Mathematica Slovaca

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Mathematica Slovaca, Vol. 39 (1989), No. 2, 149--154

Persistent URL: http://dml.cz/dmlcz/132121

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CONTINUOUS MAPPINGS AND CAUCHY SEQUENCES

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Let (X, d_X), (Y, d_Y) be pseudometric spaces and let $f\colon X \to Y$ be a mapping. A sequence in X is a mapping of the set N of all positive integers into X. It is known (see [1]) that if f is uniformly continuous, then for the Cauchy sequence S in X the sequence $f \circ S$ is Cauchy in Y. This is not true for a continuous f. We shall investigate the set of such Cauchy sequences in X the images of which are not Cauchy sequences.

Let us denote S_X the set of all constant sequences, C_X the set of all convergent sequences and F_X the set of all Cauchy sequences in X. Let $N(f) = \{S \in F_X : f \circ S \notin F_Y\}$ and let $f^*\colon X^N \to Y^N$ be a mapping defined $f^*(S) = f \circ S$ for each $S \in X^N$. For the members S and T of X^N we define $\varrho_X(S, T)$ as follows: $\varrho_X(S, S) = 0$ and $\varrho_X(S, T) = \min\{1, \inf\{\varepsilon > 0 : \exists n_\varepsilon \in N \forall m, n \geqq n_\varepsilon : d_X(S(m), T(n)) < \varepsilon\}\}$ for $S \neq T$. Further we define $\sigma_X(S, T)$ as $\sigma_X(S, T) = \min\{1, \inf\{\varepsilon > 0 : \exists n_\varepsilon \in N \forall n \geqq n_\varepsilon : d_X(S(n), T(n)) < \varepsilon\}\}$.

Remark 1. Evidently

$$\varrho_X(S, T) = \sigma_X(S, T) = \lim_{n \to \infty} d_X(S(n), T(n)) \quad \text{for} \quad S, T \in F_X.$$

It is easy to verify that (F_X, σ_X) is a complete pseudometric space (similarly as Cantor's method of a completion of a metric space) and hence also (F_X, ϱ_X) is a complete pseudometric space.

Remark 2. From the continuity of a pseudometric we get: If $S \in X^N$ converges to a and $T \in X^N$ converges to b, then $\varrho_X(S, T) = \sigma_X(S, T) = d_X(a, b)$.

Lemma 1. *Let (X, d_X) be a pseudometric space. Then (X^N, ϱ_X) is a complete pseudometric space.*

Proof. First we shall show that ϱ_X is a pseudometric on X^N. Evidently $\varrho_X(S, T) \geqq 0$, $\varrho_X(S, S) = 0$ and $\varrho_X(S, T) = \varrho_X(T, S)$ for all $S, T \in X^N$. Suppose that there are sequences S, T, P in X such that $\varrho_X(S, T) > \varrho_X(S, P) + \varrho_X(P, T)$. Then obviously $S \neq T \neq P \neq S$ and $\varrho_X(S, P) < 1$, $\varrho_X(P, T) < 1$. Let b, c be real numbers such that $\varrho_X(S, P) < b < 1$, $\varrho_X(P, T) < c < 1$ and $b + c < \varrho_X(S, T)$. Then there is a positive integer s such that for $m, n \geqq s$ we have $d_X(S(m), P(n)) < b$, $d_X(P(m), T(n)) < c$ and hence $d_X(S(m), T(n)) \leqq d_X(S(m), P(m)) + d_X(P(m), T(n)) < b + c < \varrho_X(S, T)$. However, this is a contradiction with the definition of $\varrho_X(S, T)$. Now we shall show that (X^N, ϱ_X) is a complete. Let S be a Cauchy sequence in (X^N, ϱ_X). If S has a constant subsequence, then evidently S is a convergent sequence in (X^N, ϱ_X). Now let S have no constant subsequence. Then there is a sequence P in X^N such that P is a subsequence of S and P is one-to-one. Since P is a Cauchy sequence, there is an increasing sequence (n_k) of positive integers such that

$$\forall i, j \geqq k: \varrho_X(P(n_i), P(n_j)) < 2^{-k}. \tag{1}$$

Since P is one-to-one, there is an increasing sequence (r_k) of positive integers such that

$$\forall u, v \geqq r_k: d_X(P(n_k)(u), P(n_{k+1})(v)) < 2^{-k}. \tag{2}$$

Now we define a sequence T in X as follows:

$$T(k) = P(n_k)(r_k) \quad \text{for} \quad k \in N.$$

Let $k \in N$ and let $u, p \geqq r_{k+1}$. The evidently $p > k$ and hence

$$d_X(P(n_k)(u), T(p)) = d_X(P(n_k)(u), P(n_p)(r_p) \leqq$$

$$\leqq d_X(P(n_k)(u), P(n_{k+1})(r_p) + \sum_{j=1}^{p-k-1} d_X(P(n_{k+j})(r_p), P(n_{k+j+1})(r_p)) <$$

$$< \sum_{j=0}^{p-k-1} 2^{-k-j} < \sum_{t=k}^{\infty} 2^{-t} = 2^{-k+1}.$$

From this we get $\varrho_X(P(n_k), T) < 2^{-k+1}$ for all $k \in N$. Hence the sequence $(P(n_k))$ converges to T. Since $(P(n_k))$ is a subsequence of S and S is Cauchy, the sequence S converges to T. The space (X^N, ϱ_X) is complete.

Lemma 2. *Let (X, d_X) be a pseudometric space. Then each point from $X^N - F_X$ is an isolated point in (X^N, ϱ_X).*

Proof. Let

$$o(S) = \lim_{n \to \infty} \sup \{d_X(S(k), S(m)): k, m \geqq n\}.$$

Evidently $S \in F_X$ if and only if $o(S) = 0$. It is easy to verify that for all $S, T \in \in x^N$ we have

$$\varrho_X(S, T) \geqq \min\{1, o(S)/2\}.$$

Therefore, for $S \in X^N - F_X$ we have that $\varrho_X(S, T) < \eta < o(S)/2 < 1$ implies $S = T$. Hence each point from $X^N - F_X$ is an isolated point in (X^N, ϱ_X).

Theorem 1. *Let (X, d_X), (Y, d_Y) be pseudometric spaces and let $f: X \to Y$ be a mapping. Then $N(f)$ is a boundary set in (F_X, ϱ_X).*

Proof. It is easy to see that S_X is dense in F_X. Since every constant sequence evidently belongs to $F_X - N(f)$, the set $F_X - N(f)$ is dense in F_X and therefore the set $N(f)$ is a boundary in F_X.

Theorem 2. *There are pseudometric spaces* (X, d_X), (Y, d_Y) *and a mapping* $f\colon X \to Y$ *such that the set* $N(f)$ *is residual in* (F_X, ϱ_X).

Proof. We put $X = Q \cap (0, 1)$ (the set of all rational numbers in the interval $(0, 1)$), $Y = N$, both with the usual metric. Let $f\colon X \to Y$ be a one-to-one mapping. It is easy to see that $S \in F_X - N(f)$ if and only if S is an eventually constant sequence. Hence

$$F_X - N(f) = \bigcup_{i \in f(X)} A_i,$$

where

$$A_i = \{S \in F_X : \exists k \in N: \forall n \geqq k: S(n) = f^{-1}(i)\}.$$

It is easy to verify that $\operatorname{cl}(A_i)$ (the closure of the set A_i in (X^N, ϱ_X)) is obtained in the set

$$B = \{S \in C_X : \lim_{n \to \infty} S(n) = f^{-1}(i)\}.$$

However, $\varrho_X(S, T) = 0$ for $S, T \in B$ (by Remark 2) and hence the set $\operatorname{cl}(A_i)$ has the empty interior, i.e. the set A_i is nowhere dense. Therefore $F_X - N(f)$ is a set of the first category and in view of Remark 1 the set $N(f)$ is residual in (F_X, ϱ_X).

Now we shall investigate the set $N(f)$ for a continuous mapping f. The symbol C_f denotes the set of all continuity points of f and D_f denotes the set of all discontinuity points of f.

Lemma 3. *Let* (X, d_X), (Y, d_Y) *be pseudometric spaces and let* $f\colon X \to Y$ *be a mapping. Let* $S \in X^N$ *converge to* $x \in C_f$. *Then* $S \in C_{f^*}$.

Proof. Let $\varepsilon > 0$. With respect to the continuity of f at x there exists $\delta > 0$ such that

$$d_Y(f(x), f(y)) < \varepsilon/4 \quad \text{whenever} \quad d_X(x, y) < \delta. \tag{3}$$

Let $\varrho_X(S, T) < \delta$. Then there is η, $0 < \eta < \delta$, and $n_0 \in N$ such that $d_X(S(n), T(m)) < \eta$ and $d_X(S(n), x) < \delta - \eta$ for $m, n \geqq n_0$. For $m, n \geqq n_0$ we obtain

$$d_X(T(m), x) \leqq d_X(S(n), T(m)) + d_X(S(n), x) < \delta$$

and hence according to (3)

$$d_Y(f(T(m)), f(S(n))) \leqq d_Y(f(T(m)), f(x)) + d_Y(f(x), f(S(n))) < \varepsilon/2,$$

i.e. $\varrho_Y(f^*(S), f^*(T)) \leqq \varepsilon/2 < \varepsilon$.

Lemma 4. *Let* (X, d_X), (Y, d_Y) *be pseudometric spaces and let* $f\colon X \to Y$ *be a continuous mapping. Then* $D_{f^*} \cap F_X$ *is a set of the first category in* (F_X, ϱ_X).

Proof. According to Lemma 3 we have $C_X \subset C_{f^*} \cap F_X$. The set C_X is dense in F_X and hence the set $D_{f^*} \cap F_X$ is a boundary in F_X. Since the set of all discontinuity points is an F_σ-set, $D_{f^*} \cap F_X$ is a set of the first category in F_X.

Lemma 5. *Let* (X, d_X), (Y, d_Y) *be pseudometric spaces and let* $f\colon X \to Y$ *be a mapping. Then* $N(f) \subset D_{f^*} \cap F_X$.

Proof. We shall show that $F_X \cap C_{f^*} \subset F_X - N(f)$. Let $S \in F_X \cap C_{f^*}$. Let $\varepsilon > 0$.

Then there is $\delta > 0$ such that

$$(4) \qquad \varrho_Y(f^*(S), f^*(T)) < \varepsilon/2 \quad \text{whenever} \quad \varrho_X(S, T) < \delta.$$

Since $S \in F_X$, there is $n_1 \in N$ such that $d_X(S(m), S(n)) < \delta/2$ for each $m, n \geqq n_1$. Let $T \in X^N$ be defined $T(k) = S(n_1)$ for all $k \in N$. Then for $m, n \geqq n_1$ we have $d_X(S(m), T(n)) < \delta/2$ and hence $\varrho_X(S, T) < \delta$. According to (4) we get $\varrho_Y(f^*(S), f^*(T)) < \varepsilon/2$. Hence there is $n_2 \in N$ such that

$$d_Y(f(S(n)), f(S(n_1)) < \varepsilon/2 \quad \text{whenever} \quad n \geqq n_2.$$

Thus for $m, n \geqq n_2$ we have

$$d_Y(f(S(m)), f(S(n))) \leqq d_Y(f(S(m)), f(S(n_1))) + \\ + d_Y(f(S(n_1)), f(S(n))) < \varepsilon,$$

i.e. $f^*(S) \in F_Y$. Therefore $S \in F_X - N(f)$ and

$$F_X \cap C_{f^*} \subset F_X - N(f).$$

Theorem 3. *Let (X, d_X), (Y, d_Y) be pseudometric spaces and let $f: X \to Y$ be a continuous mapping. Then $N(f)$ is a set of the first category in F_X.*

Proof. It follows from Lemma 5 and Lemma 4.

Lemma 6. *Let (X, d_X), (Y, d_Y) be pseudometric spaces. Let M be a dense subset of X and let $f: M \to Y$ be a mapping. Let $W(M, f) = \{x \in X : \text{if } S \in M^N \text{ converges to } x, \text{ then } f \circ S \in F_Y\}$. If $X - W(M, f)$ is a dense subset of X, then $N(f)$ is a dense subset of F_M.*

Proof. Let $S \in F_M - N(f)$ and let $\varepsilon > 0$. Since the set of all constant sequences is dense in F_M, there is $a \in M$ such that $\varrho_X(S, T) < \varepsilon$, where $T(n) = a$ for all $n \in N$. Let δ be a positive real number such that $K(T, \delta) \subset K(S, \varepsilon)$. From the density of $X - W(M, f)$ in X there is $b \in X - W(M, f) \cap K(a, \delta)$. Hence there is $P \in M^N$ converging to b such that $f \circ P \notin F_Y$. Therefore $P \in N(f)$. According to Remark 2 we have $\varrho_X(T, P) = d_X(a, b) < \delta$ and hence $P \in K(S, \varepsilon) \cap N(f)$; i.e. $N(f)$ is dense in F_M.

Theorem 4. *There are pseudometric spaces (X, d_X), (Y, d_Y) and a continuous mapping $f: X \to Y$ such that the set $N(f)$ is dense in (F_X, ϱ_S).*

Proof. Let $X = Q' \cap (0, 1)$ (the set of all irrational numbers from the interval $(0, 1)$) and $Y = R$, both with the usual metric. For each $n \in N$ we define $f_n: X \to Y$ as follows:

$$f_n(x) = \frac{p}{n} \cdot 2^{-n}, \quad \text{if} \quad \frac{p}{n+1} < x < \frac{p+1}{n+1}.$$

Then f_n is a continuous mapping and $|f_n(x)| \leqq 2^{-n}$ for each $x \in X$. Now we put

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

Then $f\colon X \to Y$ is a continuous mapping. Let $x \in (0, 1) \cap Q$, $x = p/q$ (where p and q are relatively prime). Since evidently all f_n are nondecreasing functions, for $a, b \in X$, $a < p/q$, $b > p/q$ we have

$$f_n(a) \leqq f_n(b) \quad \text{for all} \quad n \in N \quad \text{and}$$

$$f_{q-1}(b) - f_{q-1}(a) \geqq (q-1)^{-1} \cdot 2^{1-q}.$$

Hence also $f(b) - f(a) \geqq (q-1)^{-1} \cdot 2^{1-q}$. From this we observe that $W(X, f) = X$ and $(0, 1) - W(X, f)$ is dense in $(0, 1)$. Hence according to Lemma 6 the set $N(f)$ is dense in F_X.

Now we shall show a relation between the continuity of f and f^*. Evidently C_{f^*} is a nonempty set, unless $d_X(X) = 0$. From Lemma 4 and Lemma 2 we have:

Theorem 5. *Let (X, d_X), (Y, d_Y) be pseudometric spaces and let $f\colon X \to Y$ be a continuous mapping. Then D_{f^*} is a set of the first category in (X^N, ϱ_X).*

Theorem 6. *Let (X, d_X), (Y, d_Y) be pseudometric spaces and let $f\colon X \to Y$ be a mapping. Then f^* is a continuous mapping if and only if $N(f)$ is the empty set.*

Proof.

Necessity. It follows from Lemma 5.

Sufficiency. Let f^* be not continuous at a point $S \in X^N$. Then there are a positive number ε and a sequence (S_n) of elements of X^N such that

$$\varrho_X(S_n, S) < 1/n \quad \text{and}$$

$$\varrho_Y(f^*(S_n), f^*(S)) \geqq \varepsilon.$$

Since $\varrho_X(S_n, S) < 1/n$, there is an increasing sequence (k_n) of positive integers such that

$$l, m \geqq k_n \Rightarrow d_X(S(l), S_n(m)) < 1/n. \tag{5}$$

Since $\varrho_Y(f^*(S_n), f^*(S)) \geqq \varepsilon$, there are increasing sequences (l_n) and (m_n) of positive integers such that

$$l_n, m_n \geqq k_n \quad \text{and} \tag{6}$$

$$d_Y(f(S(l_n)), f(S_n(m_n))) \geqq \varepsilon. \tag{7}$$

We define a sequence T as follows:

$$T(2n) = S(l_n) \quad \text{and} \quad T(2n-1) = S_n(m_n) \quad \text{for} \quad n \in N.$$

In view of Lemma 2 and the discontinuity of f^* at S we see that $S \in F_X$. From this fact and (5) and (6) we observe that T is a Cauchy sequence. On the other hand with respect to (7) we see that $f \circ T$ is not a Cauchy sequence. Therefore $T \in N(f)$.

Remark 3. All theorems and lemmas in this paper are true also for σ_X, except Lemma 2 and Theorems 5 and 6.

The example $X = Y = R$ with the usual metric, $f(x) = x^2$ shows that the set D_{f^*} (with the respect to the pseudometrics σ_X and σ_Y) need not be a set of the first category (for the sequence S, where $S(n) = n$, we have $K(S, 1/4) \subset D_{f^*}$). Instead of Theorem 6 the following theorem holds:

Theorem 7. *Let* (X, d_X), (Y, d_Y) *be pseudometric spaces and let* $f: X \to Y$ *be a mapping. Then* f^* *is a continuous mapping (with respect to the pseudometrics* σ_X *and* σ_Y*) if and only if the mapping* f *is uniformly continuous.*

Proof.

Necessity. Let f be a uniformly continuous mapping and $\varepsilon > 0$. Then there is $\delta > 0$ such that $d_Y(f(a), f(b)) < \varepsilon/2$ whenever $d_X(a, b) < \delta$. Let $\sigma_X(S, T) < \delta$. Then there is $n_0 \in N$ such that $d_X(S(n), T(n)) < \delta$ for $n \geqq n_0$. Hence for $n \geqq n_0$ we have $d_Y(f(S(n)), f(T(n))) < \varepsilon/2$. From this $\sigma_Y(f^*(S), f^*(T)) \leqq \varepsilon/2 < \varepsilon$. The mapping f^* is therefore uniformly continuous and hence also continuous.

Sufficiency. Let f not be a uniformly continuous mapping. Then there are $\varepsilon > 0$ and sequences (a_n), (b_n) of elements of X such that $d_X(a_n, b_n) < 1/n$ and $d_Y(f(a_n), f(b_n)) \geqq \varepsilon$. Let $S(n) = a_n$ and $T(n) = b_n$ for each $n \in N$. Then we observe that $\sigma_X(S, T) = 0$, however, $\sigma_Y(f^*(S), f^*(T)) \geqq \varepsilon$. Therefore the mapping f^* is not continuous.

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Received Juny 27, 1986

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НЕПРЕРЫВНЫЕ ОТОБРАЖЕНИЯ И ПОСЛЕДОВАТЕЛЬНОСТИ КОШИ

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Резюме

В работе исследуется множество последовательностей Коши, образы которых при непрерывном отображении не являются последовательностями Коши.