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ON THE L^2 DISCREPANCY OF DISTANCES OF POINTS FROM A FINITE SQUENCE

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ABSTRACT. The aim of this paper is to find a quantitative proof of the following: Let $\omega = (x_n)_{n=1}^{\infty}$ be a given infinitive sequence of real numbers from $[0, 1]$ and let $\Omega = (|x_m - x_n|)_{m,m=1}^{\infty}$ be the sequence consisting of all the distances $|x_m - x_n|$, m, $n = 1, 2, \ldots$ and which are ordered such that the first N^2 terms are $(|x_m - x_n|)_{m,n=1}^{N}$ for all $N = 1, 2, \ldots$. Then ω is uniformly distributed if and only if Ω has the asymptotic distribution function $2x - x^2$.

1. Introduction

Let in what follows

$\omega_n = (x_n)_{n=1}^{N}$ be a finite sequence consisting of the N real numbers $x_1, \ldots, x_N$ from the interval $[0, 1]$ and let

$\Omega_{N^2} = (|x_m - x_n|)_{m,n=1}^{N}$ be the finite sequence consisting of the N^2 distances $|x_m - x_n|$, $1 \leqq m, n \leqq N$, in some order.

The aim of this paper is to find relations between the L^2 discrepancies of the two sequences. We shall also present a new method for finding quadrature formulae. Our main tool to do this will be the theory of the Rieman-Stieltjes integration.

In Part 3 we shall apply our results to a quantitative proof of the following probably known theorem: Let $\omega = (x_n)_{n=1}^{\infty}$ be a given infinite sequence of real numbers from $[0, 1]$ and let $\Omega = (|x_m - x_n|)_{m,n=1}^{\infty}$ be the sequence consisting of all the distances $|x_m - x_n|$, m, $n = 1, 2, \ldots$ and which are ordered such that the first N^2 terms are $(|x_m - x_n|)_{m,n=1}^{N}$ for all $N = 1, 2, \ldots$. Then ω is uniformly distributed if and only if Ω has the asymptotic distribution function $2x - x^2$. This result should be compared with the well-known theorem: Let ω be as above and let $\Omega^* = (\{x_m - x_n\})_{m,n=1}^{\infty}$ be the sequence consisting of all the fractional

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parts $\{x_m - x_n\}$ of $x_m - x_n$, $m, n = 1, 2, \dots$ and which are ordered such that the first N^2 terms are $(\{x_m - x_n\})_{m,n=1}^{N}$. Then ω and Ω^* are simultaneously uniformly distributed. This follows from the relations between the discrepancies of $\omega_N = (x_n)_{n=1}^{N}$ and $\Omega^*_{N^2} = (\{x_m - x_n\})_{m,n=1}^{N}$ was first obtained by I. M. Vinogradov [1] in 1926 and aother in 1939 by J. G. Van der Corput and C. Pisot [2].

Before we state our results we require some notation to describe the distribution of the given sequence $\omega_N = (x_n)_{n=1}^{N}$. Let

$$A([0, x), \omega_N) = \operatorname{card}\{n;\ 1 \leqq n \leqq N,\ 0 \leqq x_n < x\}.$$

By the

$$R_N(x) = A([0, x), \omega_N) - Nx,$$

$$R_{N^2}(x) = A([0, x), \Omega_{N^2}) - N^2 x,$$

$$r_{N^2}(x) = A([0, x), \Omega_{N^2}) - N^2(2x - x^2)$$

if $0 \leqq x < 1$ and

$$R_N(x) = R_{N^2}(x) = r_{N^2}(x) = 0$$

if $x < 0$ or $x \leqq 1$, we denote the remainder functions and by the L^2 discrepancies of the sequence ω_N and Ω_{N^2} we mean the integrals $\int_0^1 R_N^2(x)\,dx$, $\int_0^1 R_{N^2}^2(x)\,dx$ and $\int_0^1 r_{N^2}^2(x)\,dx$, respectively.

Our main results can now be stated.

2. Main results

First of all we shall give the expression of the L^2 discrepancy $\int_0^1 r_{N^2}^2(x)\,dx$ in terms of $R_N(x)$.

Theorem 1. *For any finite sequence $\omega_N = (x_n)_{n=1}^{N}$ in $[0, 1]$ we have*

$$\int_0^1 r_{N^2}^2(x)\,dx = \sum_{m,n=1}^{N} \int_0^1 R_N(y)\left(R_N(y - |x_m - x_n|) + R_N(y + |x_m - x_n|)\right) dy$$
$$+ 2N \sum_{n=1}^{N} \int_0^1 \int_0^1 R_N(y)\left(R_N(y - |x_n - v|) + R_N(y + |x_n - v|)\right) dy\,dv$$
$$+ N^2 \int_0^1 \int_0^1 R_N(x) R_N(y)(4|1 - x - y| + 2|x - y| - 6)\,dx\,dy.$$

Proof. With a slight modification of [3, Theorem 5.3, p. 145] we have

$$\int_0^1 R_N^2(x)\,\mathrm{d}x = \frac{1}{3}N^2 + N\sum_{n=1}^{N} x_n^2 - N\sum_{n=1}^{N} x_n + \sum_{m,n=1}^{N} - \frac{|x_m - x_n|}{2}.$$

Using Ω_{N^2} to instead of ω_N, we obtain

$$\int_0^1 R_{N^2}^2(x)\,\mathrm{d}x = \frac{1}{3}N^4 + N^2\sum_{m,n=1}^{N} |x_m - x_n|^2 - N^2\sum_{m,n=1}^{N} |x_m - x_n|$$

$$+ \sum_{m,n,r,s=1}^{N} - \frac{||x_m - x_n| - |x_r - x_s||}{2}. \tag{1}$$

Obviously, we have

$$r_{N^2}(x) = R_{N^2}(x) - N^2(x - x^2).$$

Applying the rule of integration by parts and of the familiar theory of the Riemann-Stieltjes integration, we find

$$\int_0^1 R_{N^2}(x)(x - x^2)\,\mathrm{d}x = -\int_0^1 \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\mathrm{d}R_{N^2}(x) =$$

$$= -\sum_{m,n=1}^{N}\left(\frac{|x_m - x_n|^2}{2} - \frac{|x_m - x_n|^3}{3}\right) +$$

$$+ N^2\int_0^1 \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\mathrm{d}x.$$

Summing up these results we obtain

$$\int_0^1 r_{N^2}^2(x)\,\mathrm{d}x = \frac{1}{5}N^4 + 2N^2\sum_{m,n=1}^{N} |x_m - x_n|^2 - N^2\sum_{m,n-1}^{N} |x_m - x_n| -$$

$$-\frac{2}{3}N^2\sum_{m,n=1}^{N} |x_m - x_n|^3 + \sum_{m,n,r,s=1}^{N} - \frac{||x_m - x_n| - |x_r - x_s||}{2} \tag{2}$$

Let

$$F(x, y, u, v) = \frac{1}{5} + |x - y|^2 + |u - v|^2 - \frac{|x - y| + |u - v|}{2} - \frac{|x - y|^3 + |u - v|^3}{3} -$$

$$- \frac{||x - y| - |u - v||}{2}. \tag{3}$$

Since

$$\int_0^1\int_0^1\int_0^1\int_0^1 F(x, y, u, v)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}v = 0,$$

then, in virtue of (2), we have

$$\int_0^1 r_{N^2}^2(x)\,\mathrm{d}x = \sum_{m,n,r,s=1}^{N} F(x_m, x_n, x_r, x_s) -$$

$$- N^4 \int_0^1 \int_0^1 \int_0^1 \int_0^1 F(x, y, u, v)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}v \tag{4}$$

We need some way of computing the difference on the right-hand side of (4) and this leads us to the investigation of the Riemann-Stieltjes integral with the integrands $\mathrm{d}R_N(x)$. In order to do this we divide

$$\mathrm{d}R_N(x) = \delta_N(x) - N\,\mathrm{d}x\,, \tag{5}$$

where

$$\delta_N(x) = \begin{cases} 1 & \text{if } x \in \omega_N\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

It follows from the theory of the Riemann-Stieltjes integration that we can set

$$\int_0^1 \int_0^1 \dots \int_0^1 f(x, y, \dots)\,\delta_N(x)\,\mathrm{d}y \dots = \sum_{n=1}^{N} \int_0^1 \dots \int_0^1 f(x_n, y, \dots)\,\mathrm{d}y \dots$$

whenever $f(x, y, \dots)$ is continuous for all $x, y, \dots \in [0, 1]$. To illustrate this, take $f(x, y, \dots) = f(x)$. Then

$$\int_0^1 f(x)\,\mathrm{d}R_N(x) = \int_0^1 f(x)\,(\delta_N(x) - N\,\mathrm{d}x) = \int_0^1 f(x)\,\delta_N(x) - N\int_0^1 f(x)\,\mathrm{d}x =$$

$$= \sum_{n=1}^{N} f(x_n) - N\int_0^1 f(x)\,\mathrm{d}x\,.$$

With this notation we see, e.g. in the four-dimensional case, that the problem of finding an expression of the difference

$$\sum_{m,n,r,s=1}^{N} f(x_m, x_n, x_r, x_s) - N^4 \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(x, y, u, v)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}v$$

as a linear combination of

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 f(x, y, u, v)\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}R_N(u)\,\mathrm{d}R_N(v)\,,$$

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 f(x, y, u, v)\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}R_N(u)\,\mathrm{d}v\,, \dots$$

is equivalent to the problem of finding the set of constants A, B, … such that

$$\delta_N(x)\,\delta_N(y)\,\delta_N(u)\,\delta_N(v) - N^4\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}v =$$

$$= A(\delta_N(x) - N\,\mathrm{d}x)(\delta_N(y) - N\,\mathrm{d}y)(\delta_N(u) - N\,\mathrm{d}u)(\delta_N(v) - N\,\mathrm{d}v) +$$
$$+ BN(\delta_N(x) - N\,\mathrm{d}x)(\delta_N(y) - N\,\mathrm{d}y)(\delta_N(u) - N\,\mathrm{d}u)\,\mathrm{d}v + \ldots$$

Since we are viewing f concretely as (3), we arrive at an assumption that for simplicity, let f be a symmetric by

$$f(x, y, u, v) = f(y, x, u, v) = f(u, v, x, y) \tag{6}$$

for all $x, y, u, v \in [0, 1]$. Thus, we need to find A, B, C, D, E such that

$$A\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}R_N(u)\,\mathrm{d}R_N(v) + BN\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}R_N(u)\,\mathrm{d}v +$$
$$+ CN^2\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}u\,\mathrm{d}v + DN^2\,\mathrm{d}R_N(x)\,\mathrm{d}y\,\mathrm{d}R_N(u)\,\mathrm{d}v +$$
$$+ EN^3\,\mathrm{d}R_N(x)\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}v = \delta_N(x)\,\delta_N(y)\,\delta_N(u)\,\delta_N(v) - N^4\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}v\,.$$

Substituting (5) into here and comparing the coefficients we see that $A = 1$, $B = 4$, $C = 2$, $D = 4$ and $E = 4$. Thus, the following quadrature formula we have proved:

Let f: $[0, 1]^4 \to \mathbb{R}$ be any continuous function such that satisfies (6). Then

$$\sum_{m,n,r,s=1}^{N} f(x_m, x_n, x_r, x_s) - N^4 \int_0^1\int_0^1\int_0^1\int_0^1 f(x, y, u, v)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}v =$$
$$= \int_0^1\int_0^1\int_0^1\int_0^1 f(x, y, u, v)\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}R_N(u)\,\mathrm{d}R_N(v)$$
$$+ 4N \int_0^1\int_0^1\int_0^1\int_0^1 f(x, y, u, v)\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}R_N(u)\,\mathrm{d}v$$
$$+ 2N^2 \int_0^1\int_0^1\int_0^1\int_0^1 f(x, y, u, v)\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}u\,\mathrm{d}v$$
$$+ 4N^2 \int_0^1\int_0^1\int_0^1\int_0^1 f(x, y, u, v)\,\mathrm{d}R_N(x)\,\mathrm{d}y\,\mathrm{d}R_N(u)\,\mathrm{d}v$$
$$+ 4N^3 \int_0^1\int_0^1\int_0^1\int_0^1 f(x, y, u, v)\,\mathrm{d}R_N(x)\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}v \tag{7}$$

for every finite sequence $\omega_N = (x_n)_{n=1}^N$ in $[0, 1]$.

Applying (7) to the two-dimensional case, we see that for any continuous function f: $[0, 1]^2 \to \mathbb{R}$ which satisfies $f(x, y) = f(y, x)$ for all $x, y \in [0, 1]$, we obtain

$$N^2\left(\sum_{m,n=1}^{N} f(x_m, x_n) - N^2 \int_0^1\int_0^1 f(x, y)\,\mathrm{d}x\,\mathrm{d}y\right) =$$
$$= N^2 \int_0^1\int_0^1 f(x, y)\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y) + 2N^3 \int_0^1\int_0^1 f(x, y)\,\mathrm{d}R_N(x)\,\mathrm{d}y\,. \tag{8}$$

Now we turn to the proof of the theorem. Separating the sum (3) for F into two parts

$$F(x, y, u, v) = \frac{F_1(x, y) + F_1(u, v)}{2} - \frac{||x-y| - |u-v||}{2},$$

where

$$F_1(x, y) = \frac{1}{5} - |x-y| + 2|x-y|^2 - \frac{2}{3}|x-y|^3$$

we see, by (4), that

$$\int_0^1 r_{N^2}^2(x)\,\mathrm{d}x = N^2\left(\sum_{m,n=1}^{N} F_1(x_m, x_n) - N^2\int_0^1\int_0^1 F_1(x, y)\,\mathrm{d}x\,\mathrm{d}y\right) +$$

$$+\left(\sum_{m,n,r,s=1}^{N} - \frac{||x_m - x_n| - |x_r - x_s||}{2} -\right.$$

$$\left. - N^4\int_0^1\int_0^1\int_0^1\int_0^1 - \frac{||x-y| - |u-v||}{2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}v\right). \qquad (9)$$

The first of these two differences has the representation

$$-2N^2\int_0^1\int_0^1\int_0^1\int_0^1 - \frac{||x-y| - |u-v||}{2}\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}u\,\mathrm{d}v +$$

$$-4N^3\int_0^1\int_0^1\int_0^1\int_0^1 - \frac{||x-y| - |u-v||}{2}\,\mathrm{d}R_N(x)\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}v, \qquad (10)$$

where we have used (8) and the formula

$$\int_0^1\int_0^1 ||x-y| - |u-v||\,\mathrm{d}u\,\mathrm{d}v = F_1(x, y) + \frac{2}{15} \qquad (11)$$

the proof of which follows from a direct computation. The second bracket on the right-hand side of (9) may be computd in the manner of (7). Adding these two expressions, we arrive at

$$\int_0^1 r_{N^2}^2(x)\,\mathrm{d}x = \int_0^1\int_0^1\int_0^1\int_0^1 - \frac{||x-y| - |u-v||}{2}(\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}R_N(u)\,\mathrm{d}R_N(v) +$$

$$+ 4N\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}R_N(u)\,\mathrm{d}v + 4N^2\,\mathrm{d}R_N(x)\,\mathrm{d}y\,\mathrm{d}R_N(u)\,\mathrm{d}v). \qquad (12)$$

The computation of the Rieman-Stieltjes integrals in (12) as the Riemann integrals is based upon the integration by parts and some other ideas. For example, consider now the third integral in (12). We show that

$$\int_0^1\int_0^1\int_0^1\int_0^1 -\frac{||x-y|-|u-v||}{2}\,\mathrm{d}R_X(x)\,\mathrm{d}y\,\mathrm{d}R_X(u)\,\mathrm{d}v =$$

$$= \int_0^1\int_0^1 R_X(x)\,R_X(u)\,(|1-x-u|-|x-u|)\,\mathrm{d}x\,\mathrm{d}u\,. \tag{13}$$

To see this, write

$$-\frac{||x-y|-|u-v||}{2} = -\frac{1}{2}+\sum_{k=1}^{\infty}\frac{1}{2(2k-1)}\frac{(2k)!}{(2^k k!)^2}(1-(|x-y|-|u-v|)^2)^k\,. \tag{14}$$

Since

$$\frac{1}{2(2k-1)}\frac{(2k)!}{(2k^k k!)^2} = \mathrm{O}(k^{-3/2}),$$

the series (14) is uniformly convergent in $[0, 1]^4$ and therefore we have to justify the process of taking the integral of the individual terms in the series. Moreover, by the binomial theorem

$$\int_0^1\int_0^1\int_0^1\int_0^1 (1-(|x-y|-|u-v|)^2)^k\,\mathrm{d}R_X(x)\,\mathrm{d}y\,\mathrm{d}R_X(u)\,\mathrm{d}v$$

$$= \sum_{\substack{i_1+i_2+i_3+i_4=k\\ i_1,i_2,i_3,i_4\geqq 0}}\frac{k!(-1)^{i_1+i_2}.2^{i_3}}{i_1!\,i_2!\,i_3!\,i_4!}\int_0^1\int_0^1 |x-y|^{2i_1+i_3}\,\mathrm{d}R_X(x)\,\mathrm{d}y\,.$$

$$.\int_0^1\int_0^1 |u-v|^{2i_2+i_3}\,\mathrm{d}R_X(u)\,\mathrm{d}v\,.$$

Substituing

$$\int_0^1\int_0^1 |x-y|^k\,\mathrm{d}R_X(x)\,\mathrm{d}y = -\int_0^1 R_X(x)(x^k-(1-x)^k)\,\mathrm{d}x \tag{15}$$

for all $k \geqq 0$, into the above formula and separating the sum for the resulting series into four parts we deduce at once (13).

We cannot obtain in a similar way the desired results for the first and second integral of (12). Since

$$\int_0^1\int_0^1 |x-y|^k\,\mathrm{d}R_X(x)\,\mathrm{d}R_X(y) =$$

$$= \begin{cases} 0 & \text{if } k = 0\,, \\ -2\displaystyle\int_0^1 R_N^2(x)\,\mathrm{d}x & \text{if } k = 1\,, \\ -k(k-1)\displaystyle\int_0^1\int_0^1 R_N(x)\,R_N(y)\,|x-y|^{k-2}\,\mathrm{d}x\,\mathrm{d}y & \text{if } k > 1\,, \end{cases} \tag{16}$$

the order of summation into the above mentioned series cannot be interchanging. Here we have to use another procedure.

From the decomposition (5) of $\mathrm{d}R_N(u)$ and $\mathrm{d}R_N(v)$ these two integrals can be written as

$$\int_0^1\int_0^1\int_0^1\int_0^1 -\frac{||x-y|-|u-v||}{2}\,(\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}R_N(u)\,\mathrm{d}R_N(v) +$$
$$+ 4N\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}R_N(u)\,\mathrm{d}v)$$
$$= \int_0^1\int_0^1\int_0^1\int_0^1 -\frac{||x-y|-|u-v||}{2}\,(\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\delta_N(u)\,\delta_N(v) +$$
$$+ 2N\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\delta_N(u)\,\mathrm{d}v - 3N^2\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}u\,\mathrm{d}v)\,. \tag{17}$$

The integration by parts shows that

$$\int_0^1\int_0^1 -\frac{||x-y|-|u-v||}{2}\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y) =$$
$$= \int_0^1 R_N(y)\,(R_N(y-|u-v|) + R_N(y+|u-v|))\,\mathrm{d}y - \int_0^1 R_N^2(y)\,\mathrm{d}y \tag{18}$$

and it is not difficult to calculate (17). Using (11) and (16) the third integral in (17) can be represented also in the form

$$\int_0^1\int_0^1\int_0^1\int_0^1 -\frac{||x-y|-|u-v||}{2}\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}u\,\mathrm{d}v =$$
$$= -2\int_0^1\int_0^1 R_N(x)\,R_N(y)\,(|x-y|-1)\,\mathrm{d}x\,\mathrm{d}y - \int_0^1 R_N^2(x)\,\mathrm{d}x\,. \tag{19}$$

Collecting these results, our Theorem 1 is proved.

We can use the same method as in Theorem 1 to prove that

Theorem 2. *Let f: $[0, 1]^2 \to \mathbb{R}$ be a continuous function and assume that*

$$f(x, y) = f(y, x) = f(1 - x, y) \tag{20}$$

for all $x, y \in [0, 1]$. Then, for any finite sequence $\omega_N = (x_n)_{n=1}^N$ in $[0, 1]$, we have the integral formula

$$\int_0^1 \int_0^1 f(x, y)\, \mathrm{d}r_{N^2}(x)\, \mathrm{d}r_{N^2}(y) =$$

$$= \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(|x - y|, |u - v|)\, \mathrm{d}R_N(x)\, \mathrm{d}R_N(y)\, \mathrm{d}R_N(u)\, \mathrm{d}R_N(v)\,.$$

Proof. Suppose that $f\colon [0, 1]^2 \to \mathbb{R}$ is continuous and let $f(x, y) = f(y, x)$ for all $x, y \in [0, 1]$. We begin by giving a quadrature formula for the function f, namely we have that

$$\sum_{m,n,r,s=1}^{N} f(|x_m - x_n|, |x_r - x_s|) - N^4 \int_0^1 \int_0^1 f(x, y)(2 - 2x)(2 - 2y)\, \mathrm{d}x\, \mathrm{d}y =$$

$$= \int_0^1 \int_0^1 f(x, y)\, \mathrm{d}r_{N^2}(x)\, \mathrm{d}r_{N^2}(y) + 2N^2 \int_0^1 \int_0^1 f(x, y)\, \mathrm{d}r_{N^2}(x)(2 - 2y)\, \mathrm{d}y \tag{21}$$

In the one-dimensional case we can write

$$\sum_{m,n=1}^{N} f(|x_m - x_n|) - N^2 \int_0^1 f(x)(2 - 2x)\, \mathrm{d}x = \int_0^1 f(x)\, \mathrm{d}r_{N^2}(x)\,. \tag{22}$$

The formula (21) can be proved in the following way.

Differentiating $r_{N^2}(x)$ with respect to x we have

$$\mathrm{d}r_{N^2}(x) = \sigma_{N^2}(x) - N^2(2 - 2x)\, \mathrm{d}x \tag{23}$$

where

$$\sigma_{N^2}(x) = \begin{cases} 1 & \text{if } x = |x_m - x_n| \text{ for some } m, n \leqq N, \\ 0 & \text{otherwise.} \end{cases}$$

If we now consider

$$A\, \mathrm{d}r_{N^2}(x)\, \mathrm{d}r_{N^2}(y) + B\, \mathrm{d}r_{N^2}(x)\, N^2(2 - 2y)\, \mathrm{d}y =$$

$$= \sigma_{N^2}(x)\, \sigma_{N^2}(y) - N^4(2 - 2x)(2 - 2y)\, \mathrm{d}x\, \mathrm{d}y\,,$$

replace $\mathrm{d}r_{N^2}(x)$ and $\mathrm{d}r_{N^2}(y)$ by (23), and observe by comparing the coefficients that $A = 1$ and $B = 2$, we obtain (21).

Integrals which are more suitable for computation may be derived from three following auxiliary results:

a) For continuous $f\colon [0, 1] \to \mathbb{R}$, we have

$$\int_0^1 \int_0^1 f(|x - y|)\, \mathrm{d}x\, \mathrm{d}y = \int_0^1 f(x)(2 - 2x)\, \mathrm{d}x\,. \tag{24}$$

The argument is that

$$\int_0^1 \int_0^1 |x-y|^k \, dx \, dy = \frac{2}{(k+1)(k+2)} = \int_0^1 x^k (2-2x) \, dx$$

for all $k \geqq 0$.

b) Similarly, we may deduce from (15) for continuous f: $[0, 1] \to$

$$\int_0^1 \int_0^1 f(|x-y|) \, dR_N(x) \, dy = -\int_0^1 R_N(x)(f(x) - f(1-x)) \, dx. \tag{25}$$

c) Finally, if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \; f'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1} \text{ and } f''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} \tag{26}$$

are uniformly convergent in [0, 1], then we obtain using (16)

$$\int_0^1 \int_0^1 f(|x-y|) \, dR_N(x) \, dR_N(y) = -\int_0^1 \int_0^1 R_N(x) R_N(y) f''(|x-y|) \, dx \, dy$$

$$- 2f'(0) \int_0^1 R_N^2(x) \, dx. \tag{27}$$

Returning our attention to the integrals in (21) and (22) we see, with the help (25), that their left-hand sides can be written as

$$\sum_{m,n,r,s=1}^{N} f(|x_m - x_n|, |x_r - x_s|) - N^4 \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(|x-y|, |u-v|) \, dx \, dy \, du \, dv$$

and

$$\sum_{m,n=1}^{N} f(|x_m - x_n|) - N^2 \int_0^1 \int_0^1 f(|x-y|) \, dx \, dy,$$

to which the quadrature formulae (7) and (8) may be applied, respectively. We consider first the case (22).

Substituing $f(|x-y|)$ into (8) we can expres (22) as

$$\int_0^1 f(x) \, dr_{N^2}(x) = \int_0^1 \int_0^1 f(|x-y|) \, dR_N(x) \, dR_N(y) +$$

$$+ 2N \int_0^1 \int_0^1 f(|x-y|) \, dR_N(x) \, dy \tag{28}$$

which, by virtue of (24) and (27) and assuming that the conditions of (26) are satisfied, can be represented in the form

$$\int_0^1 r_{N^2}(x) f'(x) \, dx = \int_0^1 \int_0^1 R_N(x) R_N(y) f''(|x-y|) \, dx \, dy + 2f'(0) \int_0^1 R_N^2(x) \, dx +$$

$$+\int_0^1 R_N(x)(f(x)-f(1-x))\,\mathrm{d}x\,. \tag{29}$$

In a similar way, consider the case (21). It is the function $f(|x-y|, |u-v|)$ that can be used as the funcstion $f(x, y, u, v)$ stated in (7). Assuming the symetricity (20) and applying (25) we thus obtain the alternative expression of (21), whose the second, fourth and fifth integrals are equal to zero. So we arrive at

$$\int_0^1\int_0^1 f(x, y)\,\mathrm{d}r_{N^2}(x)\,\mathrm{d}r_{N^2}(y) + 2N^2\int_0^1\int_0^1 f(x, y)\,\mathrm{d}r_{N^2}(x)(2-2y)\,\mathrm{d}y =$$

$$= \int_0^1\int_0^1\int_0^1\int_0^1 f(|x-y|, |u-v|)\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}R_N(u)\,\mathrm{d}R_N(v) +$$

$$+ 2N^2\int_0^1\int_0^1\int_0^1\int_0^1 f(|x-y|, |u-v|)\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}u\,\mathrm{d}v\,.$$

But, according to (28) and (25),

$$\int_0^1\int_0^1 f(x, y)\,\mathrm{d}r_{N^2}(x)(2-2y)\,\mathrm{d}y =$$

$$= \int_0^1\int_0^1\int_0^1\int_0^1 f(|x-y|, |u-v|)\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,\mathrm{d}u\,\mathrm{d}v\,.$$

The proof of Theorem 2 is complete.

In connection with Theorems 1, 2 and [4, (11)], where

$$\int_0^1 R_N^2(x)\,\mathrm{d}x = \int_0^1\int_0^1 -\frac{|x-y|}{2}\,\mathrm{d}R_N(x)\,\mathrm{d}R_N(y)\,,$$

the following integral identity may also be of interest.

Theorem 3. Let $\omega_N = (x_n)_{n=1}^N$ *be* N *real numbers from* $[0, 1]$. *Then*

$$\int_0^1 r_{N^2}^2(x)\,\mathrm{d}x = \int_0^1\int_0^1 -\frac{|x-y|}{2}\,\mathrm{d}r_{N^2}(x)\,\mathrm{d}r_{N^2}(y)\,.$$

Proof. First, denote

$$G(x, y) = \frac{1}{5} + x^2 + y^2 - \frac{x+y}{2} - \frac{x^3+y^3}{3} - \frac{|x-y|}{2}\,.$$

Direct computation shows that

$$\int_0^1\int_0^1 G(x, y)(2-2x)(2-2y)\,\mathrm{d}x\,\mathrm{d}y = 0\,.$$

Observing that in (3)

$$F(x, y, u, v) = G(|x - y|, |u - v|)$$

and applying the expression (4), we obtain

$$\int_0^1 r_{N^2}^2(x)\,\mathrm{d}(x) = \sum_{m,n,r,s=1}^{N} G(|x_m - x_n|, |x_r - x_s|) -$$

$$-N^4 \int_0^1 \int_0^1 G(x, y)(2 - 2x)(2 - 2y)\,\mathrm{d}x\,\mathrm{d}y$$

The result follows from (21) and from

$$\int_0^1 \int_0^1 \left(\frac{1}{5} + x^2 + y^2 - \frac{x + y}{2} - \frac{x^3 + y^3}{3}\right) \mathrm{d}r_{N^2}(x)\,\mathrm{d}r_{N^2}(y) = 0\,,$$

$$\int_0^1 \int_0^1 \left(\frac{1}{5} + x^2 + y^2 - \frac{x + y}{2} - \frac{x^3 + y^3}{3} - \frac{|x - y|}{2}\right) \mathrm{d}r_{N^2}(x)(2 - 2y)\,\mathrm{d}y = 0\,.$$

3. Application

Theorems 1 and 2 has the following application:

Theorem 4. *A necessary and sufficient condition for the sequence* $\omega = (x_n)_{n=1}^{\infty}$, $x_n \in [0, 1]$, *to be uniformly distributed in* $[0, 1]$ *is that the sequence* $\Omega = (|x_m - x_n|)_{m,n=1}^{\infty}$ *have the asymptotic distribution function* $2x - x^2$.

Proof. Befre we prove this theorem we shall need some preparation:

a) As we have already defined $r_{N^2}(x)$ in the introduction, let

$$r_N(x) = \begin{cases} A([0, x), \Omega_N) - N(2x - x^2) & \text{if } 0 \leqq x < 1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

for all $N = 1, 2, \ldots$, where the section Ω_N consisting of the first N element of Ω. Here, as already mentioned, the terms $|x_m - x_n|$ of Ω are ordered in such a way that $\Omega_{N^2} = (|x_m - x_n|)_{m,n=1}^{N}$ for all $N = 1, 2, \ldots$ and the ordering in which the terms of $\Omega_{(N-1)^2} - \Omega_{N^2}$ are given, is arbitrary.

b) The sequence Ω is said to have the asymptotic distribution function $2x - x^2$ if

$$\lim_{N \to \infty} \frac{r_N(x)}{N} = 0 \tag{30}$$

for all $x \in [0, 1]$. Since

$$|r_{N^2+n}(x) - r_{N^2}(x)| \leqq n \text{ and } \lim_{N\to\infty} ((N+1)^2 - N^2)/N^2 = 0$$

we have that this limit is equivalent to the following

$$\lim_{N\to\infty} \frac{r_{N^2}(x)}{N^2} = 0 \tag{31}$$

for all $x \in [0, 1]$.

c) Finally, (31) is equivalent to the condition that

$$\lim_{N\to\infty} \frac{1}{N^4} \int_0^1 r_{N^2}^2(x)\, \mathrm{d}x = 0\,. \tag{32}$$

The proof is based on the estimate

$$\sup_{0\leqq x\leqq 1} |r_{N^2}(x)|^3 \leqq 12N^2 \int_0^1 r_{N^2}^2(x)\, \mathrm{d}x\,, \tag{33}$$

which can obviously be compared by the well known (see [5])

$$\sup_{0\leqq x\leqq 1} |R_N(x)|^3 \leqq 3N \int_0^1 R_N^2(x)\, \mathrm{d}x\,. \tag{34}$$

From this, as a trival corollary, we have that the sequence ω is uniformly distributed if and only if

$$\lim_{N\to\infty} \frac{1}{N^2} \int_0^1 R_N^2(x)\, \mathrm{d}x = 0\,. \tag{35}$$

To get (33) we let

$$h(x) = \begin{cases} \dfrac{A([0, x), \Omega_{N^2}]}{N^2} & \text{if } 0 \leqq x < 1\,, \\ 1 & \text{if } x = 1\,. \end{cases}$$

Consider the function

$$\frac{1}{N^2} r_{N^2}(x) = h(x) - (2x - x^2)\,.$$

Here the right-hand side is the difference between these two functions, the step-function $h(x)$ and the convex function $y = 2x - x^2$. Take

$$h = \frac{1}{N^2} \sup_{0\leq x<1} |r_{N^2}(x)|$$

and assume that h is attained in $x_0 \in [0, 1]$ or $x_0 + 0$ or $x_0 - 0$. Let $y_0 = 2x_0 - x_0^2$ and consider the inverse function $x = 1 - \sqrt{1 - y}$. Evidently, as geometrical considerations show,

$$\int_{y_0}^{y_0 + h} (x - x_0)^2 \, \mathrm{d}y \leqq \frac{1}{N^4} \int_0^1 r_{N^2}^2(x) \, \mathrm{d}x .$$

The left-hand side of this inequality is equal to

$$\int_{y_0}^{y_0 + h} \left(\frac{y - y_0}{\sqrt{1 - y_0} + \sqrt{1 - y}} \right)^2 \mathrm{d}y$$

and thus $\geqq h^3/12$, as is not difficult to verify. Hence our statement c) has been proved.

We can now proceed to prove the theorem. It will therefore suffice to prove that a necessary and sufficient condition for the limit (35) is that the imit (33) holds.

From Theorem 1 we derive

$$\int_0^1 r_{N^2}^2(x) \, \mathrm{d}x \leqq 12N^2 \int_0^1 R_N^2(x) \, \mathrm{d}x \tag{36}$$

and the necessity part of the theorem is therefore proved.

In other words, we shall consider only the one-dimensional case of Theorem 2. Take $f(x) = x - x^2$ in (29). This gives

$$\int_0^1 r_{N^2}(x)(1 - 2x) \, \mathrm{d}x = 2 \int_0^1 R_N^2(x) \, \mathrm{d}x - 2\left(\int_0^1 R_N(x) \, \mathrm{d}x \right)^2 \tag{37}$$

According to [3, p. 110], we have

$$\int_0^1 r_{N^2}(x)(1 - 2x) \, \mathrm{d}x = \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \left| \sum_{n=1}^{N} \mathrm{e}^{2\pi i k x_n} \right|^2 \tag{38}$$

Moreover, by an estimate from the LeVeque [6], the right-hand side of (38) is greater or equal than

$$\frac{1}{6N} \sup_{0 \leqq x \leqq 1} |R_N(x)|^3 .$$

and therefore, an application of the Cauchy—Schwarz inequality yields

$$\int_0^1 r_{N^2}^2(x) \, \mathrm{d}x \geqq \frac{1}{12N^2} \sup_{0 \leqq x \leqq 1} |R_N(x)|^6 . \tag{39}$$

Hence, the sufficiency part is easy.

Finally, we notice that:

a) If a sequence ω is uniformly distributed in [0, 1], then, using (1) for the usual L^2 discrepancy of Ω_{N^2}, we have

$$\lim_{N \to \infty} \frac{1}{N^4} \int_0^1 R_{N^2}^2(x)\,\mathrm{d}x = \frac{1}{3} + \int_0^1 \int_0^1 (x-y)^2\,\mathrm{d}x\,\mathrm{d}y - \int_0^1 \int_0^1 |x-y|\,\mathrm{d}x\,\mathrm{d}y +$$
$$+ \int_0^1 \int_0^1 \int_0^1 \int_0^1 - \frac{||x-y|-|u-v||}{2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{30} \cdot$$

b) Let ω_N be a symmetric sequence by the

$$\omega_N = (x_n)_{n=1}^N = (1 - x_n)_{n=1}^N .$$

Then

$$\int_0^1 R_N(x)\,\mathrm{d}x = 0$$

and from (37) we derive

$$\int_0^1 r_{N^2}^2(x)\,\mathrm{d}x \geqq 12 \left(\int_0^1 R_N^2(x)\,\mathrm{d}x \right)^2 .$$

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