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ORDER PROPERTIES OF SPLITTING SUBSPACES IN AN INNER PRODUCT SPACE

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ABSTRACT. Let $E(S)$ (resp. $C(S)$) be the orthomodular poset of all splitting subspaces (resp. all complete-cocomplete subspaces) in an inner product space S. As is known, neither $E(S)$ nor $C(S)$ has to be a lattice ([PTÁK, P.—WEBER, H.: *Lattice properties of subspace families in an inner product space*, Proc. Amer. Math. Soc. **129** (2001), 2111–2117]). In this note we test $E(S)$ (resp. $C(S)$) for order properties which are "lattice-like". We show that, in general, either $E(S)$ or $C(S)$ does not have to enjoy the Riesz Interpolation Property. On the other hand, both $E(S)$ and $C(S)$ do possess the regularity property as dealt with in quantum logics (see [HARDING, J.: *Regularity in quantum logics*, Internat. J. Theoret. Phys. **37** (1998), 1173–1212]). In the final observation, we show that a very weak form of countable lattice completeness implies the (topological) completeness of S, contributing slightly to the investigations carried on in [DVUREČENSKIJ, A.: *Gleason's Theorem and Applications*, Kluwer Acad. Publ., Dordrecht-Boston-London, 1993], and elsewhere (the lattice property of $E(S)$ or $C(S)$ is known to be too weak to imply completeness of S, see [PTÁK, P.—WEBER, H.: *Lattice properties of subspace families in an inner product space*, Proc. Amer. Math. Soc. **129** (2001), 2111–2117]).

1. Notions and results

Let S be a separable inner product space ($=$ a prehilbert space) over the real or complex numbers. Let $\langle .,. \rangle$ be the inner product on S. Let us write,

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for any $A \subset S$, $A^\perp = \{b \in S : \langle a, b\rangle = 0 \text{ for any } a \in A\}$. Let us denote by $E(S)$ the set of all subspaces A of S such that $A \oplus A^\perp = S$ (by the symbol $\oplus$ we mean the orthogonal sum). The set $E(S)$ is naturally endowed with the partial ordering of inclusion $\subset$, and with the orthocomplementation $\perp$. Then the set $E(S)$ endowed with $\subset$ and $\perp$ forms an orthomodular poset ([18]), and so does $C(S)$. Obviously, if S is complete (i.e. Hilbert), then $C(S) = E(S)$ and we obtain the lattice of all projections $L(S)$, popular in quantum physics. In [18] the authors show that neither $E(S)$ nor $C(S)$ need to be lattices (though they can be so even for S incomplete). In our first result, we strengthen the cited results by showing that both $C(S)$ and $E(S)$ do not have to enjoy the Riesz Interpolation Property (the RIP for short). Recall ([5]) that an orthomodular poset $(P, \leq, \perp)$ is said to have the *Riesz Interpolation Property* if the following condition is satisfied:

> *If $a, b, c, d \in P$ and $c \leq a$, $c \leq b$, $d \leq a$, $d \leq b$, then there is an element $e \in P$ such that $e \leq a$, $e \leq b$ and $c \leq e$, $d \leq e$.*

It should be noted that a variant of this condition has first been applied in order groups ([8]) and theories of uncertainty ([6]).

Our first result says that $E(c_{00})$ does not have to enjoy RIP. Though we can also construct more sophisticated examples to this effect, and we shall do it later, we felt it desirable to discuss the RIP first in this natural example. Recall that by c_{00} we mean the (inner product) subspace of ℓ^2 whose all but finitely many coordinates are equal to 0.

THEOREM 1. *$E(c_{00})$ does not have the RIP.*

Proof. Let us first prove a few auxiliary results. (The proof technique of this is similar to that of [18] but certain modifications are needed in places. They are based on the new Proposition 6 and its interplay with Proposition 5.)

STATEMENT 1. *Let S be an inner product space and let S have a countable linear basis. Then S and c_{00} are isomorphic as inner product spaces and therefore $E(S)$ and $E(c_{00})$ are isomorphic as orthomodular posets.*

Proof. Use the Gram-Schmidt orthogonalization process in an obvious manner. $\square$

STATEMENT 2. *Take the vector $p = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right) \in \ell^2$, and put $S = c_{00} + (p)$ (thus, $S = \mathrm{Span}\left(c_{00} \cup \{p\}\right)$; S is understood as a subspace of ℓ^2). Then $E(S)$ does not satisfy the RIP.*

Statement 2 will be proved in a series of propositions.

PROPOSITION 1. *Let f_n denote the "Kronecker delta" element of S, i.e. $f_n(m) = \delta_n^m$, where $\delta_n^m = 1$ if $m = n$, $\delta_n^m = 0$ otherwise. Let $F_1 = \mathrm{Span}\{f_n : n \in \mathbb{N},\ n \text{ odd}\}$, $F_2 = \mathrm{Span}\{f_n : n \in \mathbb{N},\ n \text{ even}\}$. Then $F_1 \oplus F_2 = c_{00}$. Moreover, with the orthocomplementation $\perp$ in S, $F_1^\perp = F_2$ and $F_2^\perp = F_1$.*

Proof. Obvious. □

PROPOSITION 2. *We have: $F_1 + (p) \notin E(S)$.*

Proof. If $(F_1 + (p)) \oplus (F_1 + (p))^\perp = S = (F_1 + (p)) + F_2$, then we have $F_2 = (F_1 + (p))^\perp$ (because $(F_1 + (p))^\perp \subset F_1^\perp = F_2$). Then $p \in F_2^\perp$ — a contradiction. □

PROPOSITION 3. *Let*

$$A = \mathrm{Span}\{2n \cdot f_{2n} - (2n+2) \cdot f_{2n+2} :\ n \in \mathbb{N},\ n \text{ odd}\},$$
$$B = \mathrm{Span}\{(2n+2) \cdot f_{2n} + 2n \cdot f_{2n+2} :\ n \in \mathbb{N},\ n \text{ odd}\}.$$

Then $p \in A^\perp$, $A \oplus B = F_2$ and $(F_1 + (p) + B) \oplus A = S$. Thus $(F_1 + (p) + B) \in E(S)$.

Proof. Straightforward. □

PROPOSITION 4. *Let*

$$C = \mathrm{Span}\{(2n+2) \cdot f_{2n+2} - (2n+4) \cdot f_{2n+4} :\ n \in \mathbb{N},\ n \text{ odd}\},$$
$$D = \mathrm{Span}\{f_2\} + \mathrm{Span}\{(2n+4) \cdot f_{2n+2} + (2n+2) \cdot f_{2n+4} :\ n \in \mathbb{N},\ n \text{ odd}\}.$$

Then $p \in C^\perp$, $C \oplus D = F_2$, $C \oplus (F_1 + (p) + D) = S$ (and therefore $F_1 + (p) + D \in E(S)$) and, moreover, $B \cap D = \{o\}$.

Proof. We show that $B \cap D = \{o\}$; the rest merely requires a straightforward verification. Let $x \in B \cap D$. Then

$$x = \sum_{\substack{n=1,\\ n \text{ is odd}}}^{s} \alpha_n \big((2n+2) f_{2n} + 2n \cdot f_{2n+2}\big)$$
$$= \beta_0 f_2 + \sum_{\substack{n=1,\\ n \text{ is odd}}}^{t} \beta_n \big((2n+4) f_{2n+2} + (2n+2) f_{2n+4}\big).$$

If $x \neq o$, then we may assume that $\alpha_s \neq 0$, $\beta_t \neq 0$. Then $2s+2$ (resp. $2t+4$) is the greatest index for which the coefficient f_i in the first sum (resp. second sum) is distinct from 0. But $2s+2 = 2t+4$ and s, t are odd — a contradiction.

□

PROPOSITION 5. *Let $K = \big(F_1 + (p) + B\big)$ and $L = \big(F_1 + (p) + D\big)$. Then both K, L belong to $E(S)$ and, moreover, $K \cap L = F_1 + (p)$.*

Proof. We have shown that $K, L \in E(S)$. Suppose that $z \in K \cap L$. Then $z = x + b = y + d$, where $x, y \in F_1 + (p)$ and $b \in B$, $d \in D$. Then $b - d = y - x \in F_1 + (p)$. But $B, D \subset F_2$ and therefore $b - d = y - x \in F_2 \cap \big(F_1 + (p)\big) = \{o\}$. Thus, $b = d$ and since $B \cap D = \{o\}$, we see that $b = d = o$. We see that $K \cap L = F_1 + (p)$. □

PROPOSITION 6. *Let*

$$G = \operatorname{Span}\big\{(2n-1)f_{2n-1} - (2n+1)f_{2n+1} : \ n \in \mathbb{N}, \ \ n \ \mathit{odd}\big\}\,,$$
$$H = \operatorname{Span}\big\{(2n+1)f_{2n+1} - (2n+3)f_{2n+3} : \ n \in \mathbb{N}, \ \ n \ \mathit{odd}\big\}\,.$$

Then $G \subset F_1$, $H \subset F_1$, $p \in G^{\perp}$, $p \in H^{\perp}$, and both G, H belong to $E(S)$. Moreover, $G + H + f_1 = F_1$.

Proof. Let us show the required properties of G. Obviously, $p \in G^{\perp}$. Let $J = \operatorname{Span}\big\{(2n+1)f_{2n-1} + (2n-1)f_{2n+1} : \ n \in \mathbb{N}, \ \ n \text{ odd}\big\}$. Then $G \oplus J = F_1$ and therefore $G \oplus \big(F_2 + (p) + J\big) = S$. Thus, G belongs to $E(S)$. □

Let us now prove our Theorem 1. We want to show that $E(c_{00})$ does not enjoy the RIP. It is sufficient to prove it for $E(S)$, where $S = c_{00} + (p)$. Consider the definition of RIP and take for c the space $H + (p) + (f_1)$, where H is constructed in Proposition 6, for d the space G of Proposition 6, for a the space K of Proposition 5 and for b the space L of Proposition 5. As shown before, all spaces a, b, c, d belong to $E(S)$ ($H + (p) + (f_1) \in E(S)$ because $H \in E(S)$).

Further, $a \cap b = F_1 + (p) \notin E(S)$ (Proposition 5). But $\operatorname{Span}(c \cup d) = F_1 + (p)$ (Proposition 6) and therefore there is no $e \in E(S)$ such that $c \subset e, d \subset e$ and $e \subset a \cap b$. This proves that $E(S)$ (and therefore also $E(c_{00})$) does not satisfy the RIP. □

Let us test the RIP for $C(S)$ (obviously, $C(c_{00})$ is a (modular) lattice which is therefore RIP). Inspired by [18], let us be as ambitious as to construct an S such that $C(S) = E(S)$ and $C(S)$ does *not* satisfy the RIP.

THEOREM 2. *Let H be a Hilbert space. Then there is a dense hyperplane S in H such that $E(S)$ does not have the RIP. Since in this case $E(S) = C(S)$ ([18]) we see that the orthomodular poset $C(S)$ of complete-cocomplete subspaces of S does not have the RIP.*

Proof. Again, we use some of the technique of [18] (Statement 3). However, the final part of the proof of the Theorem 2 presents an explicit novelty — it in fact gives a simpler proof of the non-lattice property.

Let us divide the proof in a few statements.

STATEMENT 3. *Let H_1 be a Hilbert space,* $\dim H_1 = \infty$. *Then there are closed subspaces C_0, D_0 of H_1 such that $C_0 + D_0 \neq \overline{C_0 + D_0} = H_1$.*

Proof. As is known, there are closed subspaces C, D in H_1 such that $C + D$ is not closed (see e.g. the construction in [18; Proposition 2.2.5]). Then we take $C_0 = C$ and $D_0 = D + (C + D)^{\perp}$. □

Let us return to the proof of Theorem 2.

Let H be the Hilbert space obtained as the orthogonal sum of two infinite dimensional closed subspaces H_0 and H_1. Let C_0, D_0 be closed subspaces of H_1 taken such that $C_0 + D_0$ is not closed and $C_0 + D_0 \neq \overline{C_0 + D_0} = H_1$ (Statement 3). Let S_1 be a hyperplane of H_1 containing $C_0 + D_0$ and let x_1 be a vector of $H_1 \setminus S_1$. Thus, $H_1 = S_1 + (x_1)$ and S_1 is dense in H_1. Analogously, let A_0, B_0 be closed subspaces of H_0 taken such that $A_0 + B_0$ is not closed and $A_0 + B_0 \neq \overline{A_0 + B_0} = H_0$ (Statement 3). Let S_1 be a hyperplane of H_0 containing $A_0 + B_0$ and let x_0 be a vector of $H_0 \setminus S_0$. Thus, $H_0 = S_0 + (x_0)$ and S_0 is dense in H_0.

Let $S = S_0 + S_1 + (x_0 + x_1)$. Then S is a dense hyperplane of H. It is easy to see that $H_1 \cap S = S_1$ and $H_0 \cap S = S_0$. Further, S_0 is orthogonal to S_1 and the vector $x_0 + x_1$ is not orthogonal to either S_0 or S_1. Therefore $S_0^{\perp_S} = S_1$ and $S_1^{\perp_S} = S_0$.

Let $A_1 = A_0^{\perp_H}$, $A = A_0^{\perp_S} = A_1 \cap S$ and, analogously, $B_1 = B_0^{\perp_H}$, $B = B_0^{\perp_S} = B_1 \cap S$. Then, $A_1 \cap B_1 = (A_0 + B_0)^{\perp_H} = H_0^{\perp_H} = H_1$. Further, $A \cap B = A_1 \cap B_1 \cap S = H_1 \cap S = S_1$. In particular, $C_0 + D_0$ is a subspace of both A and B. Moreover, C_0, D_0 are complete and A, B are cocomplete in S. Thus, all these four spaces belong to $C(S)$. Suppose now that there is E in-between $C_0 + D_0$ and $A \cap B$. Then, E cannot be a splitting subspace of S. Indeed, if E is splitting, then E is closed in S. Since $C_0 + D_0$ is dense in $S_1 = A \cap B$, it follows that $E = S_1$. This space is closed as a complement of S_0 but it is not splitting because $S_0^{\perp_S} = S_1$ and $S_1^{\perp_S} = S_0$ but $S_0 + S_1$ is strictly smaller than S. The proof is complete. □

The previous result shows that $E(S)$ may be far from being a lattice in the sense of RIP. Let us ask how far it is in the sense of being regular (each lattice OMP is regular — see [17]). Recall that an OMP P is said to be *regular* if the following implication holds true:

> *If $a, b, c \in P$ and a, b, c are pairwise compatible, then $\{a, b, c\}$ is a distributive triple (i.e. $\{a, b, c\}$ generates a Boolean subalgebra in P).*

It should be noted that the $E(S)$ part of the following theorem can be obtained from [10] and [11] as a consequence of a rather complex algebraic reasoning. We provide a simple linear algebra based proof.

THEOREM 3. *If S is an inner product space, then both $E(S)$ and $C(S)$ are regular OMPs.*

Proof. Let us first consider $E(S)$. Let us suppose that $A, B, C \in E(S)$ are pairwise compatible. We first have (in view of $A \leftrightarrow B$) $A = (A \cap B) \cup (A \cap B^\perp) = (A \cap B) + (A \cap B^\perp)$. Then $S = A + A^\perp = (A \cap B) + (A \cap B^\perp) + A^\perp = A \cap B + A \cap B^\perp + A^\perp \cap B + A^\perp \cap B^\perp$. Let us now show that $A \cap B \leftrightarrow C$. Suppose $x \in A \cap B$. Then there is exactly one pair $x_1 \in C$, $x_2 \in C^\perp$ such that $x = x_1 + x_2$. Since $A = (A \cap C) + (A \cap C^\perp)$, $B = (B \cap C) + (B \cap C^\perp)$, we infer that $\{x_1, x_2\} \subset A \cap B$. It means that $A \cap B \subset (A \cap B) \cap C + (A \cap B) \cap C^\perp$ and since the reverse inclusion is evident, we obtain $A \cap B = (A \cap B) \cap C + (A \cap B) \cap C^\perp$. Analogously, $A \cap B^\perp = A \cap B^\perp \cap C + A \cap B^\perp \cap C^\perp$. Then $S = \sum A^{(i)} \cap B^{(j)} \cap C^{(k)}$, where the meaning of the latter symbols is $M^{(1)} = M^\perp$, $M^{(0)} = M$. We see that $\{A, B, C\}$ is a distributive triple and hence $E(S)$ is regular.

In order to show that $C(S)$ is regular, suppose that $A, B, C \in C(S)$ are pairwise compatible. Since $\{A, B, C\}$ is a distributive triple in $E(S)$, we can (in S) write $S = \sum A^{(i)} \cap B^{(j)} \cap C^{(k)}$ as before. But in the latter sum all but one summand must be complete. Thus, the last one must be complete and this finishes the proof. □

In our third note we want to mildly contribute to the study of order properties of $E(S)$ that are linked with (topological) completeness of S. This question was thoroughly analyzed in [4] and many papers ([9], [6], [2], [3] etc.). Though we also use the standard mechanism based on the Amemiya–Araki technique [1] and thus only complement the known results, our condition seems to differ from the used ones and, in the line of weak σ-completeness, looks natural. Let us say that an orthomodular poset P has the *atomistic subsequential interpolation property* if:

> *For any two sequences of atoms in P, $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, such that $a_i \le b'_j$ for any $i, j \in \mathbb{N}$, there are infinite subsequences $(a_{n_k})_{k\in\mathbb{N}}$, $(b_{n_k})_{k\in\mathbb{N}}$ such that an element $a \in P$ can be found with the following property: For each $k \in \mathbb{N}$ we have $a_{n_k} \le a$ and $b_{n_k} \le a'$.*

(The original version of this condition first appeared in measure theoretic considerations of Boolean algebras, see e.g. [7] and [21]. The condition generalizes the σ-completeness.)

THEOREM 4. *Let S be an inner product space and let $E(S)$ satisfies the atomistic subsequential interpolation property, then S is complete (i.e., S is Hilbert).*

Proof. Suppose $u \in \overline{S} \setminus S$ ($\overline{S}$ is the completion of S). Then there is $w \in S$ such that $\langle w, u\rangle \ne 0$. Let $\beta = \|u\|^2 \langle w, u\rangle^{-1}$ and define a vector $v \in \overline{S}$ by letting $v = u - \beta w$. Obviously, $u \perp v$ in $\overline{S}$. Using the standard inner product procedure (see e.g. [1]), we can construct sequences $(a_i)_{i\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ in S such that $a_i \perp b_j$ for any $i,j \in \mathbb{N}$, and $(a_i)_{i\in\mathbb{N}}$ converges to u and $(b_j)_{j\in\mathbb{N}}$ converges to v. We can view the vectors a_i, b_j as atoms of $E(S)$. Applying the atomistic subsequential property to $(a_i)_{i\in\mathbb{N}}$, $(b_j)_{j\in\mathbb{N}}$, we conclude that there are infinite subsequences $(a_{i_k})_{k\in\mathbb{N}}$, $(b_{j_k})_{k\in\mathbb{N}}$ of $(a_i)_{i\in\mathbb{N}}$, $(b_j)_{j\in\mathbb{N}}$ such that, for some $A \in E(S)$, $a_{n_k} \in A$ and $b_{n_k} \in A^\perp$ for all $k \in \mathbb{N}$. Since $A \oplus A^\perp = S$, we see that $\overline{A} \oplus \overline{A^\perp} = \overline{S}$. Then we can easily obtain a contradiction — the element βw would have two distinct orthogonal decompositions. This completes the proof. $\square$

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