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Mathematica Slovaca, Vol. 40 (1990), No. 3, 279--286

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ABSTRACT. Using the technique of additive functionals an extension of a partially ordered, separative group valued measure is obtained.

The measure extension problem is one of the basic problems of the measure theory from its beginning. In paper [3], there are fifty four references concerning the measure extension problem for vector measures. Lately some studies have been published also for measures with values in vector lattices or even in l-groups (see [1], [5], [7], /[8] and [9]). The extension process in the mentioned paper is based on using some algebraic properties of vector lattices or l-groups which are similar to some algebraic properties of real numbers.

Other techniques were used in papers [4] and [6]. In these papers the measure extension problem is transfered by means of linear functionals to the measure extension problem for real measures. In this paper we combine the technique of linear functionals with the measure extension method from [2]. As a result we extend a partially ordered, separative group valued measure. In contrast with [4] we do not assume that our group is a lattice. We differ from [6], too, because we do not use the transfinite induction and the extended measure is complete.

0. Preliminary

If $\{a_n\}$ is an increasing (decreasing) sequence of elements of a partially ordered set M and $\vee_n a_n = a$ ($\wedge_n a_n = a$), we shall write $a_n \nearrow a$ ($a_n \searrow a$). In these cases we shall write also $\lim_n a_n = a$.

We say that a sequence of elements of a partially ordered set M converges in order to x (in symbol $x_n \to x$ or $x_n \xrightarrow{o} x$) iff there exist sequences $\{u_n\}$ and $\{v_n\}$ in M with $u_n \leqq v_n$, $u_n \nearrow x$ and $v_n \searrow x$.

A partially ordered group is a set G endowed with a structure of a partially

AMS Subject Classification (1980): Primary 28B10.
Key words: Partially ordered group, Convergence group, Complete measure.

ordered space and a structure of a group satisfying the following compatibility condition:

If x, y and z are in G and $x \leqq y$, then

$$x + z \leqq y + z .$$

θ will denote the neutral element in G. By G_+ we denote the set of all non-negative elements in G.

1. Lemma. *Let G be a partially ordered group. Let* $x_n \nearrow x$ ($x_n \searrow x$) *and* $y_n \nearrow y$ ($y_n \searrow y$), *then*

(i) $x_n + y_n \nearrow x + y$ $\quad$ $(x_n + y_n \searrow x + y)$

(ii) $-x_n \searrow -x$ $\quad$ $(-x_n \nearrow -x)$.

An easy consequence of the above lemma and the definition of the order convergence is the following:

2. Theorem. *Let G be a partially order group. Then G is a convergence group with respect to the order convergence (i.e. the map* $(x, y) \mapsto x - y$ is order continuous).

As we shall deal with the extension of the measure, it is natural to assume some sort of completeness of the range space. If the range space of the measure is not complete, then, as the following example shows, the extension of the measure need not exist.

3. Example. Let $\mathscr{R}$ be a ring of subsets of reals which are finite disjoint unions of intervals with rational endpoints. Let G be the group of rational numbers, and let $\mu : \mathscr{R} \to G$ be the restriction of the Borel measure to $\mathscr{R}$. Then μ has no extension to the generated σ-ring.

Our completeness property is the following: We say that a partially ordered group G is monotone σ-complete if every monotone increasing bounded sequence $\{x_n\}$ has a limit in G, i.e.

$$\lim_n x_n = \vee_n x_n$$

exists in G.

Let f: $G \to R$ (real numbers) be a functional. We shall say that f is additive iff

$$f(x + y) = f(x) + f(y),$$

f is monotone iff $x \leqq y$ implies $f(x) \leqq f(y)$,

f is o-cóntionuous, iff $x_n \xrightarrow{o} x$ iplies $f(x_n) \to f(x)$.

It is easy to see that for an additive functional one has $f(\theta) = 0$ and $f(-x) = -f(x)$.

If G is a partially ordered group, then by the order dual of G we mean the set $G^<$ of all order continuous additive functionals on G which can be represented as a difference of two monotone additive functionals.

For our extension process, one of the basic assumptions is that $G^<$ separates points of G, i.e. for $x \in G$, $x \neq \theta$ there exists an $x^<$ in $G^<$ with $x^<(x) \neq 0$. If this is the case, we shall say that G is separative. It is easy to see that $G^<$ separates points of G if and only if the set of all monotone elements from $G^<$ (denoted by $G_+^<$) separates points of G.

To illustrate our definitions, we give now an example (see [6]).

4. Example. Let P_m be the set of all polynomials of the form

$$a_0 + a_1 x + \dots + a_{m-1} x^{m-1} + a_m x^m$$

(m fixed) with the pointwise ordering. Then P_m is a separative, monotone σ-complete group.

It is easy to see that a separative group is always a Hausdorff topological group with respect to the $G^<$-weak topology on G. Moreover, every o-continuous functional is continuous in this topology.

Let G be a partially ordered group and let $\mathscr{A}$ be an algebra of subsets of X. We shall say that $\mu\colon \mathscr{A} \to G$ is a partially ordered group valued measure on $\mathscr{A}$ if

(i) $\mu(A \cap B) + \mu(A \cup B) = \mu(A) + \mu(B)$ (additivity) for every A, $B \in \mathscr{A}$.

(ii) If A, $B \in \mathscr{A}$ and $A \subset B$, then $\mu(A) \leqq \mu(B)$.

(iii) If $A_n \searrow 0$, then $\mu(A_n) \searrow \theta$.

It is easy to see that

$$A_n \nearrow A \text{ implies } \mu(A_n) \nearrow \mu(A) \text{ and similarly}$$
$$A_n \searrow A \text{ implies } \mu(A_n) \searrow \mu(A) \text{ for every } A_n, A \text{ in } \mathscr{A}.$$

1. The construction

Throughout the rest of the paper, we shall assume that G is a partially ordered, separative, monotone σ-complete group, $\mathscr{A}$ is an algebra of subsets of the set X and $\mu\colon \mathscr{A} \to G_+$ is a measure.

Let $\mathscr{H}$ be a family of subsets of X. $\mathscr{H}_\sigma$ ($\mathscr{H}_\delta$) is a system of all sets expressible as a union (intersection) of an increasing (decreasing) sequence $\{A_k\}$ of elements in $\mathscr{H}$. Sometimes we shall write $\mathscr{H}_{\sigma\delta} = (\mathscr{H}_\sigma)_\delta$ and $\mathscr{H}_{\delta\sigma} = (\mathscr{H}_\delta)_\sigma$.

We define μ_1 on $\mathscr{A}_1 = \mathscr{A}_\sigma \cup \mathscr{A}_\delta$ and μ_2 on $\mathscr{A}_2 = \mathscr{A}_{\sigma\delta} \cup \mathscr{A}_{\delta\sigma}$ as follows. Let $A \in \mathscr{A}_1$ ($A \in \mathscr{A}_2$) and let $\{A_k\}$ be a monotone sequence of sets from $\mathscr{A}$ ($\mathscr{A}_1$) such that $\lim_k A_k = A$.
We put

$$\mu_1(A) = \lim_k \mu(A_k \qquad (\mu_2(A) = \lim_k \mu_1(A_k)).$$

We shall show that the definition of μ_1 and μ_2 does not depend on the sequence $\{A_k\}$.

Now we add to the system $\mathscr{A}_2$ all sets which differ from a set in $\mathscr{A}_2$ "a little". And on this new system we define the set function which, as we shall see, will be a reqired extension.

Denote by $\mathscr{S}_\mu(\mathscr{A})$ the system of all subsets of X for which there exist sets $A_1 \in \mathscr{A}_{\delta\sigma}$, $A_2 \in \mathscr{A}_{\sigma\delta}$, $A_1 \subset A \subset A_2$ with $\mu_2(A_2 - A_1) = \theta$ ($A_2 - A_1$ is clearly in $\mathscr{A}_{\sigma\delta}$) and put $\bar{\mu}(A) = \mu_2(A_1)$. We shall prove that the definition of $\bar{\mu}(A)$ does not depend on A_1 and A_2.

In paper [2], it was proved that if μ is a finite real measure, then $\mathscr{S}_\mu(\mathscr{A})$ is a σ-algebra and $\bar{\mu}$ is a continuous extension of μ to the $\mathscr{S}_\mu(\mathscr{A})$.

The main result of the paper is the following:

5. Theorem. *Let μ be a measure defined on an algebra $\mathscr{A}$. Let the range space of μ be a separative monotone σ-complete partially ordered group G. Then $\mathscr{S}_\mu(\mathscr{A})$ is a σ-algebra which contains $\mathscr{A}$, and $\bar{\mu}$ is the unique complete measure on $\mathscr{S}_\mu(\mathscr{A})$, which extends μ.*

2. The proofs

We shall now show that the definitions we have given above are all right. Also, we give some assertions which are necessary for the proof of Theorem 5.

The proofs of the following two lemmas are similar to the real case. For that case they can be found in [2].

6. Lemma. *μ_1 is an additive set function on $\mathscr{A}_1$.*

7. Lemma. *μ_1 is monotone on $\mathscr{A}_1$.*

The proof of the continuity of μ_1 is based on the separativity of $(G, +, \leqq)$. Note that if $f \in G_+^<$, then $f \circ \mu$ is a real measure on $\mathscr{A}$.

8. Lemma. *Let $f \in G_+^<$ and let ν be a σ-additive extension of the real measure $f \circ \mu$ from the algebra $\mathscr{A}$ to the σ-algebra generated by $\mathscr{A}$. Then $\nu(A) = f \circ \mu_1(A)$ for every A in $\mathscr{A}_1$.*

Proof. Let $A \in \mathscr{A}_\sigma$, then there exists a monotone sequence $\{A_n\} \subset \mathscr{A}$ with $A_n \nearrow A$. By the continuity of f we get $\lim_n f \circ \mu(A_n) = f(\lim_n \mu(A_n)) = f \circ \mu_1(A)$. Hence $\nu(A) = \lim_n \nu(A_n) = f \circ \mu_1(A)$. The other case is similar.

9. Lemma. *The function μ_1 is continuous on $\mathscr{A}_1$.*

Proof. If $A_n \nearrow A$ and A_n, A are in $\mathscr{A}_1$, then $\mu_1(A_n) \leqq \mu_1(A)$. By the monotone σ-completeness of $(G, +, \leqq)$, there exists an element $z \in G_+$ with $\mu_1(A_n) \nearrow z$. Let $f \in G_+^<$ and let ν be a continuous extension of the real measure $f \circ \mu_1$. Then

$$f(z) = f(\lim_n \mu_1(A_n) = \lim_n f(\mu_1(A_n)) = \lim_n \nu(A_n) = \nu(A) = f(\mu_1(A)).$$

Since $G_+^<$ separates points of G and $f(z) = f(\mu_1(A))$ for every f in $G_+^<$, we get that $z = \mu_1(A)$. The case $A_n \searrow A$ is similar.

10. Lemma. *Let A_n and B_k be in $\mathscr{A}_1$ with $A_n \searrow A$, $B_k \searrow B$ and $A \subset B$, then* $\lim_n \mu_1(A_n) \leqq \lim_k \mu_1(B_k)$.

Proof. If there exists a subsequence $\{B_{k_n}\}$ of $\{B_k\}$ with $B_{k_n} \in \mathscr{A}_\delta$, then the proof is trivial. In the other case we may assume that $B_k \in \mathscr{A}_\sigma$ for every k. In this case for every B_k there exists a sequence of sets $C_{n,k}$ in $\mathscr{A}$ such that $C_{n,k} \nearrow B_k$ $(n \to \infty)$. Put $D_{n,k} = A_n - C_{n,k}$, then $D_{n,k} \in \mathscr{A}_1$ and $D_{n,k} \searrow \emptyset$ $(n \to \infty)$. By the last lemma, $\lim_n \mu_1(D_{n,k}) = \theta$.

Since

$$\mu_1(A_n) - \mu_1(C_{n,k}) \leqq \mu_1(A_n) - \mu_1(A_n \cap C_{n,k})$$
$$\leqq \mu_1(A_n - A_n \cap C_{n,k}) = \mu_1(A_n - C_{n,k}) = \mu_1(D_{n,k})$$

and

$$\lim_n(\mu_1(A_n) - \mu_1(C_{n,k})) = \lim_n \mu_1(A_n) - \mu_1(B_k),$$

we have

$$\lim_n \mu_1(A_n) - \mu_1(B_k) = \lim_n \mu_1(D_{n,k}) = \theta.$$

This implies

$$\lim_n \mu_1(A_n) \leqq \mu_1(B_k),$$

hence

$$\lim_n \mu_1(A_n) \leqq \lim_k \mu_1(B_k).$$

In a similar way, one can get the following result:

11. Lemma. *If A_n and B_k are in $\mathscr{A}_1$ with $A_n \nearrow A$, $B_k \searrow B$ and $A \subset B$, then*

$$\lim_n \mu_1(A_n) \leqq \lim_k \mu_1(B_k).$$

Let us now turn our attention to the set function μ_2. If $E \in \mathscr{A}_2$ and $\{E_n\}$ is a monotone sequence of sets from $\mathscr{A}_1$ such that $\lim_n E_n = E$, then clearly $\lim_n \mu_1(E_n)$ exists.

According to lemmas 11, 12 and the additivity of μ_1 we have:

12. Lemma. *μ_2 is well defined.*

Similarly as in the case of the set function μ_1, one can prove the following properties of the set function μ_2.

13. Lemma. *μ_2 is an additive, monotone and continuous set function on $\mathscr{A}_2$.*

Let us note that when proving the continuity of μ_2 one has to use the following fact (similarly as in the case of μ_1): for every A in $\mathscr{A}_2$, $\nu(A) = f \circ \mu_2(A)$, where $f \in G_+^<$ and ν is a continuous extension of the real measure $f \circ \mu$ to the σ-algebra generated by $\mathscr{A}$. Now similarly as we defined μ_1 on $\mathscr{A}_1$ and μ_2 on $\mathscr{A}_2$ we can define $(f \circ \mu)_1$ and $(f \circ \mu)_2$ on $\mathscr{A}_1$ resp. $\mathscr{A}_2$, namely if $A \in \mathscr{A}_1$ $(A \in \mathscr{A}_2)$ and $\{A_k\}$ is a monotone sequence of sets from $\mathscr{A}$ $(\mathscr{A}_1)$ such that $\lim_k A_k = A$, we put

$$(f \circ \mu)_1(A) = \lim_k f \circ \mu(A) \qquad ((f \circ \mu)_2 = \lim_k (f \circ \mu)_1(A)).$$

Directly from the definition of $(f\circ\mu)_1$ and $(f\circ\mu)_2$ we get:

14. Lemma. $(f\circ\mu)_2 = f\circ\mu_2$.

Let us turn now our attention to the family $\mathscr{S}_\mu(\mathscr{A})$.

15. Lemma. *$\mathscr{S}_\mu(\mathscr{A})$ is a σ-algebra.*

Proof. We shall show that

$$\mathscr{S}_\mu(\mathscr{A}) = \cap\{\mathscr{S}_{f\ \mu}(\mathscr{A}); f\in G_+^<\}.$$

The definition of the set $\mathscr{S}_{f\ \mu}(\mathscr{A})$ is similar to the definition of $\mathscr{S}_\mu(\mathscr{A})$ (see "The construction"). Since $f\circ\mu$ is a real measure, we get (see [2]) that $\mathscr{S}_{f\ \mu}(\mathscr{A})$ is a σ-algebra for every f in $G_+^<$.

Let $A\in\mathscr{S}_\mu(\mathscr{A})$, then there exist sets $A_1\in(\mathscr{A}_{\delta\sigma}$ and $A_2\in\mathscr{A}_{\sigma\delta}$ with $A_1\subset A\subset A_2$ and $\mu_2(A_2-A_1)=\theta$. By Lemma 14, $(f\circ\mu)_2(A_2-A_1)=0$. Thus $A\in\mathscr{S}_{f\ \mu}(\mathscr{A})$ for every f in $G_+^<$, hence

$$\mathscr{S}_\mu(\mathscr{A}) \subset \cap\{\mathscr{S}_{f\ \mu}(\mathscr{A}); f\in G_+^<\}.$$

Now if $A\notin\mathscr{S}_\mu(\mathscr{A})$, then for every $A_1\in\mathscr{A}_{\delta\sigma}$ and $A_2\in\mathscr{A}_{\sigma\delta}$ with $A_1\subset A\subset A_2$ there holds $\mu_2(A_2-A_1)\neq\theta$. Since G is separative, there exists an f in $G_+^<$ with $f(\mu_2(A_2-A_1))\neq 0$. Because $(f\circ\mu)_2 = f\circ\mu_2$, we have $A\notin\mathscr{S}_{f\ \mu}(\mathscr{A})$ hence

$$\mathscr{S}_\mu(\mathscr{A}) = \cap\{\mathscr{S}_{f\ \mu}(\mathscr{A}); f\in G_+^<\}.$$

16. Corollary. $\mathscr{A}\subset\mathscr{S}_\mu(\mathscr{A})$.

Proof. Since $\mathscr{A}\subset\mathscr{S}_{f\ \mu}(\mathscr{A})$ for every $f\in G_+^<$ see [2], the proof is trivial.

17. Lemma. *$\bar\mu$ is an additive set function on $\mathscr{S}_\mu(\mathscr{A})$.*

Proof. First we show that $\bar\mu$ is well defined. Let $A\in\mathscr{S}_\mu(\mathscr{A})$ with A_1, $B_1\in\mathscr{A}_{\delta\sigma}$, A_2, $B_2\in\mathscr{A}_{\sigma\delta}$, $A_1\subset A\subset A_2$, $B_1\subset A\subset B_2$ and $\mu_2(A_2-A_1)=\mu_2(B_2-B_1)=0$. It is clear that $\mu_2(A_1)=\mu_2(A_2)$ and $\mu_2(B_1)=\mu_2(B_2)$. Since $A_1\subset B_2$ and $B_1\subset A_2$, we get $\mu_2(A_1)\leqq\mu_2(B_2)$ and $\mu_2(B_1)\leqq\mu_2(A_2)$. Hence $\mu_2(A_2)\leqq\mu_2(B_1)$ and $\mu_2(B_1)\leqq\mu_2(A_1)$, thus $\mu_2(A_1)=\mu_2(B_1)$.

Let us turn our attention to the additivity of μ_2. Let A, $B\in\mathscr{S}_\mu(\mathscr{A})$ with A_1, $B_1\in\mathscr{A}_{\delta\sigma}$, A_2, $B_2\in\mathscr{A}_{\sigma\delta}$, $A_1\subset A\subset A_2$, $B_1\subset B\subset B_2$ and $\mu_2(A_2-A_1)=\mu_2(B_2-B_1)=0$.

Since

$$\begin{aligned}\mu_2(A_2\cup B_2-A_1\cup B_1) &\leqq \mu_2((A_2-A_1)\cup(B_2-B_1))\\ &= \mu_2(A_2-A_1)+\mu_2(B_2-B_1)\\ &= 0,\end{aligned}$$

we have $\mu_2(A_1\cup B_1)=\mu_2(A_2\cup B_2)$. Similarly, $\mu_2(A_1\cap B_1) - \mu_2(A_2\cap B_2)$. Hence,

$$\bar\mu(A\cup B)+\bar\mu(A\cap B)=\mu_2(A_1\cup B_1)+\mu_2(A_1\cap B_1)$$

$$= \mu_2(A_1) + \mu_2(B_1)$$
$$= \bar{\mu}(A) + \bar{\mu}(B).$$

18. Lemma. *$\bar{\mu}$ is a monotone set function.*

Proof. Let $A, B \in \mathscr{S}_\mu(\mathscr{A})$, $A_1, B_1 \in \mathscr{A}_{\delta\sigma}$, $A_2, B_2 \in \mathscr{A}_{\sigma\delta}$ with $A_1 \subset A \subset A_2$, $B_1 \subset B \subset B_2$, $\mu_2(A_2 - A_1) = \mu_2(B_2 - B_1) = 0$ and $A \subset B$. Since $A_1 \subset B \subset B_2$, $\mu_2(A_1) \leqq \mu_2(B_2)$ and so $\bar{\mu}(A) = \mu_2(A_1) \leqq \mu_2(B_2) = \bar{\mu}(B)$.

19. Lemma. *$\bar{\mu}$ is continuous on $\mathscr{S}_\mu(\mathscr{A})$.*

Proof. It is sufficient to prove that $A_n \searrow \emptyset$ $(A_n \in \mathscr{S}_\mu(\mathscr{A}))$ implies $\bar{\mu}(A_n) \searrow 0$. By the monotone σ-completeness of G, there exists a $z = \lim_n \bar{\mu}(A_n)$. Let $f \in G_+^<$ and let ν be a continuous extension of the real measure $f \circ \mu$ from $\mathscr{A}$ to $\mathscr{S}_\mu(\mathscr{A})$. Then

$$f(z) = f(\lim_n \bar{\mu}(A_n)) = \lim_n f \circ \bar{\mu}(A_n) = \lim_n \nu(A_n) = 0,$$

where we have used the o-continuity of f and the fact that $f \circ \bar{\mu} = \nu$. Since G is separative, we get $\lim_n \bar{\mu}(A_n) = 0$.

Proof of Theorem 5. By Lemmas 14, 15, 16, 17, 18, and 19 we have that $\bar{\mu}$ is a measure on the σ-algebra $\mathscr{S}_\mu(\mathscr{A})$. It is easy to see that μ is a complete measure. If $A \in \mathscr{S}_\mu(\mathscr{A})$ $\mu(A) = 0$ and $A_1 \in \mathscr{A}_{\delta\sigma}$, $A_2 \in \mathscr{A}_{\sigma\delta}$ with $A_1 \subset A \subset A_2$ and $\mu_2(A_2 - A_1) = 0$, then $B \subset A$ implies $\emptyset \subset B \subset A_2$, hence $B \in \mathscr{S}_\mu(\mathscr{A})$. Let τ be a continuous extension of μ from $\mathscr{A}$ to $\mathscr{S}_\mu(\mathscr{A})$. Let $\{A_n\}$ be a monotone sequence in $\mathscr{A}$ with $\lim_n A_n = A$ $(A \in \mathscr{A}_1)$, then by the continuity of τ and by the definition of μ_1,

$$\tau(A) = \lim_n \tau(A_n) = \lim_n \mu(A_n) = \mu_1(A).$$

Analogically, $\tau(A) = \mu_2(A)$ for A in $\mathscr{A}_2$. If $A \in \mathscr{S}_\mu(\mathscr{A})$ with $A_1, A_2 \in \mathscr{A}_2$, $A_1 \subset A \subset A_2$ and $\bar{\mu}(A) = \mu_2(A_1) = \mu_2(A_2)$, then $\bar{\mu}(A) = \tau(A_1) = \tau(A_2)$, hence $\bar{\mu}(A) = \tau(A)$.

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Received October 20, 1987

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