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ONCE MORE ON THE DOUBLE FOURIER-HAAR SERIES

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ABSTRACT. This paper deals with a necessary and sufficient condition for integrability of the majorants Pf of the orthogonal partial sums of Fourier-Haar series.

Many authors deal with the Fourier-Haar series (see G. Alexits [1], A. M. Olevskij [4], P. L. Uljanov [5]–[7] and others).

In [7] there is proved a theorem concerning integrability of the majorants of partial sums of Fourier-Haar series for real measurable functions from the logarithmic scale of Orlicz-type function classes. In [3] this result is completed for whole scale L^p $(p \geq 1)$ of Lebesgue-integrable functions. In [2] a necessary condition on integrability of the majorants of partial sums of double Fourier-Haar series is given. The purpose of this paper is to give a sufficient condition for such integrability.

Let $J = I \times I = [0,1] \times [0,1]$ be the unit square in $\mathbb{R}^2$. Any natural number n has the following representation: $n = 2^k + i$ for $i = 0, 1, \dots, 2^k - 1$ and $k = 0, 1, \dots$. Therefore we can use the following symbols: $I_n = I_k^i = \left(i \cdot 2^{-k}, (i+1) \cdot 2^{-k}\right)$ and $I_{mn} = I_m \times I_n \subset J$. Let the Haar system of functions $h_n(x)$ have the form presented in [1] and [3]. We shall study a representation by Fourier-Haar series of a real function $f(x,y)$ measurable on J.

The orthogonal partial sum

$$S_{mn}(f; x, y) = \sum_{k=0}^{m} \sum_{l=0}^{n} b_{kl} \cdot h_k(x) \cdot h_l(y)$$

of a double Fourier-Haar series

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{kl} \cdot h_k(x) \cdot h_l(y)$$

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has the following integral representation

$$S_{mn}(f;x,y)=\begin{cases}|I_{mn}|^{-1}\cdot\iint\limits_{I_{mn}} f(t,s)\,\mathrm{d}t\,\mathrm{d}s & \text{for } (x,y)\in I_{mn}\,,\\ 0 & \text{for } (x,y)\notin I_{mn}\,.\end{cases}$$

We denote the majorant of these orthogonal partial sums by Pf,

$$(Pf)(x,y)=\sup_{m,n}|S_{mn}(f;x,y)|\,.$$

In [3] the following theorem is proved.

THEOREM 1. *Let $\Phi(u)>0$ be an increasing function defined on the interval $[0,\infty)$ satisfying the condition $\Phi(u)=o\{\log^2(u+1)\}$ for $u\to+\infty$. Then there exists a function f_0 in the Orlicz class $L\Phi(L)$ defined on the set J such that the smallest majorant of orthogonal partial sums of the double Fourier-Haar series of f_0 is not Lebesgue-integrable over J.*

The following theorem gives a complementary sufficient condition for the integrability of the majorant Pf.

THEOREM 2. *Let $\Phi(u)=\log^2(u+1)$ for nonnegative u. Let f be a real measurable function from the Orlicz class $L\Phi(L)$ on $J\subset\mathbb{R}^2$. Then the majorant Pf of orthogonal partial sums of double Fourier-Haar series of the function f is Lebesgue-integrable over J.*

Proof. For simplicity we replace $\log t$ by $\ln t$. We consider $(x,y)\in I_{mn}=(a,c)\times(b,d)$ for some positive constants a,b,c,d ($a\ge 0$, $b\ge 0$, $c\le 1$, $d\le 1$, $c>a$, $d>b$). Then we have

$$\begin{aligned}S_{mn}(f;x,y)&=|I_{mn}|^{-1}\cdot\iint\limits_{I_{mn}} f(t,s)\,\mathrm{d}t\,\mathrm{d}s\\&=|I_{mn}|^{-1}\cdot\int_a^x\int_b^y f(t,s)\,\mathrm{d}s\,\mathrm{d}t+|I_{mn}|^{-1}\cdot\int_a^x\int_y^d f(t,s)\,\mathrm{d}s\,\mathrm{d}t\\&\qquad+|I_{mn}|^{-1}\cdot\int_x^c\int_b^y f(t,s)\,\mathrm{d}s\,\mathrm{d}t+|I_{mn}|^{-1}\cdot\int_x^c\int_y^d f(t,s)\,\mathrm{d}s\,\mathrm{d}t\end{aligned}$$

We can consider $f \geq 0$ on J. Then we have the inequality

$$
\begin{aligned}
&|S_{mn}(f; x, y)| \\
&\leq \frac{1}{(x-a)\cdot(y-b)} \int_a^x \int_b^y f(t,s)\,\mathrm{d}s\,\mathrm{d}t + \frac{1}{(x-a)\cdot(d-y)} \int_a^x \int_y^d f(t,s)\,\mathrm{d}s\,\mathrm{d}t \\
&+ \frac{1}{(c-x)\cdot(y-b)} \int_x^c \int_b^y f(t,s)\,\mathrm{d}s\,\mathrm{d}t + \frac{1}{(c-x)\cdot(d-y)} \int_x^c \int_y^d f(t,s)\,\mathrm{d}s\,\mathrm{d}t .
\end{aligned}
$$

The terms of the sum on the right-hand side of this inequality are evidently nonnegative. We denote the absolute values of the integrals over the set J of these terms by A_1, A_2, A_3, A_4, respectively. We shall estimate now the integral A_1. It is easy to see that we have the equality

$$
\begin{aligned}
A_1 &= \int_0^1 \int_0^1 \frac{1}{|x-a|\cdot|y-b|} \int_a^x \int_b^y f(t,s)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \frac{1}{|y-b|} \int_b^y \int_0^1 \frac{1}{|x-a|} \int_a^x f(t,s)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y .
\end{aligned}
$$

Integrating by parts we get

$$
\begin{aligned}
&\int_0^1 \frac{1}{|x-a|} \int_a^x f(t,s)\,\mathrm{d}t\,\mathrm{d}x \\
&= \ln(1-a)\cdot \int_a^1 f(t,s)\,\mathrm{d}t + \ln a \cdot \int_0^a f(t,s)\,\mathrm{d}t + \int_0^1 f(x,s)\cdot \ln\frac{1}{|x-a|}\,\mathrm{d}x
\end{aligned}
$$

and, after some rearrangements, we obtain

$$A_1 = \int_a^1 \int_b^1 f(t,s)\cdot\Big[\ln(1-a)\cdot\ln\frac{1-b}{s-b} - \ln(1-b)\cdot\ln(t-a)\Big]\,\mathrm{d}s\,\mathrm{d}t$$
$$+\int_a^1 \int_0^b f(t,s)\cdot\Big[\ln(1-a)\cdot\ln\frac{b}{b-s} - \ln b\cdot\ln(t-a)\Big]\,\mathrm{d}s\,\mathrm{d}t$$
$$+\int_0^a \int_0^b f(t,s)\cdot\Big[\ln a\cdot\ln\frac{b}{b-s} - \ln b\cdot\ln(a-t)\Big]\,\mathrm{d}s\,\mathrm{d}t$$
$$+\int_0^a \int_b^1 f(t,s)\cdot\Big[\ln a\cdot\ln\frac{1-b}{s-b} - \ln(1-b)\cdot\ln(a-t)\Big]\,\mathrm{d}s\,\mathrm{d}t$$
$$+\int_0^1 \int_0^1 f(t,s)\cdot\ln\frac{1}{|t-a|}\cdot\ln\frac{1}{|s-b|}\,\mathrm{d}s\,\mathrm{d}t\,.$$

Then the following estimation is true.

$$A_1 < \int_0^1 \int_0^1 f(t,s)\cdot\ln\frac{1}{|t-a|}\cdot\ln\frac{1}{|s-b|}\,\mathrm{d}s\,\mathrm{d}t\,.$$

Define the following sets on J by

$$E_1 = \left\{(t,s)\in J:\ 1+f(t,s)\le \frac{1}{\sqrt{|t-a|\cdot|s-b|}}\right\},$$
$$E_2 = \left\{(t,s)\in J:\ 1+f(t,s) > \frac{1}{\sqrt{|t-a|.|s-b|}}\right\}.$$

With respect to this partition, the following estimation holds:

$$\iint_{E_1} f(t,s)\cdot\ln\frac{1}{|t-a|}\cdot\ln\frac{1}{|s-b|}\,\mathrm{d}s\,\mathrm{d}t$$
$$\le \iint_J \left(\frac{1}{\sqrt{|t-a|.|s-b|}}-1\right)\cdot\ln\frac{1}{|t-a|}\cdot\ln\frac{1}{|s-b|}\,\mathrm{d}s\,\mathrm{d}t$$
$$< \int_0^1 \frac{1}{\sqrt{|t-a|}}\cdot\ln\frac{1}{|t-a|}\,\mathrm{d}t\cdot\int_0^1 \frac{1}{\sqrt{|s-b|}}\cdot\ln\frac{1}{|s-b|}\,\mathrm{d}s\,.$$

Because of the inequalities

$$0 < \int_0^1 \frac{1}{\sqrt{|t-a|}}\cdot\ln\frac{1}{|t-a|}\,\mathrm{d}t < 8$$

the estimation

$$\iint_{E_1} f(t,s)\cdot \ln\frac{1}{|t-a|}\cdot \ln\frac{1}{|s-b|}\;\mathrm{d}s\,\mathrm{d}t < 64$$

takes place. From conditions $|t-a|<1$ and $|s-b|<1$ for $(t,s)\in J$, we obtain the following estimations in E_2

$$1 < \frac{1}{|t-a|\cdot|s-b|} < 1+f(t,s),$$

i.e.

$$\frac{1}{|t-a|} < \{1+f(t,s)\}^2\cdot|s-b| < \{1+f(t,s)\}^2$$

and

$$\frac{1}{|s-b|} < \{1+f(t,s)\}^2.$$

In the sequel we have the following inequalities.

$$\iint_{E_2} f(t,s)\cdot \ln\frac{1}{|t-a|}\cdot \ln\frac{1}{|s-b|}\;\mathrm{d}s\,\mathrm{d}t < \iint_{E_2} f(t,s)\cdot \ln^2\big[1+f(t,s)\big]^2\;\mathrm{d}s\,\mathrm{d}t$$

$$< 4\cdot \iint_{J} f(t,s)\cdot \ln^2\big[1+f(t,s)\big]\;\mathrm{d}s\,\mathrm{d}t.$$

The last integral in the previous estimation is finite according to the condition $f\in L\Phi(L)$ for $\Phi(u)=\ln^2[1+u]$. Then it follows that A_1 is nonnegative and bounded by some constant which is independent of a,b,x,y. And using the same method for A_2, A_3, A_4 we get that they are nonnegative and bounded, too, with some constants independent of a,b,c,d,x,y. According to this conclusion we obtain that the integral

$$\iint_J |S_{mn}(f;x,y)|\;\mathrm{d}x\,\mathrm{d}y$$

is bounded by some nonnegative constant which is independent of m,n and according to the Fatou's lemma we obtain that the integral

$$\iint_J (Pf)(x,y)\;\mathrm{d}x\,\mathrm{d}y$$

is bounded. This completes the proof of the Theorem 2. □

Conclusion. Theorem 1 and Theorem 2 give a necessary and sufficient condition for integrability of the majorants Pf of orthogonal partial sums of double Fourier-Haar series.

REFERENCES

[1] ALEXITS, G.: *Convergence Problems of Orthogonal Series*, Academiai Kiadó, Budapest, 1961.

[2] KOVÁČIK, O.: *On double Fourier-Haar series*, Izv. Vyssh. Uchebn. Zaved. Mat. **11** (1991), 36 38. (Russian)

[3] KOVÁČIK, O.: *On the majorants of partial sums of Fourier-Haar series*, Práce a štú lie VŠDS v Žiline, séria matematicko-fyzikálna **5** (1985), 75–84. (Slovak)

[4] OLEVSKIJ, A. M.: *Fourier Series with Respect to General Orthogonal Systems*, Springer-Verlag, Berlin-Heidelberg, 1975.

[5] ULJANOV, P. L: *Haar series and related questions*. In: Colloq. Math. Soc. János Bolyai 49, North-Holland, Amsterdam, 1987, pp. 57–96.

[6] ULJANOV, P. L.: *On Haar series*, Mat. Sb. **63** (1964), 356–391. (Russian)

[7] ULJANOV, P. L.: *On Haar series with monotone coefficients*, Izv. Akad. Nauk SSSR **32** (1968), 925–950. (Russian)

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