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SPECIAL STRUCTURES OF MIXED LINEAR MODELS WITH NUISANCE PARAMETERS

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Introduction

A mixed linear model is characterized by a triple $(\boldsymbol{Y}, \mathbf{X}\boldsymbol{\beta}, \Sigma(\boldsymbol{\vartheta}))$; $\boldsymbol{Y}$ is an n-dimensional random vector, $\mathbf{X}$ a known $n \times k$ matrix, $\boldsymbol{\beta}$ an unknown k-dimensional vector, $\boldsymbol{\beta} \in \mathscr{R}^k$ (k-dimensional Euclidean space), $\Sigma(\boldsymbol{\vartheta})$ a covariance matrix of the random vector $\boldsymbol{Y}$, $\Sigma(\boldsymbol{\vartheta}) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$. The symmetric matrices $\mathbf{V}_1, \ldots, \mathbf{V}_p$ are known, the vector $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_p)'$ of variance components is unknown, $\boldsymbol{\vartheta} \in \underline{\vartheta} \subset \mathscr{R}^p$ and the topological interior of the set $\underline{\vartheta}$ is not empty.

In the following the vector $\boldsymbol{\beta}$ is usually considered in the form $\boldsymbol{\beta} = (\boldsymbol{\Theta}', \boldsymbol{\varkappa}')'$, where $\boldsymbol{\Theta}$ is a k_1-dimensional vector of necessary parameters and $\boldsymbol{\varkappa}$ is a k_2-dimensional vector of nuisance parameters. Therefore the matrix $\mathbf{X}$ is usually written in the form $(\mathbf{A}, \mathbf{S})$, where $\mathbf{A}$ corresponds to the necessary parameters and $\mathbf{S}$ to the nuisance parameters.

The model $(\boldsymbol{Y}, \mathbf{X}\boldsymbol{\beta}, \Sigma(\boldsymbol{\vartheta}))$ is called regular if the rank of the matrix $\mathbf{X}$ is $R(\mathbf{X}) = k_1 + k_2 = k$ (i.e. $\mathscr{M}(\mathbf{A}) \cap \mathscr{M}(\mathbf{S}) = \{\mathbf{0}\}$, where $\mathscr{M}(\mathbf{A})$ denotes the column space of the matrix $\mathbf{A}$ and $\mathscr{M}(\mathbf{S})$ is of analogous meaning) and the set $\underline{\vartheta}$ possesses the property $\boldsymbol{\vartheta} \in \underline{\vartheta} \Rightarrow \Sigma(\boldsymbol{\vartheta})$ is positive definite.

Two typical situations occur in the process of estimating the parameter $\boldsymbol{\beta}$. Either there exists the uniformly best linear unbiased estimator of $\boldsymbol{\beta}$, when the values of the parameters $\vartheta_1, \ldots, \vartheta_p$ are not required to be known for obtaining the mentioned estimator of $\boldsymbol{\beta}$ or knowledge of some of them is required for obtaining a locally best linear unbiased estimator of $\boldsymbol{\beta}$. In the latter case the variance components have to be estimated before estimating the parameter $\boldsymbol{\beta}$.

As far as possible preference is given to invariant estimators (realizations of such an estimator do not depend on $\boldsymbol{\beta}$). However, in the former case the parameters $\vartheta_1, \ldots, \vartheta_p$ need not be known for estimating the $\boldsymbol{\beta}$, their knowledge is necessary for determining the covariance matrix of its estimator.

As in the model $(\boldsymbol{Y}, \mathbf{X}\boldsymbol{\beta}, \Sigma(\boldsymbol{\vartheta}))$ the observation vector $\boldsymbol{Y}$ (according to the assumption) is influenced by the nuisance parameter $\boldsymbol{\varkappa}$, the transformed model

($\mathbf{T}Y$, $\mathbf{A}\Theta$, $\mathbf{T}\Sigma(\vartheta)\mathbf{T}'$) is considered, where $\mathbf{T}$ is a transformation (elimination) matrix with the properties $\mathbf{TA} = \mathbf{A}$ and $\mathbf{TS} = \mathbf{0}$.

Moreover, the transformation by the matrix $\mathbf{T}$ has to preserve full information on the necessary parameter Θ and the necessary variance components, i.e. the estimators of Θ obtained from the original and the transformed model (irrespective of their being determined without means of the variance components in the former case or with means of some of them in the latter case) have to be the same and simultaneously the estimators of the covariance matrix characterizing the obtained estimators of Θ have to be the same.

Therefore it is reasonable to seek for such structures of mixed linear models which ensure the existence of the transformation mentioned, or to investigate the influence of a given structure of a model on some obtained estimators.

In the following the normality of the random vector Y is assumed.

1. Definitions and auxiliary statements

The mean value of a random variable (vector) ξ under a given parameter of its probability distribution β is denoted $E(\xi|\beta)$; analogously $\mathrm{Var}(\xi|\vartheta)$ denotes its dispersion (covariance matrix).

Definition 1.1. *The* ϑ_0-LMVQUIE (*the locally minimum variance quadratic unbiased invariant estimator*) *of a function* $g(\vartheta) = \mathbf{f}'\vartheta$, $\vartheta \in \underline{\vartheta}$, *is* $Y'\mathbf{U}Y$, $\mathbf{U} = \mathbf{U}'$ *if*

(1) $\forall\{\beta \in \mathscr{R}^k\}\ (Y - \mathbf{X}\beta)'\mathbf{U}(Y - \mathbf{X}\beta) = Y'\mathbf{U}Y$ (*invariance*),

(2) $\forall\{\vartheta \in \underline{\vartheta}\}\ E(Y'\mathbf{U}Y|\vartheta) = \mathbf{f}'\vartheta$ *and*

(3) $\forall\{\mathbf{W} = \mathbf{W}' : \mathbf{W}$ *fulfils* (1)&(2)$\}$ $\mathrm{Var}(Y'\mathbf{U}Y|\vartheta_0) \leq \mathrm{Var}(Y'\mathbf{W}Y|\vartheta_0)$.

In what follows $\hat{\vartheta}(Y, \vartheta_0)$ *denotes the* ϑ_0-LMVQUIE of ϑ.

Definition 1.2. *The* ϑ_0-LBLUE (*the locally best linear unbiased estimator*) *of a function* $h(\beta) = \mathbf{p}'\beta$, $\beta \in \mathscr{R}^k$, *is* $\mathbf{L}'Y$ *if*

(1) $\forall\{\beta \in \mathscr{R}^k\}\ E(\mathbf{L}'Y|\beta) = \mathbf{p}'\beta$ *and*

(2) $\forall\{\mathbf{s} \in \mathscr{R}^n$: $\mathbf{s}$ *fulfils* (1)$\}$ $\mathrm{Var}(\mathbf{L}'Y|\vartheta_0) \leq \mathrm{Var}(\mathbf{s}'Y|\vartheta_0)$.

The symbol $\hat{\beta}(Y, \vartheta_0)$ *denotes the* ϑ_0-LBLUE *of* β; *the* UBLUE *means the uniformly* (*with respect to* ϑ) *best linear unbiasead estimator.*

The generalized Moore-Penrose inverse of a matrix $\mathbf{X}$ is denoted $\mathbf{X}^+(\mathbf{XX}^+\mathbf{X} = \mathbf{X}, \mathbf{X}^+\mathbf{XX}^+ = \mathbf{X}^+, \mathbf{XX}^+ = (\mathbf{XX}^+)'$ (transposition), $\mathbf{X}^+\mathbf{X} = (\mathbf{X}^+\mathbf{X})')$; $\otimes$ means the Kronecker multiplication of matrices; let $\{\mathbf{S}\}_{i,j} = S_{i,j}$ be the (i, j)th element of $\mathbf{S}$ and $\mathbf{S} = \mathbf{S}'$, then $\mathrm{vech}(\mathbf{S}) = (S_{1,1}, S_{1,2}, \ldots, S_{1,n}; S_{2,2}, S_{2,3}, \ldots, S_{2,n}; \ldots, S_{n-1,n-1}, S_{n-1,n}; S_{n,n})'$. If $\mathbf{N}$ is a positive semidefinite $n \times n$ matrix and $\mathbf{A}$ an arbitrary $n \times k$ matrix, $\mathscr{M}(\mathbf{A}) \subset \mathscr{M}(\mathbf{N})$, then $\mathbf{P}_{\mathbf{A}}^{\mathbf{N}} = \mathbf{A}(\mathbf{A}'\mathbf{NA})^+\mathbf{A}'\mathbf{N}$, $\mathbf{M}_{\mathbf{A}}^{\mathbf{N}} =$

$= \mathbf{I} - \mathbf{P}_\mathbf{A}^\mathbf{N}$ ($\mathbf{I}$ is an identity matrix); for $\mathbf{N} = \mathbf{I}$ the notation $\mathbf{P}_\mathbf{A}$ is used, i.e. $\mathbf{P}_\mathbf{A} = \mathbf{P}_\mathbf{A}^\mathbf{I}$.

Lemma 1.3. *Let* $(Y, (\mathbf{A}, \mathbf{S})\begin{pmatrix}\Theta\\ \varkappa\end{pmatrix}, \Sigma(\vartheta))$ be a regular model. If an elimination matrix $\mathbf{T}$ (i.e. $\mathbf{TA} = \mathbf{A}, \mathbf{TS} = \mathbf{O}$) *has a form* $\mathbf{I} - \mathbf{SC}$, *then* $\hat{\beta}(Y, \vartheta_0) = \hat{\beta}(\mathbf{T}Y, \vartheta_0)$, *where* $\hat{\beta}(Y, \vartheta_0)$ *is the* ϑ_0-LBLUE *of* β *in the original model and* $\hat{\beta}(\mathbf{T}Y, \vartheta_0)$ *is the* ϑ_0-LBLUE *in the transformed model* $(\mathbf{T}Y, \mathbf{A}\Theta, \mathbf{T}\Sigma(\vartheta)\mathbf{T}')$.

Proof. Cf. Corollary 2.4 in [3].

In the framework of the considered model $(Y, X\beta, \Sigma(\vartheta))$ the symbol $\mathbf{K}_0$ denotes a $p \times p$ matrix whose (i, j)th element is $\{\mathbf{K}_0\}_{i,j} = \mathrm{Tr}(\mathbf{V}_i\mathbf{V}_j)$ and $\mathbf{K}^{(I)}$ denotes a $p \times p$ matrix with the (i, j)th element $\{\mathbf{K}^{(I)}\}_{i,j} = \mathrm{Tr}(\mathbf{M}_\mathbf{X}\mathbf{V}_i\mathbf{M}_\mathbf{X}\mathbf{V}_j)$, $i, j = 1, \ldots, p$.

Lemma 1.4. *If the matrices* $\mathbf{V}_1, \ldots, \mathbf{V}_p$ *are linearly independent, then* $\mathbf{K}_0$ *is regular.*

Proof. The matrix $\mathbf{K}_0$ is the Gram matrix of the p-tuple of the elements $\mathbf{V}_1, \ldots, \mathbf{V}_p$ in the Hilbert space of symmetric $p \times p$ matrices with the inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{Tr}(\mathbf{AB})$.

Lemma 1.5. Let the matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ be symmetric and p.s.d. Then $\mathcal{M}(\mathbf{S}_1, \mathbf{S}_2) = \mathcal{M}(\mathbf{S}_1 + \mathbf{S}_2)$.

Proof. See p. 126 in [7].

Lemma 1.6. The LMVQUIE of a function $g(\vartheta) = \mathbf{f}'\vartheta$, $\vartheta \in \underline{\vartheta}$, exists iff $\mathbf{f} \in \mathcal{M}(\mathbf{K}^{(I)})$.

Proof. Cf [8].

Lemma 1.7. *Let* $\mathbf{V}$ *be an arbitrary symmetric p.d. matrix of the type* $n \times n$ *and* $\mathbf{K} = \mathbf{V} - \mathbf{VA}(\mathbf{A}'\mathbf{VA})^{-1}\mathbf{A}'\mathbf{V}$, *where* $\mathbf{A}$ *is the matrix from the regular model* $(Y, (\mathbf{A}, \mathbf{S})\begin{pmatrix}\Theta\\ \varkappa\end{pmatrix}, \Sigma(\vartheta))$. *If* $\mathbf{T} = \mathbf{M}_\mathbf{S}^\mathbf{K}$, *then the model* $(\mathbf{M}_\mathbf{S}^\mathbf{K} Y, \mathbf{A}\Theta, \mathbf{M}_\mathbf{S}^\mathbf{K}\Sigma(\vartheta)\mathbf{M}_\mathbf{S}^{\mathbf{K}'})$ *enables us to construct the* ϑ_0-LBLUE *of* Θ *and the* ϑ_0-LMVQUIE *of each function* $g(\vartheta) = \mathbf{f}'\vartheta$, $\vartheta \in \underline{\vartheta}$, *possessing the* ϑ_0-LMVQUIE *in the original model. Moreover,* $\hat{\Theta}(Y, \vartheta_0) = \hat{\Theta}(\mathbf{T}Y, \vartheta_0)$ *and* $\widehat{\mathbf{f}'\vartheta}(Y, \vartheta_0) = \widehat{\mathbf{f}'\vartheta}(\mathbf{T}Y, \vartheta_0)$.

Proof. Cf. Theorems 2.1 and 2.2 in [5].

Definition 1.8. *An* $n \times m$ *matrix* $\mathbf{G}$ *is a minimum* $\mathbf{N}$*-seminorm g-inverse of an* $m \times n$ *matrix* $\mathbf{A}$ *if* $\mathbf{AGA} = \mathbf{A}\ \&\ \forall\{y \in \mathcal{M}(\mathbf{A})\}\forall\{x: \mathbf{A}x = y\}\ \|\mathbf{G}y\|_\mathbf{N} \le \|x\|_\mathbf{N}$. *Here* $\|x\|_\mathbf{N} = \sqrt{x'\mathbf{N}x}$ *and* $\mathbf{N}$ *is a p.s.d matrix of the type* $n \times n$. *The matrix* $\mathbf{G}$ *is denoted by* $\mathbf{A}^-_{m(\mathbf{N})}$.

Lemma 1.9. *Let* $\mathbf{N}$ *be an* $n \times n$ *p.s.d. symmetric matrix and* $\mathbf{A}$ *be an arbitrary* $m \times n$ *matrix.*

a) *If* $\mathcal{M}(\mathbf{A}') \subset \mathcal{M}(\mathbf{N})$, *then* $\mathbf{N}^{-}\mathbf{A}'(\mathbf{A}\mathbf{N}^{-}\mathbf{A}')^{-}$ *is a minimum* $\mathbf{N}$*-seminorm g-inverse of the matrix* $\mathbf{A}$.

b) *If* $R(\mathbf{A}_{m,n}) = m$ and $\mathcal{M}(\mathbf{A}') \subset \mathcal{M}(\mathbf{N})$, then $(\mathbf{A}\mathbf{N}^{-}\mathbf{A}')^{-} = (\mathbf{A}\mathbf{N}^{-}\mathbf{A}')^{-1}$.

Proof. a) Cf. [7].

b) $\mathcal{M}(\mathbf{A}') \subset \mathcal{M}(\mathbf{N}) \Leftrightarrow \exists \{\mathbf{E}_{n,m}\} \mathbf{A}' = \mathbf{N}\mathbf{E}$. Thus $\mathbf{A}\mathbf{N}^{-}\mathbf{A}' = \mathbf{E}'\mathbf{N}\mathbf{E}$ and $R(\mathbf{A}\mathbf{N}^{-}\mathbf{A}') = R(\mathbf{E}'\mathbf{N}\mathbf{E})$. As $\mathbf{N}$ is p.s.d., there exists $\mathbf{J}_{n,R(\mathbf{N})}$ such that $\mathbf{N} = \mathbf{J}\mathbf{J}'$. Thus $R(\mathbf{E}'\mathbf{N}\mathbf{E}) = R(\mathbf{E}'\mathbf{J}) = R(\mathbf{E}'\mathbf{N}) = R(\mathbf{A}) = m$.

Lemma 1.10. *Let* $\mathbf{S}|\,.\,|$ *be a* $p \times p$ *matrix, the* (i, j)*th element of which has the form* $\{\mathbf{S}|\mathbf{A}|\}_{i,j} = \mathrm{Tr}(\mathbf{A}\mathbf{V}_i\mathbf{A}\mathbf{V}_j)$, $i, j = 1, \ldots, p$; $\mathbf{A}$ *is an arbitrary* $n \times n$ *matrix. Then in the mixed linear model*

a) $\mathcal{M}(\mathbf{K}^{(I)}) = \mathcal{M}[\mathbf{S}|(\mathbf{M}_{\mathbf{X}}\Sigma_0\mathbf{M}_{\mathbf{X}})^{+}|]$;

b) $\mathcal{M}[\mathbf{K}_0(m-1) + \mathbf{K}^{(I)}] = \mathcal{M}[(m-1)\mathbf{S}|\Sigma_0^{-1}| + \mathbf{S}|(\mathbf{M}_{\mathbf{X}}\Sigma_0\mathbf{M}_{\mathbf{X}})^{+}|]$,

where $\Sigma_0 = \sum_{i=1}^{p} \vartheta_{0,i}\mathbf{V}_i$, $\boldsymbol{\vartheta}_0 = (\vartheta_{0,1}, \ldots, \vartheta_{0,p})' \in \underline{\boldsymbol{\vartheta}}$.

Proof. Cf [8].

Lemma 1.11. *Let* $\left(\boldsymbol{Y}, (\mathbf{A}, \mathbf{S})\begin{pmatrix}\boldsymbol{\Theta}\\ \boldsymbol{\varkappa}\end{pmatrix}, \Sigma(\boldsymbol{\vartheta})\right)$ *be a regular model. Let* $\mathbf{T} = \mathbf{I} - \mathbf{S}\mathbf{C}$, $\mathbf{T}\mathbf{A} = \mathbf{A}$, $\mathbf{T}\mathbf{S} = \mathbf{O}$. *Then* $\mathbf{T} = \mathbf{M}_{\mathbf{S}}^{\mathbf{K}} - \mathbf{S}\mathbf{U}\mathbf{M}_{\mathbf{A}}^{\mathbf{V}}\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\mathbf{M}_{\mathbf{A}}^{\mathbf{V}}$, *where* $\mathbf{V}$ *is an arbitrary but fixed symmetric* $n \times n$ *p.d. matrix,* $\mathbf{K} = \mathbf{V} - \mathbf{V}\mathbf{A}(\mathbf{A}'\mathbf{V}\mathbf{A})^{-1}\mathbf{A}'\mathbf{V}$ *and* $\mathbf{U}$ *is an arbitrary* $k_2 \times n$ *matrix. Moreover the matrix* $\mathbf{S}'\mathbf{K}\mathbf{S}$ *is regular.*

Proof. The regularity of $\mathbf{S}'\mathbf{K}\mathbf{S}$ is implied by Theorem 2.5 in [3]. As $\mathbf{S}$ is of the full rank in columns, $\mathbf{T}\mathbf{A} = \mathbf{A} \Leftrightarrow \mathbf{C}\mathbf{A} = \mathbf{O}$ and $\mathbf{T}\mathbf{S} = \mathbf{O} \Leftrightarrow \mathbf{C}\mathbf{S} = \mathbf{I}$; $\mathbf{C}\mathbf{A} = \mathbf{O} \Leftrightarrow \mathbf{C} = \mathbf{Z}\mathbf{M}_{\mathbf{A}}^{\mathbf{V}}$, where $\mathbf{V}$ is an arbitrary but fixed symmetric $n \times n$ p.d. matrix and $\mathbf{Z}$ is an arbitrary $k_2 \times n$ matrix. $\mathbf{Z}$ has to fulfil the equation $\mathbf{Z}\mathbf{M}_{\mathbf{A}}^{\mathbf{V}}\mathbf{S} = \mathbf{I}$ the solution of which is $\mathbf{Z} = (\mathbf{S}'\mathbf{K}\mathbf{S})^{-1}\mathbf{S}'\mathbf{V} + \mathbf{U}(\mathbf{I} - \mathbf{P}_{\mathbf{M}_{\mathbf{A}}^{\mathbf{V}}\mathbf{S}}^{\mathbf{V}})$, where $\mathbf{U}$ is an arbitrary $k_2 \times n$ matrix ($(\mathbf{S}'\mathbf{K}\mathbf{S})^{-1}\mathbf{S}'\mathbf{V}$ is a particular solution of the equation $\mathbf{Z}\mathbf{M}_{\mathbf{A}}^{\mathbf{V}}\mathbf{S} = \mathbf{I}$). As $\mathbf{P}_{\mathbf{M}_{\mathbf{A}}^{\mathbf{V}}\mathbf{S}}^{\mathbf{V}} = \mathbf{M}_{\mathbf{A}}^{\mathbf{V}}\mathbf{P}_{\mathbf{S}}^{\mathbf{K}}$, we obtain $\mathbf{C} = (\mathbf{S}'\mathbf{K}\mathbf{S})^{-1}\mathbf{S}'\mathbf{K} + \mathbf{U}(\mathbf{I} - \mathbf{M}_{\mathbf{A}}^{\mathbf{V}}\mathbf{P}_{\mathbf{S}}^{\mathbf{K}})\,.$ $.\,\mathbf{M}_{\mathbf{A}}^{\mathbf{V}} \Rightarrow \mathbf{S}\mathbf{C} = \mathbf{P}_{\mathbf{S}}^{\mathbf{K}} + \mathbf{U}(\mathbf{I} - \mathbf{M}_{\mathbf{A}}^{\mathbf{V}} + \mathbf{M}_{\mathbf{A}}^{\mathbf{V}}\mathbf{M}_{\mathbf{S}}^{\mathbf{K}})\mathbf{M}_{\mathbf{A}}^{\mathbf{V}} = \mathbf{P}_{\mathbf{S}}^{\mathbf{K}} + \mathbf{U}\mathbf{M}_{\mathbf{A}}^{\mathbf{V}}\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\mathbf{M}_{\mathbf{A}}^{\mathbf{V}} \Rightarrow \mathbf{T} =$ $= \mathbf{M}_{\mathbf{S}}^{\mathbf{K}} - \mathbf{S}\mathbf{U}\mathbf{M}_{\mathbf{A}}^{\mathbf{V}}\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\mathbf{M}_{\mathbf{A}}^{\mathbf{V}}$

Remark 1.12. As $\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\mathbf{M}_{\mathbf{A}}^{\mathbf{L}} = \mathbf{M}_{\mathbf{X}}^{\mathbf{V}}$ and $\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\mathbf{M}_{\mathbf{A}}^{\mathbf{V}} = \mathbf{M}_{\mathbf{S}}^{\mathbf{K}}$ (Cf. Theorem 2.5 in [3]), where $\mathbf{L} = \mathbf{V} - \mathbf{V}\mathbf{S}(\mathbf{S}'\mathbf{V}\mathbf{S})^{-1}\mathbf{S}'\mathbf{V}$, the elimination matrix $\mathbf{T}$ can be written in the form $\mathbf{T} = \mathbf{M}_{\mathbf{S}}^{\mathbf{K}} - \mathbf{S}\mathbf{U}\mathbf{M}_{\mathbf{A}}^{\mathbf{V}}\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}$.

Remark 1.13. In the following the transformation $\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}$ is considered only; namely, the term $\mathbf{S}\mathbf{U}\mathbf{M}_{\mathbf{A}}^{\mathbf{V}}\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\boldsymbol{Y}$ is of no use in the invariant estimation of the vector $\boldsymbol{\vartheta}$.

2. Structures generated by replications

The aim of this section is to study such structures of linear mixed models which ensure the existence of the ϑ_0-LMVQUIE of the whole vector ϑ. Then, with respect to Lemmas 1.7 and 1.11, there exists a suitable transformation $\mathbf{T} = \mathbf{M}_{\mathbf{S}}^{\mathbf{K}}$ ($\mathbf{K} = \mathbf{V} - \mathbf{V}\mathbf{A}(\mathbf{A}'\mathbf{V}\mathbf{A})^{-1}\mathbf{A}'\mathbf{V}$, $\mathbf{V}$ being an arbitrary symmetric p.d. matrix of the type $n \times n$) eliminating nuisance parameters. The basis for obtaining the explicit formulae is the following lemma.

Lemma 2.1. *Let in the mixed linear model* $(\boldsymbol{Y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}(\vartheta))$ (*it need not be regular*) $g(\vartheta) = \boldsymbol{f}'\vartheta$, $\vartheta \in \underline{\vartheta}$, *be a function such that* $\boldsymbol{f} \in \mathscr{M}(\mathbf{K}^{(I)})$. *Then there exists the* ϑ_0-LMVQUIE *of it and if* $\mathscr{M}(\mathbf{X}) \subset \mathscr{M}(\Sigma_0)$, *the estimator has the form* $\widehat{\boldsymbol{f}'\vartheta}(\boldsymbol{Y}, \vartheta_0) = \sum_{i=1}^{p} \lambda_i \boldsymbol{Y}'(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+\mathbf{V}_i(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+\boldsymbol{Y}$, *where* $\lambda = (\lambda_1, \ldots, \lambda_p)'$ *is a solution of the equation* $\mathbf{S}|(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+|\,\lambda = \boldsymbol{f}$.

Proof. Cf. [8].

Theorem 2.2. *Let* $\left(\underline{\boldsymbol{Y}}, (\underline{\mathbf{A}}, \underline{\mathbf{S}})\begin{pmatrix}\boldsymbol{\Theta}\\ \boldsymbol{\varkappa}\end{pmatrix}, \sum_{i=1}^{p} \vartheta_i\underline{\mathbf{V}}_i\right)$ *be an m-times replicated regular model* $\left(\boldsymbol{Y}, (\mathbf{A}, \mathbf{S})\begin{pmatrix}\boldsymbol{\Theta}\\ \boldsymbol{\varkappa}\end{pmatrix}, \sum_{i=1}^{p} \vartheta_i\mathbf{V}_i\right)$ (*i.e.* $\underline{\boldsymbol{Y}} = (\boldsymbol{Y}_1', \ldots, \boldsymbol{Y}_m')'$, $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m$ *are i.i.d. random vectors,* $\underline{\mathbf{A}} = \boldsymbol{1} \otimes \mathbf{A}$, $\boldsymbol{1} = (1, \ldots, 1)' \in \mathscr{R}^m$, $\underline{\mathbf{S}} = \boldsymbol{1} \otimes \mathbf{S}$, $\underline{\mathbf{V}}_i = \mathbf{I} \otimes \mathbf{V}_i$, $i = 1, \ldots, m$). *Let* $\mathbf{P}_m = \boldsymbol{1}\boldsymbol{1}'/m$, $\mathbf{M}_m = \mathbf{I} - \mathbf{P}_m$, $\mathbf{V}$ *be an arbitrary symmetric p.d.matrix of the type* $n \times n$, $\mathbf{K} = \mathbf{V} - \mathbf{V}\mathbf{A}(\mathbf{A}'\mathbf{V}\mathbf{A})^{-1}\mathbf{A}'\mathbf{V}$ *and* $\mathbf{T} = (\mathbf{P}_m \otimes \mathbf{M}_{\mathbf{S}}^{\mathbf{K}} + \mathbf{M}_m \otimes \mathbf{I})$, *i.e.* $\mathbf{T}\underline{\boldsymbol{Y}} = \boldsymbol{1} \otimes \mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\bar{\mathbf{Y}} + [(\boldsymbol{Y}_1 - \bar{\boldsymbol{Y}})', \ldots, (\boldsymbol{Y}_m - \bar{\boldsymbol{Y}})']'$, *where* $\bar{\boldsymbol{Y}} = (1/m) \sum_{i=1}^{m} \boldsymbol{Y}_i$. *Then*

a) $\hat{\boldsymbol{\Theta}}(\underline{\boldsymbol{Y}}, \vartheta_0) = \hat{\boldsymbol{\Theta}}(\mathbf{T}\underline{\boldsymbol{Y}}, \vartheta_0)$,
b) $\hat{\vartheta}(\underline{\boldsymbol{Y}}, \vartheta_0) = \hat{\vartheta}(\mathbf{T}\underline{\boldsymbol{Y}}, \vartheta_0)$.

Proof. a) The matrix $\mathbf{T} = \mathbf{I} - \underline{\mathbf{S}}[\underline{\mathbf{S}}'(\mathbf{I} \otimes \mathbf{K})\underline{\mathbf{S}}]^{-1}\underline{\mathbf{S}}'(\mathbf{I} \otimes \mathbf{K})$ is of the form $\mathbf{I} - \underline{\mathbf{S}}\mathbf{C}$. With respect to Lemma 1.3 this is sufficient for the validity of a).

b) The model $\left(\underline{\boldsymbol{Y}}, (\underline{\mathbf{A}}, \underline{\mathbf{S}})\begin{pmatrix}\boldsymbol{\Theta}\\ \boldsymbol{\varkappa}\end{pmatrix}, \sum_{i=1}^{p} \vartheta_i\underline{\mathbf{V}}_i\right)$ is regular. Therefore the matrix $\underline{\mathbf{K}}^{(I)}$ having in the replicated model the form $\underline{\mathbf{K}}^{(I)} = (m-1)\mathbf{K}_0 + \mathbf{K}^{(I)}$ ($\mathbf{K}_0$, $\mathbf{K}^{(I)}$ being matrices for the model $\left(\boldsymbol{Y}, (\mathbf{A}, \mathbf{S})\begin{pmatrix}\boldsymbol{\Theta}\\ \boldsymbol{\varkappa}\end{pmatrix}, \sum_{i=1}^{p} \vartheta_i\mathbf{V}_i\right)$) is regular with respect to Lemmas 1.4 and 1.5. With respect to Lemma 1.6 the $\hat{\vartheta}(\underline{\boldsymbol{Y}}, \vartheta_0)$ exists. If $\underline{\mathbf{V}} = \mathbf{I} \otimes \mathbf{V}$, then $\underline{\mathbf{K}} = \mathbf{M}_m \otimes \mathbf{V} + \mathbf{P}_m \otimes \mathbf{K}$ and $\mathbf{T}$ can be expresed in the form $\mathbf{M}_{\underline{\mathbf{S}}}^{\underline{\mathbf{K}}}$, which with respect to Lemma 1.7 proves the assertion b.

Remark 2.3. The explicit expression for the ϑ_0-LBLUE of $\boldsymbol{\Theta}$ is

$$\hat{\boldsymbol{\Theta}}(\mathbf{T}\underline{\boldsymbol{Y}}, \vartheta_0) = \left[(\underline{\mathbf{A}}')^-_{m\left(\mathbf{M}_{\underline{\mathbf{S}}}^{\underline{\mathbf{K}}}\underline{\Sigma}_0\mathbf{M}_{\underline{\mathbf{S}}}^{\underline{\mathbf{K}}'}\right)}\right]'\mathbf{M}_{\underline{\mathbf{S}}}^{\underline{\mathbf{K}}}\underline{\boldsymbol{Y}} = \{(\boldsymbol{1}' \otimes \mathbf{A}')\,[\mathbf{P}_m \otimes (\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}\Sigma_0\mathbf{M}_{\mathbf{S}}^{\mathbf{K}'}) +$$

$$+ \mathbf{M}_m \otimes \Sigma_0]^+ (\mathbf{1} \otimes \mathbf{A})\}^{-1} (\mathbf{1}' \otimes \mathbf{A}') [\mathbf{P}_m \otimes (\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ + \mathbf{M}_m \otimes \Sigma_0]^+ .$$

$$.\{\mathbf{1} \otimes \mathbf{M}_S^K \bar{\mathbf{Y}} + [(\mathbf{Y}_1 - \bar{\mathbf{Y}})', \dots, (\mathbf{Y}_m - \bar{\mathbf{Y}})']'\} = [\mathbf{A}'(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ \mathbf{A}]^{-1}.$$

$$. \mathbf{A}'(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ \bar{\mathbf{Y}};$$

Lemma 1.9 and the inclusion $\mathcal{M}(\mathbf{A}) \subset \mathcal{M}(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})$ were used here. The same expression $\hat{\Theta}(\underline{Y}, \vartheta_0) = (\mathbf{I}, \mathbf{O})(\underline{\mathbf{X}}'\underline{\Sigma}_0\underline{\mathbf{X}})^{-1}\underline{\mathbf{X}}'\underline{\Sigma}_0^{-1}\underline{Y}$ can be obtained for $\hat{\Theta}(\underline{Y}, \vartheta_0)$ in the model before its transformation (in the model after the transformation the matrix $\mathbf{A}'(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ \mathbf{A}$ of the dimension $k_1 \times k_1$ has to be inverted while a substantially larger matrix $\mathbf{X}'\Sigma_0^{-1}\mathbf{X}$ of the dimension $(k_1 + k_2) \times (k_1 + k_2)$ has to be inverted in the model before the transformation). Instead of $(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+$ the matrix $\Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{S}(\mathbf{S}'\Sigma_0^{-1}\mathbf{S})^{-1}\mathbf{S}'\Sigma_0^{-1}$ may be used for calculating $\hat{\Theta}(\mathbf{T}\underline{Y}, \vartheta_0)$.

The explicit expression for $\hat{\vartheta}(\underline{Y}, \vartheta_0)$ is of the form

$$\hat{\vartheta}(\underline{Y}, \vartheta_0) = [\mathbf{S}|\Sigma_0^{-1}|(m - 1) + \mathbf{S}|(\mathbf{M}_X \Sigma_0 \mathbf{M}_X)^+|]^{-1} \hat{\gamma},$$

$$\hat{\gamma} = (\hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_p)',$$

$$\hat{\gamma}_i = (m - 1) \operatorname{Tr}(\hat{\Sigma} \Sigma_0^{-1} \mathbf{V}_i \Sigma_0^{-1}) + m\bar{\mathbf{Y}}'(\mathbf{M}_X \Sigma_0 \mathbf{M}_X)^+ \bar{\mathbf{Y}}, \quad i = 1, \dots, p,$$

$$\hat{\Sigma} = [1/(m - 1)] \sum_{i=1}^{m} (\mathbf{Y}_i - \bar{\mathbf{Y}})(\mathbf{Y}_i - \bar{\mathbf{Y}})', \quad \bar{\mathbf{Y}} = (1/m) \sum_{i=1}^{m} \mathbf{Y}_i,$$

$$(\mathbf{M}_X \Sigma_0 \mathbf{M}_X)^+ = \Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{X}(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1} \quad \text{(Cf. [1])}.$$

The estimator $\hat{\vartheta}(\mathbf{T}\underline{Y}, \vartheta_0)$ can be expressed in the following way

$$\hat{\vartheta}(\mathbf{T}\underline{Y}, \vartheta_0) = [\mathbf{S}|(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ (m - 1) + \mathbf{S}|(\mathbf{M}_A \mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'} \mathbf{M}_A)^+|]^{-1} \hat{\gamma},$$

$$\hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_p)',$$

$$\hat{\gamma}_i = (m - 1) \operatorname{Tr}[\hat{\Sigma}(\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+ \mathbf{M}_S^K \mathbf{V}_i \mathbf{M}_S^{K'} (\mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'})^+] +$$

$$+ m(\mathbf{M}_S^K \bar{\mathbf{Y}})' (\mathbf{M}_A \mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'} \mathbf{M}_A)^+ \mathbf{M}_S^K \mathbf{V}_i \mathbf{M}_S^{K'} (\mathbf{M}_A \mathbf{M}_S^K \Sigma_0 \mathbf{M}_S^{K'} \mathbf{M}_A)^+ \mathbf{M}_S^K \bar{\mathbf{Y}},$$
$$i = 1, \dots, p;$$

the relationships $(\mathbf{M}_{\underline{X}} \underline{\Sigma}_0 \mathbf{M}_{\underline{X}})^+ = \mathbf{M}_m \otimes \Sigma_0^{-1} + \mathbf{P}_m \otimes (\mathbf{M}_X \Sigma_0 \mathbf{M}_X)^+$ and $(\mathbf{M}_{\underline{X}} \underline{\Sigma}_0 . \mathbf{M}_{\underline{X}})^+ \underline{\mathbf{V}}_i (\mathbf{M}_{\underline{X}} \underline{\Sigma}_0 \mathbf{M}_{\underline{X}})^+ = \mathbf{M}_m \otimes \Sigma_0^{-1}\mathbf{V}_i \Sigma_0^{-1} + \mathbf{P}_m \otimes (\mathbf{M}_X \Sigma_0 \mathbf{M}_X)^+ \mathbf{V}_i (\mathbf{M}_X \Sigma_0 \mathbf{M}_X)^+$ were used.

Another structure, occurring frequently in engineering experiments, is generated by replications of a regular model $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \Sigma(\vartheta))$:

$$(\underline{Y}, \underline{X}\boldsymbol{\beta}, \underline{\Sigma}(\vartheta)), \tag{2.1}$$

where $\underline{Y} = (\underline{Y}_1', \underline{Y}_2', \dots, \underline{Y}_m')'$, $\underline{Y}_1, \dots, \underline{Y}_m$ are stochastically independent random vectors, $E(\underline{Y}_i|\boldsymbol{\beta}) = (\mathbf{1}_{r_i} \otimes \mathbf{A}_i, \quad \mathbf{1}_{r_i} \otimes \mathbf{S}_i) \begin{pmatrix} \Theta \\ \varkappa \end{pmatrix} = (\mathbf{1}_{r_i} \otimes \mathbf{X}_i)\boldsymbol{\beta}, \quad \operatorname{Var}(\underline{Y}_i|\vartheta) =$

$= \sum_{s=1}^{p} \vartheta_s(\mathbf{I}_{r_i} \otimes \mathbf{V}_s^{(i)})$, $\mathbf{1}_{r_i} = (1, \ldots, 1)' \in \mathscr{R}^{r_i}$, $\underline{Y}_i = (Y'_{i,1}, Y'_2, \ldots, Y'_{i,r_i})'$ and $Y_{i,1}, \ldots, Y_{i,r_i}$ are i.i.d. n_i-dimensional random vectors. The matrix $\mathbf{X}_i$ consists of those rows of the matrix $\mathbf{X}$ which were replicated just r_i-times; analogously $\mathbf{V}_s^{(i)}$ consists of those elements of the matrix $\mathbf{V}_s$ which are determined as the points of intersections of the rows and columns corresponding to the matrix $\mathbf{X}_i$. It is assumed that $r_1 > 1$ and $\mathbf{X}_1 = \mathbf{X}$.

Theorem 2.4. *There exists the* ϑ_0-LMVQUIE *of* $\boldsymbol{\vartheta}$ *in the model* (2.1) *and has the form*

$$\hat{\boldsymbol{\vartheta}}(\underline{Y}, \vartheta_0) = \mathbf{S}^{*-1}(\hat{\gamma}_1, \ldots, \hat{\gamma}_p)',$$

$$\mathbf{S}^* = \sum_{i=1}^{p} \{(r_i - 1)\,\mathbf{S}|\Sigma_{0,i}^{+}| + \mathbf{S}|\Sigma_{0,i}^{+} - r_i\,\Sigma_{0,i}^{+}\mathbf{X}_i\mathbf{N}^{-1}\mathbf{X}_i'\Sigma_{0,i}^{+}| -$$

$$- \mathbf{S}|r_i\,\Sigma_{0,i}^{+}\mathbf{X}_i\mathbf{N}^{-1}\mathbf{X}_i'\Sigma_{0,i}^{+}|\} + \mathbf{T},$$

$$\{\mathbf{T}\}_{s,t} = \sum_{i=1}^{m}\sum_{j=1}^{m} r_i r_j \operatorname{Tr}(\Sigma_{0,i}^{+}\mathbf{X}_i\mathbf{N}^{-1}\mathbf{X}_j'\Sigma_{0,j}^{+}\mathbf{V}_s^{(j)}\,\Sigma_{0,j}^{+}\mathbf{X}_j\mathbf{N}^{-1}\mathbf{X}_i'\Sigma_{0,i}^{+}\mathbf{V}_t^{(i)}), \quad s, t = 1, \ldots, p,$$

$$\hat{\gamma}_s = \sum_{i=1}^{m} [\operatorname{Tr}(\hat{\Sigma}_i\,\Sigma_{0,i}^{+}\mathbf{V}_s^{(i)}\,\Sigma_{0,i}^{+})(r_i - 1) + r_i(\bar{Y}_i - \mathbf{X}_i\hat{\hat{\boldsymbol{\beta}}})'\,\Sigma_{0,i}^{+}\mathbf{V}_s^{(i)}\,\Sigma_{0,i}^{+}(\bar{Y}_i - \mathbf{X}_i\hat{\hat{\boldsymbol{\beta}}})],$$

$$s = 1, \ldots, p,\; \bar{Y}_i = (1/r_i)\sum_{j=1}^{r_i} Y_{i,j},$$

$$\hat{\Sigma}_i = [1/(r_i - 1)]\sum_{j=1}^{r_i}(Y_{i,j} - \bar{Y}_i)(Y_{i,j} - \bar{Y}_i)',\; \Sigma_{0,i} = \sum_{j=1}^{p}\vartheta_{0,j}\mathbf{V}_j^{(i)},$$

$$\{\mathbf{S}|\Sigma_{0,i}^{+}|\}_{s,t} = \operatorname{Tr}(\mathbf{V}_s^{(i)}\,\Sigma_{0,i}^{+}\mathbf{V}_t^{(i)}\,\Sigma_{0,i}^{+}),\; \mathbf{N} = \sum_{i=1}^{m} r_i\mathbf{X}_i'\Sigma_{0,i}^{+}\mathbf{X}_i,\; \hat{\hat{\boldsymbol{\beta}}} = \mathbf{N}^{-1}\sum_{i=1}^{m} r_i\mathbf{X}_i'\Sigma_{0,i}^{+}\bar{Y}_i.$$

Proof. First it has to be proved that $\mathbf{S}|(\mathbf{M}_{\underline{X}}\,\underline{\Sigma}_0\mathbf{M}_{\underline{X}})^{+}| = \mathbf{S}^*$. With respect to Lemmas 1.10 and 1.4 and to the assumption $r_1 > 1$, $\mathbf{X}_1 = \mathbf{X}$, the matrix $\mathbf{S}^*$ is obviously regular and thus the $\hat{\boldsymbol{\vartheta}}(\underline{Y}, \vartheta_0)$ has to exist.

Using the relationship

$$(\mathbf{M}_{\underline{X}}\,\underline{\Sigma}_0\mathbf{M}_{\underline{X}})^{+} = \underline{\Sigma}_0^{-1} - \underline{\Sigma}_0^{-1}\underline{\mathbf{X}}(\underline{\mathbf{X}}'\,\underline{\Sigma}_0^{-1}\underline{\mathbf{X}})^{-1}\underline{\mathbf{X}}'\,\underline{\Sigma}_0^{-1} = \begin{pmatrix} \mathbf{B}_{1,1}, & \ldots, & \mathbf{B}_{1,m} \\ \ldots & , & , \\ \mathbf{B}_{m,1}, & \ldots, & \mathbf{B}_{m,m} \end{pmatrix},$$

where

$$\mathbf{B}_{i,i} = \mathbf{I} \otimes \Sigma_{0,i}^{-1} - r_i\mathbf{P}_{r_i} \otimes \Sigma_{0,i}^{-1}\mathbf{X}_i\mathbf{N}^{-1}\mathbf{X}_i'\Sigma_{0,i}^{-1},$$

$$\mathbf{N} = \sum_{i=1}^{m} r_i \mathbf{X}_i' \Sigma_{0,i}^{-1} \mathbf{X}_i, \quad \mathbf{P}_{r_i} = (1/r_i)\, \mathbf{1}_{r_i} \mathbf{1}_{r_i}',$$

$$\mathbf{B}_{i,j} = -\mathbf{1}_{r_i} \mathbf{1}_{r_j}' \otimes \Sigma_{0,i}^{-1} \mathbf{X}_i \mathbf{N}^{-1} \mathbf{X}_j' \Sigma_{0,j}^{-1}, \quad i \neq j,\ i, j = 1, \ldots, m,$$

after a simple but time-consuming calculation we obtain the relationship

$$(\underline{Y}_1', \ldots, \underline{Y}_m') [\underline{\Sigma}_0^{-1} - \underline{\Sigma}_0^{-1} \underline{X} (\underline{X}' \underline{\Sigma}_0^{-1} \underline{X})^{-1} \underline{X}' \underline{\Sigma}_0^{-1}] \underline{V}_s [\underline{\Sigma}_0^{-1} - \underline{\Sigma}_0^{-1} \underline{X} (\underline{X}' \underline{\Sigma}_0^{-1} \underline{X})^{-1} .$$

$$.\underline{X}' \underline{\Sigma}_0^{-1}] (\underline{Y}_1', \ldots, \underline{Y}_m')' = \sum_{i=1}^{m} (r_i - 1) \operatorname{Tr}(\hat{\Sigma}_i \Sigma_{0,i}^{-1} \mathbf{V}_s^{(i)} \Sigma_{0,i}^{-1}) + \sum_{i=1}^{m} r_i \bar{Y}_i' \Sigma_{0,i}^{-1} \mathbf{V}_s^{(i)} \Sigma_0^{-1} \bar{Y}_i +$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{m} r_i r_j \bar{Y}_i' (\Sigma_{0,i}^{-1} \mathbf{X}_i \mathbf{N}^{-1} \sum_{k=1}^{m} r_k \mathbf{X}_k' \Sigma_{0,k}^{-1} \mathbf{V}_s^{(k)} \Sigma_{0,k}^{-1} \mathbf{X}_k \mathbf{N}^{-1} \mathbf{X}_j' \Sigma_{0,j}^{-1} -$$

$$- \Sigma_{0,i}^{-1} \mathbf{V}_s^{(i)} \Sigma_{0,i}^{-1} \mathbf{X}_i \mathbf{N}^{-1} \mathbf{X}_j' \Sigma_{0,j}^{-1} - \Sigma_{0,i}^{-1} \mathbf{X}_i \mathbf{N}^{-1} \mathbf{X}_j' \Sigma_{0,j}^{-1} \mathbf{V}_s^{(j)} \Sigma_{0,j}^{-1})\, \bar{Y}_j.$$

The right side does not change if $\bar{Y}_i - \mathbf{X}_i \hat{\hat{\beta}}$, $i = 1, \ldots, m$, is substituted for $\bar{Y}_i$, $i = 1, \ldots, m$. Then the right side attains the form $\sum_{i=1}^{m} [(r_i - 1) \operatorname{Tr}(\hat{\Sigma}_i \Sigma_{0,i}^{-1} \mathbf{V}_s^{(i)} . \Sigma_{0,i}^{-1}) + r_i (\bar{Y}_i - \mathbf{X}_i \hat{\hat{\beta}})' \Sigma_{0,i}^{+} \mathbf{V}_s^{(i)} \Sigma_{0,i}^{+} (\bar{Y}_i - \mathbf{X}_i \hat{\hat{\beta}})]$. Now, respecting Lemma 2.1, the proof can be finished in a standard but rather time-consuming way.

Remark 2.5. In many cases it is reasonable to utilize for a numerical determination of the ϑ_0-LMVQUIE of ϑ the fact that the matrices $\underline{\mathbf{V}}_1, \ldots, \mathbf{V}_p$ in the model (2.1) or in the model after an elimination transformation are sparse. The following theorem can be useful here.

Theorem 2.6. *Let in the regular model* $(Y, \mathbf{X}\beta, \Sigma(\vartheta))$ *the matrix* $\Sigma(\vartheta)$ *be of the form*

$$\Sigma(\vartheta) = \begin{pmatrix} \Sigma_1(\vartheta^{(1)}), & \mathbf{0}, & \ldots, & \mathbf{0} \\ \mathbf{0}, & \Sigma_2(\vartheta^{(2)}), & \ldots, & \mathbf{0} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{0}, & \mathbf{0}, & \ldots, & \Sigma_m(\vartheta^{(m)}) \end{pmatrix} = \sum_{s=1}^{m} \sum_{i=1}^{r_s} \vartheta_i^{(s)} \mathbf{e}_s^{(m)} \mathbf{e}_s^{(m)\prime} \otimes \mathbf{H}_i^{(s)},$$

where $\mathbf{e}_s^{(m)} = (0_1, \ldots, 0_{s-1}, 1_s, 0_{s+1}, \ldots, 0_m)'$ *and* $\mathbf{H}_i^{(s)}$ *is a known symmetric* $n_i \times n_i$ *matrix;* $\vartheta = (\vartheta_1^{(1)}, \ldots, \vartheta_{p_1}^{(1)}, \vartheta_1^{(2)}, \ldots, \vartheta_{p_2}^{(2)}, \ldots, \vartheta_1^{(m)}, \ldots, \vartheta_{p_m}^{(m)})'$. *Let a function* $g(\vartheta) = \mathbf{f}'\vartheta$, $\vartheta \in \underline{\vartheta}$, *have the property* $\mathbf{f} \in \mathscr{M}(\mathbf{K}^{(I)})$. *Then the* ϑ_0-LMVQUIE *of* $g(.)$ *is*

$$\widehat{\mathbf{f}'\vartheta} = \sum_{s=1}^{m} \sum_{t=1}^{p_s} \lambda_t^{(s)} \mathbf{v}_s' \Sigma_{0,s}^{-1} \mathbf{H}_t^{(s)} \Sigma_{0,s}^{-1} \mathbf{v}_s,$$

where $\mathbf{v}_s = Y_s - \mathbf{X}_s \hat{\beta}(Y, \vartheta_0)$,

$$\hat{\beta}(Y, \vartheta_0) = \mathbf{N}_0^{-1} \sum_{s=1}^{m} \mathbf{X}_s' \Sigma_{0,s}^{-1} Y_s, \ \mathbf{N}_0 = \sum_{s=1}^{m} \mathbf{X}_s' \Sigma_{0,s}^{-1} \mathbf{X}_s, \ \Sigma_{0,s} = \sum_{t=1}^{p_s} \vartheta_{0,t}^{(s)} \mathbf{H}_t^{(s)},$$

the decomposition of $\mathbf{X}$ *into* $(\mathbf{X}_1', \ldots, \mathbf{X}_m')'$ *corresponds to the decomposition of* Σ_0 *into* $\Sigma_{0,1}, \ldots, \Sigma_{0,m}$ *and the vector* $\lambda = (\lambda_1^{(1)}, \ldots, \lambda_{p_1}^{(1)}, \ldots, \lambda_{p_m}^{(m)})'$ *is a solution of the equation* $\mathbf{S}|(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+|\lambda = \mathbf{f}$. *The matrix* $\mathbf{S}|(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+|$ *is given by the formula*

$$\mathbf{S}|(\mathbf{M}_\mathbf{X}\Sigma_0\mathbf{M}_\mathbf{X})^+| = \begin{pmatrix} \mathbf{S}_{1,1}, & \ldots, & \mathbf{S}_{1,m} \\ \ldots & \ldots & \ldots \\ \mathbf{S}_{m,1}, & \ldots, & \mathbf{S}_{m,m} \end{pmatrix},$$

where

$$\mathbf{S}_{i,i} = \mathbf{S}|\Sigma_{0,i}^{-1} - \Sigma_{0,i}^{-1}\mathbf{X}_i\mathbf{N}_0^{-1}\mathbf{X}_i'\Sigma_{0,i}^{-1}|, \quad i = 1, \ldots, m,$$

$$\{\mathbf{S}_{i,j}\}_{t,u} = \mathrm{Tr}(\Sigma_{0,j}^{-1}\mathbf{X}_j\mathbf{N}_0^{-1}\mathbf{X}_i\Sigma_{0,i}^{-1}\mathbf{H}_t^{(i)}\Sigma_{0,i}^{-1}\mathbf{X}_i\mathbf{N}_0^{-1}\mathbf{X}_j\Sigma_{0,j}^{-1}\mathbf{H}_u^{(j)}), \quad i, j = 1, \ldots, m,$$
$$t = 1, \ldots, p_i, \; u = 1, \ldots, p_j, \; i \neq j.$$

Proof. Regarding Lemma 2.1 we proceed in the same way as in the proof of Theorem 2.4. Because of its being tedious and lenghty, it is omitted.

Remark 2.7. The numbers m and p are significantly smaller than the number $n = n_1 + \ldots + n_m$ and the same holds in many cases for the dimensions of matrices $\mathbf{H}_t^{(s)}$. Thus the calculation of the elements of the matrices $\mathbf{S}_{i,j}$ requires significantly less time and occupies significantly less of the memory of computers than the calculation respecting directly Lemma 2.1.

A further important structure of a mixed linear model arises in such experiments of technical sciences the aim of which is to determine positions of stable and non-stable points of the investigated constructions in m epochs when the model has the form

$$\left(\begin{pmatrix} \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_m \end{pmatrix}; (\mathbf{1}_m \otimes \mathbf{X}_1, \mathbf{I}_m \otimes \mathbf{X}_2)\begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2^{(1)} \\ \vdots \\ \boldsymbol{\beta}_2^{(m)} \end{pmatrix}; \sum_{i=1}^{p} \vartheta_i(\mathbf{I} \otimes \mathbf{V}_i)\right), \tag{2.2}$$

$\boldsymbol{\beta}_1$ being a k_1-dimensional vector of coordinates of stable points, $\boldsymbol{\beta}_2^{(j)}$ a k_2-dimensional vector of coordinates of non-stable points in the jth epoch.

Theorem 2.8. a) *The* (i, j)*th element of the matrix* $\mathbf{K}^{(I)}$ *for the regular model* (2.2) *is*

$$\{\mathbf{K}^{(I)}\}_{i,j} = (m-1)\,\mathrm{Tr}(\mathbf{M}_{\mathbf{X}_2}\mathbf{V}_i\mathbf{M}_{\mathbf{X}_2}\mathbf{V}_j) + \mathrm{Tr}(\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}}\mathbf{V}_i\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}}\mathbf{V}_j),$$

$$i, j = 1, \ldots, p, \; \mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}} = \mathbf{M}_{\mathbf{X}_2} - \mathbf{M}_{\mathbf{X}_2}\mathbf{X}_1(\mathbf{X}_1'\mathbf{M}_{\mathbf{X}_2}\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{M}_{\mathbf{X}_2}.$$

b) *If a function* $g(\vartheta) = \mathbf{f}'\vartheta$, $\vartheta \in \underline{\vartheta}$, *fulfils the condition* $\mathbf{f} \in \mathcal{M}(\mathbf{K}^{(I)})$, *then the* ϑ_0-LMVQUIE *of it is* $\widehat{\mathbf{f}'\vartheta} = \mathbf{f}'\hat{\vartheta}$,

$$\hat{\gamma} = (\hat{\gamma}_1, \ldots, \hat{\gamma}_p)',$$

$$\hat{\gamma}_i = (m-1)\operatorname{Tr}[\mathbf{W}_v(\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+\mathbf{V}_i(\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+] +$$

$$+ m\bar{\boldsymbol{v}}'(\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}}\Sigma_0\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}})^+\mathbf{V}_i(\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}}\Sigma_0\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}})^+\bar{\boldsymbol{v}}, \quad i = 1, \ldots, p,$$

$$\mathbf{W}_v = [1/(m-1)]\sum_{i=1}^{m}(\boldsymbol{v}_i - \bar{\boldsymbol{v}})(\boldsymbol{v}_i - \bar{\boldsymbol{v}})', \quad \bar{\boldsymbol{v}} = (1/m)\sum_{i=1}^{m}\boldsymbol{v}_i,$$

$$(\boldsymbol{v}_1', \ldots, \boldsymbol{v}_m')' = (\boldsymbol{Y}_1', \ldots, \boldsymbol{Y}_m')' - (\boldsymbol{1}\otimes\mathbf{X}_1, \mathbf{I}\otimes\mathbf{X}_2)(\tilde{\boldsymbol{\beta}}_1', \ldots, \tilde{\boldsymbol{\beta}}_2^{(m)\prime})'$$

and $\tilde{\boldsymbol{\beta}}_1, \tilde{\boldsymbol{\beta}}_2^{(1)}, \ldots, \tilde{\boldsymbol{\beta}}_2^{(m)}$ *are arbitrary unbiased linear estimators of the vector parameters* $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2^{(1)}, \ldots, \boldsymbol{\beta}_2^{(m)}$. *The vector* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)'$ *is a solution of the equation* $[(m-1)\mathbf{S}|(\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+| + \mathbf{S}|\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}}|]\boldsymbol{\lambda} = \boldsymbol{f}$.

Proof. a) According to Lemma 2.1 the (i, j)th element of the matrix $\mathbf{K}^{(I)}$ for the model (2.2) is $\operatorname{Tr}(\mathbf{M}_{\underline{\mathbf{X}}}\underline{\mathbf{V}}_i\mathbf{M}_{\underline{\mathbf{X}}}\underline{\mathbf{V}}_j)$. The matrix $\mathbf{M}_{\underline{\mathbf{X}}}$ can be expressed as

$$\mathbf{P}_m\otimes\mathbf{I} + \mathbf{M}_m\otimes\mathbf{I} - (\boldsymbol{1}\otimes\mathbf{X}_1, \mathbf{I}\otimes\mathbf{X}_2)\begin{pmatrix} m\otimes\mathbf{X}_1'\mathbf{X}_1, & \boldsymbol{1}'\otimes\mathbf{X}_1'\mathbf{X}_2 \\ \boldsymbol{1}\otimes\mathbf{X}_2'\mathbf{X}_1, & \mathbf{I}\otimes\mathbf{X}_2'\mathbf{X}_2 \end{pmatrix}^{-1}\begin{pmatrix} \boldsymbol{1}'\otimes\mathbf{X}_1' \\ \mathbf{I}\otimes\mathbf{X}_2' \end{pmatrix}.$$

If the known formula

$$\begin{pmatrix} \mathbf{A}, & \mathbf{B} \\ \mathbf{B}', & \mathbf{C} \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbf{A}-\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}, & -(\mathbf{A}-\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1} \\ -\mathbf{C}^{-1}\mathbf{B}'(\mathbf{A}-\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}, & \mathbf{C}^{-1}+\mathbf{C}^{-1}\mathbf{B}'(\mathbf{A}-\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1} \end{pmatrix}$$

is utilized, we obtain

$$\mathbf{M}_{\underline{\mathbf{X}}} = \mathbf{P}_m\otimes\mathbf{C}_{1,\mathbf{X}_1\mathbf{M}_{\mathbf{X}_2}} + \mathbf{M}_m\otimes\mathbf{M}_{\mathbf{X}_2}.$$

Thus $\mathbf{M}_{\underline{\mathbf{X}}}\underline{\mathbf{V}}_i\mathbf{M}_{\underline{\mathbf{X}}}\underline{\mathbf{V}}_j = \mathbf{M}_m\otimes\mathbf{M}_{\mathbf{X}_2}\mathbf{V}_i\mathbf{M}_{\mathbf{X}_2}\mathbf{V}_j + \mathbf{P}_m\otimes\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}}\mathbf{V}_i\mathbf{C}_{1,\mathbf{X}_1\mathbf{M}_{\mathbf{X}_2}}\mathbf{V}_j$ and $\operatorname{Tr}(\mathbf{M}_{\underline{\mathbf{X}}}\underline{\mathbf{V}}_i\mathbf{M}_{\underline{\mathbf{X}}}\underline{\mathbf{V}}_j) = (m-1)\{\mathbf{S}|\mathbf{M}_{\mathbf{X}_2}|\}_{i,j} + \{\mathbf{S}|\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}}|\}_{i,j}$.

b) Regarding Lemma 2.1 and applying the same procedure as in a) we obtain

$$(\mathbf{M}_{\underline{\mathbf{X}}}\underline{\Sigma}_0\mathbf{M}_{\underline{\mathbf{X}}})^+ = [(\mathbf{M}_m\otimes\mathbf{M}_{\mathbf{X}_2} + \mathbf{P}_m\otimes\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}})(\mathbf{M}_m\otimes\Sigma_0 + \mathbf{P}_m\otimes\Sigma_0).$$

$$.(\mathbf{M}_m\otimes\mathbf{M}_{\mathbf{X}_2} + \mathbf{P}_m\otimes\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}})]^+ = (\mathbf{M}_m\otimes\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2} +$$

$$+ \mathbf{P}_m\otimes\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}}\Sigma_0\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}})^+ = \mathbf{M}_m\otimes(\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+ +$$

$$+ \mathbf{P}_m\otimes(\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}}\Sigma_0\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}})^+.$$

If the equality

$$(\boldsymbol{Y}_1', \ldots, \boldsymbol{Y}_m')(\mathbf{M}_m\otimes\mathbf{U}_1 + \mathbf{P}_m\otimes\mathbf{U}_2)(\boldsymbol{Y}_1', \ldots, \boldsymbol{Y}_m')' =$$

$$= (m-1)\operatorname{Tr}\left\{[1/(m-1)]\sum_{i=1}^{m}(\boldsymbol{Y}_i - \bar{\boldsymbol{Y}})(\boldsymbol{Y}_i - \bar{\boldsymbol{Y}})'\mathbf{U}_1\right\} + m\bar{\boldsymbol{Y}}'\mathbf{U}_2\bar{\boldsymbol{Y}}$$

and the invariance of the estimator are taken into account, the proof can easily be finished.

Remark 2.9. Another expression for $(\mathbf{M}_{\underline{X}}\underline{\Sigma}_0\mathbf{M}_{\underline{X}})^+$ can be obtained in the following way:

$$(\mathbf{M}_{\underline{X}}\underline{\Sigma}_0\mathbf{M}_{\underline{X}})^+ = \underline{\Sigma}_0^{-1} - \underline{\Sigma}_0^{-1}\underline{\mathbf{X}}(\underline{\mathbf{X}}'\underline{\Sigma}_0^{-1}\underline{\mathbf{X}})^{-1}\underline{\mathbf{X}}\,\underline{\Sigma}_0^{-1} = \mathbf{M}_m \otimes (\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+ +$$
$$+ \mathbf{P}_m \otimes \{(\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+ - (\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+\mathbf{X}_1[\mathbf{X}_1'(\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+\mathbf{X}_1]^{-1}\mathbf{X}_1' .$$
$$. (\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+\},$$

where

$$(\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+ = \Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{X}_2(\mathbf{X}_2'\Sigma_0^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\Sigma_0^{-1}$$

and

$$(\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+ - (\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+\mathbf{X}_1[\mathbf{X}_1'(\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+\mathbf{X}_1]^{-1}\mathbf{X}_1'(\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+ =$$
$$= \mathbf{C}_{1,\mathbf{X}_1,(\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+}.$$

Thus $(\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}}\Sigma_0\mathbf{C}_{1,\mathbf{X}_1,\mathbf{M}_{\mathbf{X}_2}})^+ = \mathbf{C}_{1,\mathbf{X}_1,(\mathbf{M}_{\mathbf{X}_2}\Sigma_0\mathbf{M}_{\mathbf{X}_2})^+}$.

3. Sensitivity and invariance

Consider the special structure of the linear mixed model $(\boldsymbol{Y}, \mathbf{X}\boldsymbol{\beta}, \Sigma(\boldsymbol{\vartheta}))$ with $\Sigma(\boldsymbol{\vartheta})$ of the form $\Sigma(\boldsymbol{\vartheta}) = \sigma^2\mathbf{I} + \mathbf{XGX}' + \mathbf{Z}\Delta\mathbf{Z}'$, where $\boldsymbol{\vartheta} = (\sigma^2, [\mathrm{vech}(\mathbf{G})]', [\mathrm{vech}(\Delta)]')'$, $\mathbf{Z}'\mathbf{X} = \mathbf{O}$, $\mathcal{M}(\mathbf{Z}) = \mathrm{Ker}(\mathbf{X}')$. If next in accordance with [6] $\mathbf{I} \in \{\Sigma(\boldsymbol{\vartheta}) : \boldsymbol{\vartheta} \in \underline{\vartheta}\}$, then $\forall\{\boldsymbol{\vartheta} \in \underline{\vartheta}\}\ \hat{\boldsymbol{\beta}}(\boldsymbol{Y}, \boldsymbol{\vartheta}) = (\mathbf{X}'\Sigma^{-1}(\boldsymbol{\vartheta})\mathbf{X})^{-1}\mathbf{X}'\Sigma^{-1}(\boldsymbol{\vartheta})\boldsymbol{Y} = (\mathbf{X}'\mathbf{X})^{-1}. .\mathbf{X}'\boldsymbol{Y}$, i.e. there exists the UBLUE of the vector $\boldsymbol{\beta}$.

This example shows that a determination of some locally best estimator of $\boldsymbol{\beta}$ does not always require to estimate $\boldsymbol{\vartheta}$. (However, when $\mathrm{Var}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{Y}|\boldsymbol{\vartheta}] = = \sigma^2(\mathbf{X}'\mathbf{X})^{-1} + \mathbf{G}$ is to be estimated, then it is quite clear that some function of $\boldsymbol{\vartheta}$ must be estimated.)

For the mentioned structure it is typical that the parameter $\boldsymbol{\beta}$ is uniformly non-sensitive on $\boldsymbol{\vartheta}$ (see further). It will be shown that a structure with some non-sensitiveness of $\boldsymbol{\beta}$ on $\boldsymbol{\vartheta}$ implies the existence of an invariant estimator of $\boldsymbol{\vartheta}$. Further, it will be shown that in this case the $(\boldsymbol{\beta}_0, \boldsymbol{\vartheta}_0)$-LMVLQUE (the locally minimum variance linear quadratic unbiased estimator) of a function $g(\boldsymbol{\vartheta}) = \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\vartheta}$, becomes the $\boldsymbol{\vartheta}_0$-LMVQUIE. (Let us recall that the $(\boldsymbol{\beta}_0, \boldsymbol{\vartheta}_0)$-LMVLQUE has the form

$$\sum_{i=1}^{p} \lambda_i(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0)'[\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{X}(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1}\mathbf{X}(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}.$$

$$.\mathbf{X}'\Sigma_0^{-1}](\boldsymbol{Y}-\mathbf{X}\boldsymbol{\beta}_0),$$

where the vector $\boldsymbol{\lambda}=(\lambda_1, \ldots, \lambda_p)'$ is a solution of the equation $(\mathbf{S}|\Sigma_0^{-1}| - \mathbf{S}|\Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}|)\boldsymbol{\lambda} = \boldsymbol{f}$.) Analogously the ϑ_0-mLMVQE (the modified locally minimum variance quadratic estimator) given by the formula

$$\sum_{i=1}^{p}\lambda_i(\boldsymbol{Y}-\mathbf{X}\hat{\boldsymbol{\beta}})'[\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{X}(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1}\mathbf{X}(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}.$$

$$.\mathbf{X}'\Sigma_0^{-1}](\boldsymbol{Y}-\mathbf{X}\hat{\boldsymbol{\beta}}),$$

where $\hat{\boldsymbol{\beta}}=(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1}\boldsymbol{Y}$ and $\boldsymbol{\lambda}$ is the same as in the $(\boldsymbol{\beta}_0, \vartheta_0)$-LMVLQUE, becomes the ϑ_0-LMVQUIE.

Definition 3.1. *A parameter β_i is ϑ_0-locally non-sensitive on a variance component ϑ_j if* $\forall\{\boldsymbol{y}\in\mathscr{R}^n\}\,\partial\hat{\beta}_i(\boldsymbol{y}, \boldsymbol{\vartheta})/\partial\vartheta_j|_{\boldsymbol{\vartheta}=\boldsymbol{\vartheta}_0}=0$.

Lemma 3.2. *In a regular mixed model a parameter β_i is ϑ_0-locally non-sensitive on ϑ_j iff*

$$\{(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1}\}_{i.}\mathbf{V}_j\Sigma_0^{-1}\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}}=\boldsymbol{0};$$

the vector $\boldsymbol{\beta}$ is ϑ_0-locally non-sensitive on ϑ_j iff

$$\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_j\Sigma_0^{-1}\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}}=\mathbf{0}\Leftrightarrow\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_j\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}\prime}=\mathbf{0}.$$

Proof. $\partial\hat{\boldsymbol{\beta}}(\boldsymbol{y}, \boldsymbol{\vartheta})/\partial\vartheta_j=[\partial(\mathbf{X}'\Sigma^{-1}(\boldsymbol{\vartheta})\mathbf{X})^{-1}/\partial\vartheta_j]\mathbf{X}'\Sigma^{-1}(\boldsymbol{\vartheta})\boldsymbol{y}+(\mathbf{X}'\Sigma^{-1}(\boldsymbol{\vartheta})\mathbf{X})^{-1}\mathbf{X}'.$ $.[\partial\Sigma^{-1}(\boldsymbol{\vartheta})/\partial\vartheta_j]\boldsymbol{y}$; as $\partial(\mathbf{X}'\Sigma^{-1}(\boldsymbol{\vartheta})\mathbf{X})^{-1}/\partial\vartheta_j=(\mathbf{X}'\Sigma^{-1}(\boldsymbol{\vartheta})\mathbf{X})^{-1}\mathbf{X}'\Sigma^{-1}(\boldsymbol{\vartheta})\mathbf{V}_j\Sigma^{-1}(\boldsymbol{\vartheta}).$ $\mathbf{X}(\mathbf{X}'\Sigma^{-1}(\boldsymbol{\vartheta})\mathbf{X})^{-1}$ and $\partial\Sigma^{-1}(\boldsymbol{\vartheta})/\partial\vartheta_j=-\Sigma^{-1}(\boldsymbol{\vartheta})\mathbf{V}_j\Sigma^{-1}(\boldsymbol{\vartheta})$, it can be easily obtained that $\partial\hat{\boldsymbol{\beta}}(\boldsymbol{y}, \boldsymbol{\vartheta})/\partial\vartheta_j|_{\boldsymbol{\vartheta}=\boldsymbol{\vartheta}_0}=-(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1}\mathbf{V}_j\Sigma_0^{-1}(\boldsymbol{y}-\mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{y}, \boldsymbol{\vartheta}_0))$, where $\hat{\boldsymbol{\beta}}(\boldsymbol{y}, \boldsymbol{\vartheta}_0)=(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1}\boldsymbol{y}$. As $R(\mathbf{X})=k<n$ and $\boldsymbol{y}-\mathbf{X}\hat{\boldsymbol{\beta}}(\boldsymbol{y}, \boldsymbol{\vartheta}_0)=\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}}\boldsymbol{y}$, the equivalence $\forall\{\boldsymbol{y}\in\mathscr{R}^n\}\,\partial\hat{\boldsymbol{\beta}}(\boldsymbol{y}, \boldsymbol{\vartheta})/\partial\vartheta_j|_{\boldsymbol{\vartheta}=\boldsymbol{\vartheta}_0}=\boldsymbol{0}\Leftrightarrow\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_j\Sigma_0^{-1}.$ $.\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}}=\mathbf{0}$ is obvious.

Corollary 3.3. *If* $\mathscr{M}(\mathbf{V}_j)\subset\mathscr{M}(\mathbf{X})$, *then* $\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_j\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}\prime}=\mathbf{0}$ (*obviously* $\mathbf{V}_i=\mathbf{XUX}'$, $\mathbf{U}=\mathbf{U}'$ *and* $\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{XUX}'\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}\prime}=\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{XUO}=\mathbf{O}$).

Remark 3.4. If $\mathscr{M}(\mathbf{V}_i)\subset\mathscr{M}(\mathbf{X})$, then the ith column and the ith row of the matrix $\mathbf{K}^{(I)}$ are zero. With respect to Lemma 1.10 the same is true for the matrix $\mathbf{S}|(\mathbf{M}_{\mathbf{X}}\Sigma_0\mathbf{M}_{\mathbf{X}})^+|$. Therefore there does not exists the ϑ_0-LMVQUIE for ϑ_i.

Definition 3.5. *A parameter β_i in a regular mixed model is uniformly non-sensitive on the variance component ϑ_j if* $\forall\{\boldsymbol{\vartheta}\in\underline{\vartheta}\}\forall\{\boldsymbol{y}\in\mathscr{R}^n\}\,\partial\hat{\beta}_i(\boldsymbol{y}, \boldsymbol{\vartheta})/\partial\vartheta_j=0$.

Remark 3.6. If $\Sigma = \sigma^2 \mathbf{I} + \mathbf{XGX}' + \mathbf{Z}\Delta\mathbf{Z}'$, $\vartheta = \{\sigma^2, [\mathrm{vech}(\mathbf{G})]', [\mathrm{vech}(\Delta)']\}'$, then the vector $\boldsymbol{\beta}$ is obviously uniformly non-sensitive on $\underline{\vartheta}$.

Lemma 3.7. *If* $\mathbf{I} \in \left\{\Sigma(\vartheta) : \Sigma(\vartheta) = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i,\ \mathbf{V}_i,\ \vartheta \in \underline{\vartheta}\right\}$ *and* $\boldsymbol{\beta}$ *is uniformly non-sensitive on* $\underline{\vartheta}$, *then in the regular model* $\mathbf{V}_i = \mathbf{P_X S}_i \mathbf{P_X} + \mathbf{M_X S}_i \mathbf{M_X}$, $\mathbf{S}_i = \mathbf{S}_i'$, $i = 1, \ldots, p$, *and* $\Sigma(\vartheta) = \sigma^2 \mathbf{I} + \mathbf{XGX}' + \mathbf{Z}\Delta\mathbf{Z}'$, $\mathbf{Z}'\mathbf{X} = \mathbf{O}$, $\mathcal{M}(\mathbf{Z}) = \mathrm{Ker}(\mathbf{X}')$.

Proof. $\forall\{\vartheta \in \underline{\vartheta}\}\ \mathbf{P}_{\mathbf{X}}^{\Sigma^{-1}} \mathbf{V}_i \Sigma^{-1} \mathbf{M}_{\mathbf{X}}^{\Sigma^{-1}} = \mathbf{O} \Rightarrow \mathbf{P_X V}_i \mathbf{M_X} = \mathbf{O} \Rightarrow \mathbf{V}_i = = \mathbf{Z}_i - \mathbf{P_X Z}_i \mathbf{M_X} = \mathbf{Z}_i' - \mathbf{M_X Z}_i' \mathbf{P_X} \Rightarrow \mathbf{V}_i = (1/2)(\mathbf{Z}_i + \mathbf{Z}_i') - \mathbf{P_X}(1/2).(\mathbf{Z}_i + \mathbf{Z}_i').\mathbf{M_X} - \mathbf{M_X}(1/2)(\mathbf{Z}_i' + \mathbf{Z}_i)\mathbf{P_X}$. If $(1/2)(\mathbf{Z}_i + \mathbf{Z}_i') = \mathbf{S}_i$, then $\mathbf{V}_i = (\mathbf{P_X} + \mathbf{M_X})\mathbf{S}_i.(\mathbf{P_X} + \mathbf{M_X}) - \mathbf{P_X S}_i \mathbf{M_X} - \mathbf{M_X S}_i \mathbf{P_X} = \mathbf{P_X S}_i \mathbf{P_X} + \mathbf{M_X S}_i \mathbf{M_X}$. Thus $\Sigma(\vartheta) = \mathbf{P_X}.\sum_{i=1}^{p} \vartheta_i \mathbf{S}_i \mathbf{P_X} + \mathbf{M_X} \sum_{i=1}^{p} \vartheta_i \mathbf{S}_i \mathbf{M_X}$. Let σ^2, $\mathbf{G}$, $\Delta(\mathbf{G} = \mathbf{G}', \Delta = \Delta')$ be arbitrary. It is to be shown that there exists a matrix $\mathbf{U} = \sum_{i=1}^{p} \vartheta_i \mathbf{S}_i$ *such that* $\mathbf{P_X U P_X} + \mathbf{M_X U M_X} = \sigma^2 \mathbf{I} + \mathbf{XGX}' + \mathbf{Z}\Delta\mathbf{Z}'$. Let $\mathbf{U} = \sigma^2 \mathbf{I} + \mathbf{H}$, $\mathbf{H} = \mathbf{H}'$. Then $\mathbf{P_X U P_X} + \mathbf{M_X U M_X} = \sigma^2 \mathbf{I} + \mathbf{P_X H P_X} + \mathbf{M_X H M_X}$; thus the matrix $\mathbf{H}$ must be a solution of the equation

$$\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{H}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' + \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{H}\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' = \mathbf{XGX}' + \mathbf{Z}\Delta\mathbf{Z} \quad (3.1)$$

If $\mathbf{H}$ fulfils the equations

$$(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{H}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} = \mathbf{G}, \quad (3.2)$$

$$(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{H}\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1} = \Delta, \quad (3.3)$$

then it is a solution of (3.1). The class of all solutions of (3.2) is $\{(\mathbf{X}')^{+}\mathbf{X}'\mathbf{XGX}'\mathbf{XX}^{+} + \mathbf{T} - (\mathbf{X}')^{+}\mathbf{X}'\mathbf{TXX}^{+} : \mathbf{T} \text{ arbitrary}\}$. The choice $\mathbf{T} = \mathbf{Z}\Delta\mathbf{Z}'$ makes $\mathbf{H}$ a solution of (3.3) as well. Thus such an $\mathbf{H}$ is a solution of 3.1.

Remark 3.8. The assertion of Lemma 3.7 can be found in [6] or in [4] (Theorem 5.7.3); here another way of proving it is given. If $\mathbf{I} \notin \{\Sigma(\vartheta) : \vartheta \in \underline{\vartheta}\}$, then it can easily be shown that $\mathbf{V}_i = \mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}} \mathbf{S}_i \mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}\prime} + \mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}} \mathbf{S}_i \mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}\prime}$, $\mathbf{S}_i = \mathbf{S}_i'$, $\Sigma_0 = \Sigma(\vartheta_0)$, $\vartheta_0 \in \underline{\vartheta}$.

Theorem 3.9. *If the vector* $\boldsymbol{\beta}$ *is* ϑ_0*-locally non-sensitive on the variance component* ϑ_j, *then* $\hat{\gamma}_i = (\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta})'[\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{X}(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1}\mathbf{X}.(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1}](\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta})$, *which is a term of the* $(\boldsymbol{\beta}, \vartheta_0)$-LMVLQUE *and it equals for each* $\boldsymbol{\beta} \in \mathcal{R}^k$ $\hat{\gamma}_i^{(I)} = \boldsymbol{Y}'(\mathbf{M_X}\Sigma_0\mathbf{M_X})^{+}\mathbf{V}_i(\mathbf{M_X}\Sigma_0\mathbf{M_X})^{+}\boldsymbol{Y}$ *which is a term of the* ϑ_0-LMVQUIE.

Proof. $(\mathbf{M_X}\Sigma_0\mathbf{M_X})^{+}\mathbf{V}_i(\mathbf{M_X}\Sigma_0\mathbf{M_X})^{+} = [\Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{X}(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1}]\mathbf{V}_i.[\Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{X}(\mathbf{X}'\Sigma_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma_0^{-1}] = \Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1}\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}} - \Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_i\Sigma_0^{-1}\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}} =$

$= \Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1}\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}} = \Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1} - (\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}} + \mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}})'\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}} = \Sigma_0^{-1}\mathbf{V}_i\,.$
$.\,\Sigma_0^{-1} - \mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}\prime}\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}$, because of $\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_i\Sigma_0^{-1}\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}} = \mathbf{O}$.

Theorem 3.10. *a) If the vector* $\boldsymbol{\beta}$ *is* ϑ_0*-locally non-sensitive on the whole vector* ϑ, *then* $\mathbf{S}|\mathbf{M}_{\mathbf{X}}\Sigma_0\mathbf{M}_{\mathbf{X}})^+| = \mathbf{S}|\Sigma_0^{-1}| - \mathbf{S}|\Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}|$.
b) *Let* $\vartheta = (\vartheta_1', \vartheta_2')'$. *If the vector* $\boldsymbol{\beta}$ *is* ϑ_0*-locally nonsensitive on the subvector* ϑ_1, *then*

$$\mathbf{S}|\mathbf{M}_{\mathbf{X}}\Sigma_0\mathbf{M}_{\mathbf{X}})^+| = \begin{pmatrix}\mathbf{A}, & \mathbf{B}\\ \mathbf{B}', & \mathbf{C}\end{pmatrix}$$

and

$$\mathbf{S}|\Sigma_0^{-1}| - \mathbf{S}|\Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}| = \begin{pmatrix}\mathbf{A}, & \mathbf{B}\\ \mathbf{B}', & \mathbf{D}\end{pmatrix}.$$

The decomposition of these matrices corresponds to the decomposition of the vector ϑ *into* ϑ_1 *and* ϑ_2.

Proof. a) $\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}\prime} = \mathbf{O}$, $i = 1, \ldots, p \Rightarrow \mathrm{Tr}[(\mathbf{M}_{\mathbf{X}}\Sigma_0\mathbf{M}_{\mathbf{X}})^+\mathbf{V}_i\,.$
$.\,(\mathbf{M}_{\mathbf{X}}\Sigma_0\mathbf{M}_{\mathbf{X}})^+\mathbf{V}_j] = \mathrm{Tr}[\Sigma_0^{-1}\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_i\Sigma_0^{-1}(\mathbf{I} - \mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}})\mathbf{V}_j] = \mathrm{Tr}[\Sigma_0^{-1}(\mathbf{I} - \mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}})\mathbf{V}_i\,.$
$.\,\Sigma_0^{-1}\mathbf{V}_j] = \mathrm{Tr}(\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1}\mathbf{V}_j) - \mathrm{Tr}[\Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_i\Sigma_0^{-1}(\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}} + \mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}})\mathbf{V}_j] = \mathrm{Tr}(\Sigma_0^{-1}\mathbf{V}_i\,.$
$.\,\Sigma_0^{-1}\mathbf{V}_j) - \mathrm{Tr}(\Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_i\Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_j)$.
b) Let $\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_i\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}\prime} = \mathbf{0}$. Then $\mathrm{Tr}(\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1}\mathbf{V}_j - \Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_i\Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_j) =$
$= \mathrm{Tr}[\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1}\mathbf{V}_j - \Sigma_0^{-1}(\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}} + \mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}})\mathbf{V}_i\Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_j] = \mathrm{Tr}(\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1}\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_j) =$
$= \mathrm{Tr}[\Sigma_0^{-1}(\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}} + \mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}})\mathbf{V}_i\Sigma_0^{-1}\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_j] = \mathrm{Tr}(\Sigma_0^{-1}\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_i\Sigma_0^{-1}\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}}\mathbf{V}_j)$.

Corollary 3.11. *All functions* $g(\vartheta) = \mathbf{f}'\vartheta$, $\vartheta \in \underline{\vartheta}$, *with the property*

$$\begin{pmatrix}\mathbf{A}, & \mathbf{B}\\ \mathbf{B}', & \mathbf{D}\end{pmatrix}\begin{pmatrix}\lambda_1\\ \mathbf{0}\end{pmatrix} = \mathbf{f}$$

possess $(\boldsymbol{\beta}, \vartheta_0)$-LMVLQUE*s identical with* ϑ_0-LMVQUIE*s for each* $\boldsymbol{\beta} \in \mathscr{R}^k$. *These functions can be invariantly estimated in the model after a transformation of the type* $\mathbf{M}_{\mathbf{S}}^{\mathbf{K}}$ ($\mathbf{V}$ *arbitrary) and the estimators in the original and in the transformed model are identical.*

As the ϑ_0-mLMVQE is usually used instead of the $(\boldsymbol{\beta}_0, \vartheta_0)$-LMVLQUE, the following theorem can be of some interest.

Theorem 3.12. *Let* $g(\vartheta) = \mathbf{f}'\vartheta$, $\vartheta \in \underline{\vartheta}$, *where* $\mathbf{f} \in \mathscr{M}(\mathbf{S}|\Sigma_0^{-1}| - \mathbf{S}|\Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}|)$ *and let the vector* $\boldsymbol{\beta}$ *be* ϑ_0*-locally non-sensitive on* ϑ. *Then the* ϑ_0-mLMVQE *of the function* $g(.)$. *is unbiased and is identical with the* ϑ_0-LMVQUIE.

Proof. The relationships $\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}} \mathbf{V}_i \mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}\prime} = \mathbf{O}$ and $Y - \mathbf{X}\hat{\beta} = \mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}} Y$ applied in the expression $\tau_g(Y, \vartheta_0) = \sum_{i=1}^{p} \lambda_i (Y - \mathbf{X}\hat{\beta})' (\Sigma_0^{-1}\mathbf{V}_i\Sigma_0^{-1} - \Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}} \mathbf{V}_i .$ $. \mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}\prime} \Sigma_0^{-1}) (Y - \mathbf{X}\hat{\beta})$ give $\tau_g(Y, \vartheta_0) = \sum_{i=1}^{p} \lambda_i Y'(\mathbf{M}_{\mathbf{X}}\Sigma_0\mathbf{M}_{\mathbf{X}})^{+}\mathbf{V}_i(\mathbf{M}_{\mathbf{X}}\Sigma_0\mathbf{M}_{\mathbf{X}})^{+} Y.$ As the vector λ for the ϑ_0-mLMVQE is a solution of the equation $(\mathbf{S}|\Sigma_0^{-1}| - \mathbf{S}|\Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}}|)\lambda = \mathbf{f}$ and the vector λ for the ϑ_0-LMVQUIE is a solution of the equation $\mathbf{S}|(\mathbf{M}_{\mathbf{X}}\Sigma_0\mathbf{M}_{\mathbf{X}})^{+}| \lambda = \mathbf{f}$, the assertion is obvious with respect to Theorem 3.10.

The unbiasedness can be proved directly as well. The bias of the ϑ_0-mLMVQE is $b(\vartheta) = E[\tau_g(Y, \vartheta_0)|\vartheta] - \mathbf{f}'\vartheta = -2\lambda'\{\mathrm{Tr}[(\mathbf{M}_{\mathbf{X}}\Sigma_0\mathbf{M}_{\mathbf{X}})^{+}\mathbf{V}_1\Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}} \Sigma], \ldots,$ $\ldots, \mathrm{Tr}[(\mathbf{M}_{\mathbf{X}}\Sigma_0\mathbf{M}_{\mathbf{X}})^{+}\mathbf{V}_p\Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}} \Sigma]\}'$; $\mathbf{M}_{\mathbf{X}}^{\Sigma_0^{-1}} \mathbf{V}_i \Sigma_0^{-1}\mathbf{P}_{\mathbf{X}}^{\Sigma_0^{-1}} = \mathbf{O}$ obviously implies $b(\vartheta) = 0,\ \vartheta \in \underline{\vartheta}$.

4. Estimators of variance components in the structure with a uniform non-sensitiveness of β on ϑ

In the regular linear mixed model $(Y, \mathbf{X}\beta, \Sigma(\vartheta) = \sigma^2\mathbf{I} + \mathbf{X}\mathbf{G}\mathbf{X}' + \mathbf{Z}\Delta\mathbf{Z}')$, $\beta \in \mathscr{R}^k$, $\vartheta = (\sigma^2, [\mathrm{vech}(\mathbf{G})]', [\mathrm{vech}(\Delta)]')' \in \underline{\vartheta}$ there exists the UBLUE of a function $h(\beta) = \mathbf{p}'\beta$, $\beta \in \mathscr{R}^k$, but there does not exist an unbiased and invariant estimator of its dispersion $\mathrm{Var}[\mathbf{p}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'Y|\sigma^2, \mathbf{G}, \Delta] = \mathrm{Tr}\{\mathbf{p}\mathbf{p}'[\sigma^2(\mathbf{X}'\mathbf{X})^{-1} + \mathbf{G}]\}$. (If $\mathbf{U}\mathbf{X} = \mathbf{O}$ — invariance, then $E(Y'\mathbf{U}Y|\sigma^2, \mathbf{G}, \Delta) = \sigma^2\mathrm{Tr}(\mathbf{U}) + \mathrm{Tr}(\mathbf{Z}'\mathbf{U}\mathbf{Z}) \neq$ $\neq \mathrm{Tr}\{\mathbf{p}\mathbf{p}'[\sigma^2(\mathbf{X}'\mathbf{X})^{-1} + \mathbf{G}]\}$.) If a replication of the experiment is possible, then the structure of the replicated model ensures the existence of the sought estimator.

Theorem 4.1. *Let* $(\underline{Y}, (\mathbf{1}\otimes\mathbf{X})\beta, \mathbf{I}\otimes(\sigma^2\mathbf{I} + \mathbf{X}\mathbf{G}\mathbf{X}' + \mathbf{Z}\Delta\mathbf{Z}'))$ *be a regular replicated linear mixed model and let* $g(\sigma^2, \mathbf{G}) = \mathrm{Tr}\{\mathbf{F}[\sigma^2(\mathbf{X}'\mathbf{X})^{-1} + \mathbf{G}]\}$; *then there exists the uniformly minimum variance quadratic unbiased invariant estimator of the function* $g(.)$ *and has the form* $\mathrm{Tr}[\hat{\Sigma}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{F}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']$, *where* $\hat{\Sigma} = [1/(m-1)] \sum_{i=1}^{p} (Y_i - \bar{Y})(Y_i - \bar{Y})'$, $\bar{Y} = (1/m)\sum_{i=1}^{m} Y_i$, $\underline{Y} = (Y_1', \ldots, Y_m')'$.

Proof. As, in accordance with [2], for every unbiased estimator of the form $\underline{Y}'\mathbf{U}_{nm,nm}\underline{Y}$ there exists an unbiased estimator of the form

$$\tau_g(\underline{Y}) = \underline{Y}'(\mathbf{M}_m \otimes \mathbf{U}_1 + \mathbf{P}_m \otimes \mathbf{U}_2)\underline{Y} = (m-1)\mathrm{Tr}(\hat{\Sigma}\mathbf{U}_1) + m\bar{Y}'\mathbf{U}_2\bar{Y},$$

where

$$(m-1)\mathrm{Tr}(\mathbf{U}_1) + \mathrm{Tr}(\mathbf{U}_2) = \mathrm{Tr}[\mathbf{F}(\mathbf{X}'\mathbf{X})^{-1}],$$

$$(m-1)\,\mathbf{X}'\mathbf{U}_1\mathbf{X} + \mathbf{X}'\mathbf{U}_2\mathbf{X} = \mathbf{F},$$
$$(m-1)\,\mathbf{Z}'\mathbf{U}_1\mathbf{Z} + \mathbf{Z}'\mathbf{U}_2\mathbf{Z} = \mathbf{O}$$

(unbiasedness) and $\mathbf{U}_2\mathbf{X} = \mathbf{O}$ (invariance), whose dispersion is not larger than that of $\underline{Y}'\mathbf{U}_{nm,nm}\underline{Y}$, we confine ourselves to the latter form.

The matrices $\mathbf{U}_1$ and $\mathbf{U}_2$ satisfying these conditions can be obtained in the following way: $\mathbf{U}_1 = \mathbf{U}_1' \,\&\, (m-1)\mathbf{X}'\mathbf{U}_1\mathbf{X} = \mathbf{F} \Leftrightarrow \mathbf{U}_1 = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{F}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'/ /(m-1) + \mathbf{W}_1 - \mathbf{P_X}\mathbf{W}_1\mathbf{P_X}, \mathbf{W}_1 = \mathbf{W}_1'$, $\mathbf{U}_2 = \mathbf{U}_2' \,\&\, \mathbf{U}_2\mathbf{X} = \mathbf{O} \Leftrightarrow \mathbf{U}_2 = \mathbf{P_Z}\mathbf{W}_2\mathbf{P_Z}$, $\mathbf{W}_2 = \mathbf{W}_2'$ and $(m-1)\,\mathbf{Z}'\mathbf{W}_1\mathbf{Z} + \mathbf{Z}'\mathbf{W}_2\mathbf{Z} = \mathbf{O}$, $(m-1)\,\mathrm{Tr}(\mathbf{W}_1\mathbf{P_Z}) + \mathrm{Tr}(\mathbf{W}_2\mathbf{P_Z}) = = \mathbf{0}$.

The variance of the estimator is

$$\begin{aligned}
&\mathrm{Var}[\tau_g(\underline{Y})|\vartheta_0] = (m-1)\,\mathrm{Tr}(\mathbf{U}_1\Sigma_0\mathbf{U}_1\Sigma_0) + \mathrm{Tr}(\mathbf{U}_2\Sigma_0\mathbf{U}_2\Sigma_0) = (m-1)\,. \\
&.\{[\sigma_0^4/(m-1)^2]\,\mathrm{Tr}[\mathbf{F}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{F}(\mathbf{X}'\mathbf{X})^{-1}] + [\sigma_0^2/(m-1)^2]\,\mathrm{Tr}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{F}\mathbf{G}_0\mathbf{F}] + \\
&+ \mathrm{Tr}(\mathbf{W}_1^2) - \mathrm{Tr}(\mathbf{P_X}\mathbf{W}_1\mathbf{P_X}\mathbf{W}_1) + \mathrm{Tr}(\mathbf{W}_1\mathbf{X}\mathbf{G}_0\mathbf{X}'\mathbf{W}_1) - \mathrm{Tr}(\mathbf{P_X}\mathbf{W}_1\mathbf{X}\mathbf{G}_0\mathbf{X}'\mathbf{W}_1) + \\
&+ \mathrm{Tr}(\mathbf{W}_1\mathbf{Z}\,\Delta_0\mathbf{Z}'\mathbf{W}_1) + [\sigma_0^2/(m-1)^2]\,\mathrm{Tr}[\mathbf{F}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{F}\mathbf{G}_0] + \mathrm{Tr}(\mathbf{X}\mathbf{G}_0\mathbf{X}'\mathbf{W}_1^2) - \\
&- \mathrm{Tr}(\mathbf{W}_1\mathbf{P_X}\mathbf{W}_1\mathbf{X}\mathbf{G}_0\mathbf{X}') + \mathrm{Tr}(\mathbf{W}_1\mathbf{Z}\,\Delta_0\mathbf{Z}'\mathbf{W}_1\mathbf{X}\mathbf{G}_0\mathbf{X}') + \mathrm{Tr}(\mathbf{W}_1^2\mathbf{Z}\,\Delta_0\mathbf{Z}') + \\
&+ \mathrm{Tr}(\mathbf{W}_1\mathbf{X}\mathbf{G}_0\mathbf{X}'\mathbf{W}_1\mathbf{Z}\,\Delta_0\mathbf{Z}') + \mathrm{Tr}(\mathbf{Z}\,\Delta_0\mathbf{Z}'\mathbf{W}_1\mathbf{Z}\,\Delta_0\mathbf{Z}')\} + \sigma_0^4\,\mathrm{Tr}(\mathbf{M_X}\mathbf{W}_2\mathbf{M_X}\mathbf{W}_2) + \\
&+ \sigma_0^2\,\mathrm{Tr}(\mathbf{M_X}\mathbf{W}_2\mathbf{Z}\,\Delta_0\mathbf{Z}'\mathbf{W}_2) + \sigma_0^2\,\mathrm{Tr}(\mathbf{M_X}\mathbf{W}_2\mathbf{M_X}\mathbf{W}_2\mathbf{Z}\,\Delta_0\mathbf{Z}') + \\
&+ \mathrm{Tr}(\mathbf{M_X}\mathbf{W}_2\mathbf{Z}\,\Delta_0\mathbf{Z}'\mathbf{W}_2\mathrm{Z}\,\Delta_0\mathbf{Z}').
\end{aligned}$$

The Lagrange method of indefinite multipliers is used in order to find out the minimum of this dispersion under the conditions $(m-1)\mathbf{Z}'\mathbf{W}_1\mathbf{Z} + \mathbf{Z}'\mathbf{W}_2\mathbf{Z} = \mathbf{O}$ and $(m-1)\,\mathrm{Tr}(\mathbf{W}_1\mathbf{P_Z}) + \mathrm{Tr}(\mathbf{W}_2\mathbf{P_Z}) = 0$;

$$\begin{aligned}
\Phi(\mathbf{W}_1, \mathbf{W}_2) = {} & \mathrm{Var}[\tau_g(\underline{Y})|\sigma_0^2, \Delta_0] - 2\lambda[(m-1)\,\mathrm{Tr}(\mathbf{W}_1\mathbf{P_Z}) + \mathrm{Tr}\,(\mathbf{W}_2\mathbf{P_Z})] - \\
& - 2\mathrm{Tr}\{\gamma'\mathbf{Z}'[(m-1)\mathbf{W}_1 + \mathbf{W}_2]\,\mathbf{Z}\},
\end{aligned}$$

where λ (a scalar) and γ' (a matrix) are indefinite multipliers.

$$\begin{aligned}
&\partial\Phi(\mathbf{W}_1, \mathbf{W}_2)/\partial\mathbf{W}_1 = \mathbf{O} \Rightarrow (m-1)\,(4\mathbf{W}_1 - 4\mathbf{P_X}\mathbf{W}_1\mathbf{P_X} + 2\mathbf{X}\mathbf{G}_0\mathbf{X}'\mathbf{W}_1 + 2\mathbf{W}_1\mathbf{X}\mathbf{G}_0\mathbf{X}' - \\
&- 2\mathbf{X}\mathbf{G}_0\mathbf{X}'\mathbf{W}_1\mathbf{P_X} - 2\mathbf{P_X}\mathbf{W}_1\mathbf{X}\mathbf{G}_0\mathbf{X}' + 2\mathbf{Z}\,\Delta_0\mathbf{Z}'\mathbf{W}_1 + 2\mathbf{W}_1\mathbf{Z}\,\Delta_0\mathbf{Z}' + 2\mathbf{W}_1\mathbf{X}\mathbf{G}_0\mathbf{X}' + \\
&+ 2\mathbf{X}\mathbf{G}_0\mathbf{X}'\mathbf{W}_1 - 2\mathbf{P_X}\mathbf{W}_1\mathbf{X}\mathbf{G}_0\mathbf{X}' - 2\mathbf{X}\mathbf{G}_0\mathbf{X}'\mathbf{W}_1\mathbf{P_X} + 4\mathbf{Z}\,\Delta_0\mathbf{Z}'\mathbf{W}_1\mathbf{X}\mathbf{G}_0\mathbf{X}' + \\
&+ 4\mathbf{X}\mathbf{G}_0\mathbf{X}'\mathbf{W}_1\mathbf{Z}\,\Delta_0\mathbf{Z}' + 2\mathbf{W}_1\mathbf{Z}\,\Delta_0\mathbf{Z}' + 2\mathbf{Z}\,\Delta_0\mathbf{Z}'\mathbf{W}_1 + 4\mathbf{Z}\,\Delta_0\mathbf{Z}'\mathbf{W}_1\mathbf{Z}\,\Delta_0\mathbf{Z}') - \\
&- 2\lambda(m-1)\,2\mathbf{P_Z} - 2[(m-1)\,\mathbf{Z}\gamma'\mathbf{Z}' + (m-1)\,\mathbf{Z}\gamma\mathbf{Z}'] = \mathbf{O};
\end{aligned}$$

$$\partial\Phi(\mathbf{W}_1, \mathbf{W}_2)/\partial\mathbf{W}_2 = \mathbf{O} \Rightarrow 4\sigma_0^4\mathbf{M_X}\mathbf{W}_2\mathbf{M_X} + 2\sigma_0^2\mathbf{Z}\,\Delta_0\mathbf{Z}'\mathbf{W}_2\mathbf{M_X} + 2\sigma_0^2\mathbf{M_X}\mathbf{W}_2\mathbf{Z}\,\Delta_0\mathbf{Z}' +$$

$$+ 2\sigma_0^2 \mathbf{M_X W_2 Z} \Delta_0 \mathbf{Z' M_X} + 2\sigma_0^2 \mathbf{M_X Z} \Delta_0 \mathbf{Z' W_2 M_X} + 2\mathbf{Z} \Delta_0 \mathbf{Z' W_2 Z} \Delta_0 \mathbf{Z' M_X} +$$
$$+ 2\mathbf{M_X Z} \Delta_0 \mathbf{Z' W_2 Z} \Delta_0 \mathbf{Z'} - 4\lambda \mathbf{P_Z} - 2\mathbf{Z}\gamma' \mathbf{Z'} - 2\mathbf{Z}\gamma \mathbf{Z'} = \mathbf{O}.$$

As $\mathbf{W}_1 = \mathbf{0}$, $\mathbf{W}_2 = 0$, $\lambda = -1/2$ and $\gamma + \gamma' = (\mathbf{Z'Z})^{-1}$ satisfy the given equations we see that $\mathbf{U}_1 = \mathbf{X}(\mathbf{X'X})^{-1}\mathbf{F}(\mathbf{X'X})^{-1}\mathbf{X'}/(m-1)$, $\mathbf{U}_2 = \mathbf{O}$ and thus $\tau_g(\underline{Y}) = (m-1)\,\mathrm{Tr}[\hat{\Sigma}\mathbf{X}(\mathbf{X'X})^{-1}\mathbf{F}(\mathbf{X'X})^{-1}\mathbf{X'}/(m-1)]$. This estimator does not depend on ϑ, therefore it is uniform.

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ОСОБЫЕ СТРУКТУРЫ СМЕШАННЫХ ЛИНЕЙНЫХ МОДЕЛЕЙ С МЕШАЮЩИМИ ПАРАМЕТРАМИ

Lubomír Kubáček

Резюме

Рассмотрены смешанные линейные модели с мешающими параметрами в среднем значении наблюдаемого вектора. Особые структуры позволяют исключить мешающие параметры без затраты информации о полезных параметрах и о вариационных компонентах. Эти структуры порождены либо какими-то повторениями, либо так называемой нечувствительностью полезных параметров на вариационные параметры. В структуре с равномерной нечувствительностью найдена оценка вариационных компонентов.