iota, \lambda<2^{\mathfrak{c}}$. Then there exists no homomorphism $f\colon \lim G_\iota\to \lim G_\lambda$.

Proof: Let $f\colon \lim G_\iota\to\lim G_\lambda$ be a homomorphism, by Proposition 1 f is $1-1$. One can easily prove $\chi(\lim G_\iota)=\chi(G_\iota)$ and for each subgraph $G^*\subseteq\lim G_\iota$, $\chi(G^*)=4$ there holds $\chi(G^*)=\chi(G^*\cap G)$. But H (see step 1) was a colour critical graph; consequently f maps each copy of H which has to belong to G_ι into a copy of H in G_λ. Thus f induces a monomorphism $\bar{f}\colon G_\iota\to G_\lambda$, which is a contradition with step 4. This proved Theorem A.

Remark: The assumption $\chi(H)=4$ in 1 is not necessary. One can lower it to $\chi(H)=3$ (which of course is the best possible one) by a more elaborate construction of rigid graphs. The aim of this paper is to apply well-known constructions.

It is also possible to prove that for every group $\mathscr{G}$ there exists a graph $G\in\mathbf{Tied}(k)$ such that the group Aut G of all automorphisms of G is isomorphic to $\mathscr{G}$ and other similar results (e.g. for the monoid of endomorphisms).

§3. Other structures

3.1. A friendship graph $G=(V, E)$ is defined by the following condition: for every pair x, y of different vertices of G there exists exactly one vertex z such that $[x, z] \in E$ and $[y, z] \in E$. A complete description of finite friendship graphs is given in [6]: on each set with an odd number of vertices there exists exactly one such graph and there is no such graph with an even number of vertices. In the infinite case the situation is entirely different:

Theorem B. *On each infinite set of cardinality* $\boldsymbol{c}$ *there exists a family* G_ι, $\iota < 2^{\boldsymbol{c}}$, *of friendship graphs such that given* $\iota \neq \lambda < 2^{\boldsymbol{c}}$ *there exists no monomorphism* $f: G_\iota \to G_\lambda$.

Outline of a proof of Theorem B: The proof is very similar to the above proof of Theorem A, therefore, we only stress the differences between these two proofs.

As regards step 1, it suffices to take a rectangle free graph wich contains each edge in a cycle and which does not contain a vertex of degree 2 (this may be established, e.g., again via a chromatic number). Steps 2, 3, and 4 remain unchanged.

As regard step 5, we have to define the construction of $\lim G$. As can be expected this is the "firendship" version of the above procedure: Put $G = G_0 = (V_0, E_0)$. Given $G_n = (V_n, E_n)$ we define G_{n+1} by adjoining a path of the length 2 a to any pair of vertices of G_n which was not in G_n connected by a path of the length 2. One can check that $\lim G = \bigcup(G_n ; n \in \mathbf{N})$ is a friendship graph. Using this we prove Theorem B quite analogously to the proof above.

3.2. Let k, p, λ be positive integer, $k \geqslant p \geqslant 2$. Let $\boldsymbol{c}$ be a cardinal number.

A *k-graph* is a couple (V, E) where $e \subseteq V$, $|e| = k$ for any $e \in E$. The elements of E are called edges.

A *tactical configuration* with parameters $(\boldsymbol{c}, k, \lambda, p)$ is a k-graph (V, E), $|V| = \boldsymbol{c}$, with the property that each p-element subset P of V belongs to exactly λ edges thus $(p \leqslant k)$.

It is difficult to construct a finite tactical configuration with given parameters (this is the basic question of the theory of block designs and related subjects). We prove:

Theorem C. *Let* $\boldsymbol{c}$ *be an infinite cardinal. For every* k, λ, *and* p *there exists* $2^{\boldsymbol{c}}$ *tactical configurations of the type* $(\boldsymbol{c}, k, \lambda, p)$ *such that there is no monomorphism between then.*

The proof is again very similar to the above one. The crucial point is the following:

Let $G = (V, E)$ be a k-graph which satisfies $|e \cap e'| \leqslant 1$ whenever $e \neq e' \in E$. Let $|V| = \boldsymbol{c}$, and let k, λ, p be fixed. Define a tactical configuration $\lim G$ with the parameters $\boldsymbol{c}, k, \lambda. p$ as follows: Put $G_0 = (V_0, E_0) = G$. Let $G_n = (V_n, E_n)$ be given. For any p-element subset P of V_n determine the number $\lambda(P)$ of edges of E_n which contain P. If $\lambda(P)<\lambda$, then adjoin to G_n $\lambda-\lambda(P)$ edges which contain P and otherwise are disjoint. Do this simultaneously for all p-element subsets of V_n and call the resulting k-graph G_{n+1}. It may be veryfied that $\lim G = \bigcup(G_n ; n \in \mathbf{N})$ is a $(\boldsymbol{c}, k, \lambda, p)$ tactical configuration.

Now let $\boldsymbol{c}$, k, and p be fixed. It suffices to start from a tactical configuration H with the parameters $(n, k, 1, 2)$ for a convenient finite $n>k$ (thus H is a nontrivial block design; it is known that such an n exists). As in the proof of Theorem A we obtain oriented graphs H_ι, $\iota<2^c$, and by the šíp-construction we define the k-graphs $G_\iota=H_\iota*(H, a, b)$, where a, b are arbitrary different vertices of H. The šíp-construction for the k-graphs is defined quite analogously to that for graphs. It can be proved (and this is simpler than in Theorem A) that the $(\boldsymbol{c}, k, \lambda, p)$ tactical configurations $\lim G_\iota$, $\iota<2^c$, are non-isomorphic and that there is no monomorphism between them. We leave out the details.

Remark: The method of this paper may be further applied to „precise objects" which are defined by a local condition.

A further example is provided by "infinite strongly regular graphs" [4], graphs where each two vertices have exactly k common neighbours.

Finally, let us remark that Theorems B, C may be proved with "homomorphisms" instead of "monomorphisms" (see the above statements) by using a refinement of the above proofs.

REFERENCES

[1] BANNAI, E.—ITO, T.: On finite Moore graphs. J. Fac. Sci. Univ. Tokyo Sect. I-A 20, 1973, 191—208.

[2] BOSÁK, J.: On the k-index of graphs. Discrete Math., 1, 1971, 133—146.

[3] BOSÁK, J.—KOTZIG, A.—ZNÁM, Š.: Strongly geodetic graphs. J. Comb. Th., 5, 1968, 170—176.

[4] BOSE, R. C.: Strongly regular graphs, partial geometries and partially balanced designs. Pacific J. Math., 13, 1963, 389—419.

[5] DAMERELL, R. M.: On Moore graphs. Proc. Cambridge Philos. Soc., 74, 1973, 227—236.

[6] ERDÖS, P.—RÉNYI, A.—SÓS, V. T.: On a problem of graph theory. Studia Sci. Math. Hunger, 1, 1966, 215—236.

[7] ERDÖS, P.: Graph theory and probability. Canad. J. Math., 11, 1959, 34—38.

[8] HEDRLÍN, Z.—PULTR, A.: Symmetric relations (undirected graphs) with given semigroups. Monatsh. Math., 69, 1965, 318—322.

[9] HELL, P.—NEŠETŘIL, J.: Graphs and k-societes. Canad. Math. Bull., 13, 1970, 375—381.

[10] HELL, P.—NEŠETŘIL, J.: On edge sets of rigid and corigid graphs to appear in Math. Nachr.

[11] HOFFMAN, A. J.—SINGLETON, R. R.: On Moore graphs with diameters 2 and 3. IBM J. Res. Develop., 4, 1960, 497—504.

[12] NEŠETŘIL, J.: On symmetric and antisymmetric relations. Monath. Math., 76, 1972, 323—327.

[13] VOPĚNKA, P.—PULTR, A.—HEDRLÍN, Z.: A rigid relation exists on any set. Comment. Math. Univ. Carolinae, 6, 1965, 149—155.

Received August 18, 1976

Katedra základnej a aplikovanej algebry
mat.-fyz. fakulty Karlovej Univerzity
Sokolovská 83
186 00 Praha

БЕСКОНЕЧНЫЕ ТОЧНЫЕ ОБЪЕКТЫ

Ярослав Нешетршил

Резюме

В частности, в работе доказывается, что для каждой бесконечной мощности α существует 2^α взаимно не изоморфных

(1) графов Мура с параметрами (α, k), $k \geq 2$:

(2) тактических конфигураций с параметрами (α, k, λ, p).

Предлагается метод для доказательства аналогичных теорем.