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ON ORIENTED METRIC SPACES

I. L. REILLY—M. K. VAMANAMURTHY

1. Introduction

In a recent paper [3] Bodjanova has reconsidered some of the basic properties of the topological spaces induced by not necessarily symmetric distance functions, termed oriented metrics by Bodjanova but more usually known as quasi-metrics. These topics have received periodic attention from topologists, from the classical 1931 paper [16] of Wilson to the present day. We refer the reader to the papers of Albert [1], Riberio [13], Balanzat [2], Di Concilio [4], Kelly [8], Nedev and Choban [10], Patty [11], Reilly, Subrahmanyam and Vamanamurthy [12], Sion and Zelmer [14], Fletcher and Lindgren [7], and Dutta, Das and Majumdar [6].

As far as applications of quasi-metrics are concerned, Bodjanova [3] has mentioned the cost of transport in hilly regions. Similar examples are the shortest-time distance and the minimum-energy distance, and they have relevance when we consider such things as topography, prevailing winds, river or ocean currents, or man-made barriers to travel such as one-way street systems. Waterman, Smith and Beyer [15] have considered some quasi-metrics of biological origin, and Domiaty [5] has discussed the relevance of quasi-metrics to the structure of space-time.

The main topics studied by Bodjanova [3] are the topologies induced by quasi-metrics with respect to their convergence, completeness and compactness properties. The paper provides several especially nice examples to illustrate the theory it develops. Some of these ideas have been considered as far back as Wilson's early paper [16], and in several subsequent papers. Wilson [16] seems to be the first to distinguish between left and right convergence of a sequence in a quasi-metric space with his definition of u-limit points and l-limit points. More recently, Kelly [8] has studied the two topologies determined for a set X by a pair p and q of conjugate quasi-pseudo-metrics on X, and hence regained some of the symmetry of the metric situation by his consideration of quasi-metric bitopological spaces. Kelly's definition of a Cauchy sequence in a quasi-metric space [8, Definition 2.10] is equivalent to Bodjanova's definition of an r-fundamental sequence [3, Definition 4.6]. Dutta, Das and Majumdar [6] and Majumdar [9] have used the terms r-fundamental and d-fundamental for sequences described by Bodjanova [3] as r-fundamental and l-fundamental. Di Concilio [4] has used a stronger concept of the Cauchy sequence to discuss the same notions. All of these definitions as well as some others are considered in the paper [12] of Reilly, Subrahmanyam and Vamanamurthy, who are concerned with largely the same questions as those which have stimulated Bodjanova [3].

The purpose of this paper is to introduce some new concepts relevant to Bodjanova's work, to improve some of the theorems of [3], and to correct a few minor errors we have observed in [3]. Unless otherwise stated, we will follow the terms and notation used by Bodjanova [3], except that we follow the example of Wilson [16] and use the term quasi-metric where Bodjanova [3] uses oriented metric. In particular, for $x \in X$, in a quasi-metric space (X, ϱ) and for $\varepsilon > 0$, we let $L_\varepsilon(x) = \{y \in X : \varrho(y, x) < \varepsilon\}$ and $R_\varepsilon(x) = \{y \in X : \varrho(x, y) < \varepsilon\}$. As usual, **R** denotes the reals with the usual topology and **N** is the set of natural numbers.

2. Definitions and Results

In the following definitions (X, ϱ) is an arbitrary quasi-metric space and $\{x_n\}$ is a sequence of points in X.

Definition 1. *The sequence $\{x_n\}$ is l-convergent (r-convergent) to a point a in X if $\varrho(x_n, a) \to 0$ as $n \to \infty$ ($\varrho(a, x_n) \to 0$ as $n \to \infty$) in* **R**.

Definition 2. *The sequence $\{x_n\}$ is l-Cauchy (r-Cauchy) if $\varrho(x_n, x_m) \to 0$ ($\varrho(x_m, x_n) \to 0$) as $m \geqslant n \to \infty$.*

Definition 3. *(X, ϱ) is called (r, l)-complete if each r-Cauchy sequence in X is l-convergent to some point in X.*

Similar definitions can be given for (l, l)-, (l, r)-, and (r, r)-completeness.

Definition 4. *(X, ϱ) is called (r, l)-sequentially compact if each sequence of points in X has a subsequence which is r-Cauchy and l-convergent.*

Again there are three similar definitions of (l, l)-, (l, r)-, and (r, r)-sequential compactness.

Definition 5. *(X, ϱ) is called l-sequentially compact (r-sequentially compact) if each sequence of points in X has a subsequence which is l-convergent (r-convergent).*

Definition 6. *(X, ϱ) is said to be l-compact (r-compact) if every l-open (r-open) cover of X has a finite subcover.*

A similar definition of l-countable (r-countable) compactness of (X, ϱ) can be stated.

Definition 7. *Let* $\varepsilon > 0$. *An* $l - \varepsilon$ — *net in* (X, ϱ) *is a subset* A_ε *of* X *such that* $X \subset \cup\{L_\varepsilon(x) \colon x \in A_\varepsilon\}$.

Definition 8. (X, ϱ) *is said to be* l*-totally bounded if* (X, ϱ) *has a finite* $l - \varepsilon$ — *net for each* $\varepsilon > 0$.

Corresponding to Definitions 7 and 8, there are notions of $r - \varepsilon$ — net (where $R_\varepsilon(x)$ replaces $L_\varepsilon(x)$) and r-total boundedness.

We observe that Bodjanova [3, Definition 4.6] uses the term l-fundamental for what we prefer to call an l-Cauchy sequence. Our notion of (l, l)-complete coincides with the l-completeness of Bodjanova [3, Definition 4.11]. However the concept of (r, l)-completeness does not appear in [3]. While the usual notions of l- and r-sequential compactness appear in Bodjanova [3, especially section 5], the ideas of (r, l)-sequential compactness of our Definition 4 are not discussed in [3]. We remind the reader that in a quasi-metric space l-convergence of a sequence does not imply that the sequence is l-Cauchy [3, Example 4.8]. A Comparison of Definitions 7 and 8 above with [3, Definition 5.4] whows that our concepts of l-total boundedness are equivalent.

The following example distinguishes between (l, l)-complete and (r, l)-complete quasi-metric spaces.

Example 1. Let $X = (1, 2]$ and define a quasi-metric ϱ on X by

$$\varrho(x, y) = \begin{cases} y - x & \text{if} \quad x \leqslant y \\ x & \text{if} \quad x > y. \end{cases}$$

Let $\{x_n\}$ be any l-Cauchy sequence in X. If there is a constant subsequence, we are done. Otherwise, by taking a subsequence if necessary, we may assume that $\{x_n\}$ is strictly monotone. If $\{x_n\}$ is monotone decreasing, then $m > n$ implies $x_m < x_n$ and hence that $\varrho(x_n, x_m) = x_n > 1$, which does not have limit 0 in $\mathbf{R}$, contradicting the l-Cauchyness of $\{x_n\}$. Thus $\{x_n\}$ is monotone increasing, and hence $x_n \to a \leqslant 2$ in $\mathbf{R}$, so that $\varrho(x_n, a) = a - x_n \to 0$ in $\mathbf{R}$. Therefore $\{x_n\}$ is l-convergent to a and (X, ϱ) is (l, l)-complete.

However, the sequence $\{y_n\}$, where $y_n = 1 + \dfrac{1}{n}$, is r-Cauchy, since $m \geqslant n$ implies $\varrho(y_m, y_n) = \dfrac{1}{n} - \dfrac{1}{m} \to 0$. Howevery for all points $x \in X$ there is an integer n_0 such that $y_n < x$ for all $n \geqslant n_0$. Thus for $n \geqslant n_0$ we have $\varrho(y_n, x) = x - y_n$, so that $\varrho(y_n, x) \to x - 1 > 0$, in $\mathbf{R}$. Thus $\{y_n\}$ is not l-convergent to x and hence (X, ϱ) is not (r, l)-complete.

The first result we state follows immediately from the definitions.

Proposition 1. *If* (X, ϱ) *is* (r, l)*-sequentially compact, then it is* l*-sequentially compact.* $\square$

Proposition 2. *An l-Cauchy sequence in (X, ϱ) is l-convergent (r-convergent) if and only if it has a subsequence which is l-convergent (r-convergent).*

Proof: The 'only if' part is obvious and we need only prove the 'if' part.

(1) Let $\{x_n\}$ be l-Cauchy and $\{y_k\}$ be a subsequence l-convering to a. Let $\varepsilon > 0$ be given. Then there exists an n_0, $k_0 \in \mathbf{N}$ such that

$$\varrho(x_n, x_m) < \varepsilon \quad \text{and} \quad \varrho(y_k, a) < \varepsilon$$

for all $m \geqslant n \geqslant n_0$ and $k \geqslant k_0$. Let $n \geqslant n_0$ and $k_1 = n + k_0$. Then $\varrho(x_n, a) \leqslant \varrho(x_n, y_{k_1}) + \varrho(y_{k_1}, a) < 2\varepsilon$. Thus $\{x_n\}$ l-converges to a.

(2) Let $\{x_n\}$ be l-Cauchy and $\{y_k\}$ be a subsequence r-converging to a. Let $\varepsilon > 0$ be given. Then there exists n_0, $k_0 \in \mathbf{N}$ such that

$$\varrho(x_n, x_m) < \varepsilon \quad \text{and} \quad \varrho(a, y_k) < \varepsilon$$

for all $m \geqslant n \geqslant n_0$ and $k \geqslant k_0$. Let $k_1 = n_0 + k_0$ and $m \geqslant n(k_1)$, where $x_{n(k_1)} = y_{k_1}$. Then $\varrho(a, x_m) \leqslant \varrho(a, y_{k_1}) + \varrho(y_{k_1}, x_m) < 2\varepsilon$. Thus $\{x_m\}$ r-converges to a. □

We note that Proposition 2 is a generalization of Theorem 4.13 of [3], and that there is an analogous result for the case of an r-Cauchy sequence.

One of the interesting examples discussed by Bodjanova [3] appears throughout her paper as Examples 1.3, 2.3, 3.2, 3.7(b), 4.7, 4.8 and 4.12(b). Let (M, ϱ), where $M = E \cup F \cup G$, be the space described in Example 1.3 of [3, page 278]. It seems that Bodjanova's characterization of $R_\varepsilon(g_c)$ is incorrect [3, Example 2.3]. $R_\varepsilon(g_c)$ is a larger set than claimed, namely

$$R_\varepsilon(g_c) = \{g_c\} \cup \{g_x : x \in (1, c) \cap (c - \varepsilon, c)\} \cup \{e_x : x \in (c - \varepsilon, 1]\} \cup \{f_x : x(c - \varepsilon, 1]\}.$$

In Example 3.7(b) of her paper Bodjanova shows that the following proposition holds. If a subset $A \subset M$ is r-dense in M, then $E \cup F \subset A$. This result can be improved as follows: A is r-dense in M if and only if $E \cup F \cup B \subset A$, where $B = \{g_x : x \in D\}$ and D is a subset dense in $(1, 2]$ with respect to the usual topology.

In [3, Example 4.12(c)] the claim that the space (M, ϱ) of Example 1.4 is neither l-complete not r-complete is not correct. In fact, (M, l) is l-complete but not r-complete, as shown below. Let $\{x_n\}$ be any l-Cauchy sequence in M. Thus for each $\varepsilon > 0$ there is an integer n_0 such that $m > n \geqslant n_0$ implies $\varrho(x_n, x_m) < \varepsilon$. If a subsequence $\{y_k\}$ of $\{x_n\}$ lies in $(1, 2]$, then $\varrho(y_k, y_l) = y_k > 1$. Hence $\{x_n\}$ lies eventually in $(0, 1]$, so that $\varrho(x_n, x_m) = x_n$. Thus $\{x_n\}$ is l-Cauchy in M if and only if $x_n \to 0$ in the reals, $\mathbf{R}$, and so if $a \in M$, we have $\varrho(x_n, a) = x_n$ and hence $\varrho(x_n, a) \to 0$ as $n \to \infty$. Thus $[x_n]L = M$, and (M, ϱ) is l-complete. However, the sequence $\left\{\frac{1}{n}\right\}$ is r-Cauchy but not r-convergent in M. Hence (M, ϱ) is not r-complete. □

Our next result that total boundedness is not hereditary in quasimetric spaces illustrates the pathology possible in quasi-metric spaces when compared with metric spaces.

Proposition 3. *A subspace of an l- (r-) totally bounded quasi-metric space need not be l- (r-) totally bounded.*

Proof. Let (M, ϱ) be the quasi-metric space of Remark 5.10 of Bodjanova [3]. So $M=\left\{\frac{1}{n}\colon n\in\mathbf{N}\right\}$ and $\varrho\left(\frac{1}{n}, 1\right)=\frac{1}{n}$, $\varrho\left(1, \frac{1}{n}\right)=1$ and $\varrho\left(\frac{1}{n}, \frac{1}{m}\right)=1+\frac{1}{n}$ for all $n\neq m\neq 1$. If $0<\varepsilon<1$, then $L_\varepsilon(1)=\left\{\frac{1}{n}\colon n\geqslant n_0+1\right\}$ where $n_0\leqslant\frac{1}{\varepsilon}<n_0+1$, and $L_\varepsilon\left(\frac{1}{n}\right)=\left\{\frac{1}{n}\right\}$ for $n\geqslant 2$. Hence the set $\left\{\frac{1}{n}\colon 1\leqslant n\leqslant n_0\right\}$ is a finite $l-\varepsilon$ — net for (M, ϱ), which is therefore l-totally bounded. However, the subspace $Y=\left\{\frac{1}{n}\colon n\geqslant 2\right\}$ is l-open and l-discrete, and hence cannot be l-totally bounded. □

Our next result has the same conclusion as Theorem 5.7 of Bodjanova [3], but a considerably different hypothesis. The proof is a simple modification of that of [3].

Proposition 4. *If each sequence of points in (M, ϱ) has an l-cauchy (respectively, r-Cauchy) subsequence, then (M, ϱ) is r-totally (or, respectively, l-totally) bounded.*

It is a classical result for metric spaces that completeness and total boundedness are together equivalent to compactness. Theorem 5.9 of Bodjanova [3] is one quasi-metric partial analogue of this metric theorem. Others are given in Theorems 11, 12 and 13 of [12]. We now consider this question in the light of the concepts we have introduced here.

Theorem 1. *If (X, ϱ) is (r, l)-sequentially compact, then it is l-totally bounded and (r, l)-complete.*

Proof. The l-total boundedness of (X, ϱ) follows from Proposition 4.

Let $\{x_n\}$ be an r-Cauchy sequence in X. Then there is a subsequence $\{y_k\}$ of $\{x_n\}$ which is l-convergent to some point b of X. Hence, by Proposition 2, $\{x_n\}$ l-converges to b and (X, ϱ) is (r, l)-complete. □

To observe that the converse of Theorem 1 is false we can employ the space (M, ϱ) of Remark 5.10 of Bodjanova [3]. The sequence $\left\{\frac{1}{n}\right\}$ in M is l-convergent to 1, thus (M, ϱ) is (r, l)-complete. We have shown in Proposition 3 that (M, ϱ) is l-totally bounded. However, the sequence $\left\{\frac{1}{n}\right\}$ has no r-Cauchy subsequence, so that (M, ϱ) is not (r, l)-sequentially compact.

Theorem 2. *If (X, ϱ) is l-compact, then it is l-totally bounded and (r, l)-complete.*

Proof. Let $\varepsilon > 0$. Then $\{L_\varepsilon(x): x \in X\}$ is an l-open cover of X and hence has a finite subcover. Thus (X, ϱ) is l-totally bounded. Next since (X, ϱ) is l-compact it is l-countably compact, and therefore, being first countable, it is l-sequentially compact. Hence, if $\{x_n\}$ is any r-Cauchy sequence in X, there is a subsequence $\{y_k\}$ of $\{x_n\}$ which is l-convergent and the result follows from Proposition 2. □

As a partial converse of Theorem 2 we have the following result.

Theorem 3. *If (X, ϱ) is (r, l)-complete and if each subspace is l-totally bounded, then (X, ϱ) is l-compact.*

Proof. We suppose that (X, ϱ) is not l-compact. Hence there is an l-open cover $\mathscr{G}$ of X which has no finite subcover. Let $\varepsilon_k = \frac{1}{2^k}$ for each positive integer k. Since X is l-totally bounded, it is the finite union of l-balls of radius ε_1. Thus at least one of these balls, say $B_1 = L_{\varepsilon_1}(x_1)$, cannot be covered by finitely many members of $\mathscr{G}$. Now B_1 is l-totally bounded, and so is covered by finitely many l-balls of radius ε_2 with centres in B_1. Again, at least one of them, say $B_2 = L_{\varepsilon_2}(x_2)$, cannot be covered by finitely many members of $\mathscr{G}$. By induction, we obtain a sequence of balls $B_n = L_{\varepsilon_n}(x_n)$. In particular $\varrho(x_2, x_1) < \varepsilon_1$, $\varrho(x_3, x_2) < \varepsilon_2$, ..., $\varrho(x_{n+1}, x_n) < \varepsilon_n$, and so on. Now let $\varepsilon > 0$ and choose n_0 such that $\varepsilon_{n_0} = \frac{1}{2^{n_0}} < \frac{\varepsilon}{2}$. Then for $m \geqslant n \geqslant n_0$ we have

$$\begin{aligned}
\varrho(x_m, x_n) &\leqslant \varrho(x_m, x_{m-1}) + \varrho(x_{m-1}, x_{m-2}) + \ldots + \varrho(x_{n+1}, x_n) \\
&< \varepsilon_{m-1} + \varepsilon_{m-2} + \ldots + \varepsilon_n \\
&= \varepsilon_n \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{m-n-1}}\right) \\
&< 2\varepsilon_n \\
&< \varepsilon.
\end{aligned}$$

Thus $\{x_n\}$ is r-Cauchy, and so l-converges to a point $b \in X$. Then there is a $G_0 \in \mathscr{G}$ such that $b \in G_0$. We choose $\delta > 0$ such that $L_\delta(b) \subset G_0$. Thus there is an integer k_0 such that $n \geqslant k_0$ implies $\varrho(x_n, b) < \frac{\delta}{3}$. We choose $k_1 \geqslant k_0$ such that $\varepsilon_{k_1} = \frac{1}{2^{k_1}} < \frac{\delta}{3}$. We claim that $L_{\varepsilon_{k_1}}(x_{k_1}) \subset G_0$: For if we let $x \in L_{\varepsilon_{k_1}}(x_{k_1})$, then $\varrho(x, b) \leqslant \varrho(x, x_{k_1}) + \varrho(x_{k_1}, b) < \frac{\delta}{3} + \frac{\delta}{3} < \delta$. Hence $x \in L_\delta(b) \subset G_0$. Thus $L_{\varepsilon_{k_1}}(x_{k_1})$ has been covered by a single member of $\mathscr{G}$, which contradicts the fact that it could not be covered by finitely many members of $\mathscr{G}$. Thus (X, ϱ) is l-compact. □

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ОБ ОРИЕНТИРОВАННЫХ МЕТРИЧЕСКИХ ПРОСТРАНСТВАХ

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Резюме

Основные свойства топологических пространств, индуцированных не обязательно симметричными функциями расстояния, были рассмотрены в недавно опубликованной работе: С. Бодйанова, «Некоторые понятия математического анализа в ориентированных метрических пространствах», Math. Slovaca 31, 1981, 277—289.

В этой работе введены некоторые новые понятия, связанные с работой С. Бодйановой, и получены некоторые результаты относительно левой и правой компактности, секвенциальной компактности, вполне ограниченности и полноты в этих пространствах.