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CONSTRUCTING AND FORBIDDING AUTOMORPHISMS IN LIFTED MAPS

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(*Communicated by Martin Škoviera*)

ABSTRACT. We use an algebraic description of surfaces with boundary to study covering maps. The focus is on the relationship between automorphisms in the base and lifted maps. We show how to introduce and/or prohibit additional automorphisms in the lift. Using this control we give proofs of MacBeath's Theorem, Grünbaum's Conjecture, and settle a conjecture of Nedela and Škoviera.

1. Introduction

We study covering maps between surfaces with boundaries. We are particularly interested in controlling the level of symmetry in the lifted map. Our surfaces are described in terms of holey maps: a generalization of a graph embedding which arises naturally when taking quotients. Our covering maps are described in terms of an algebraic technique called extended voltage assignments.

A holey map is a graph with semiedges embedded in a surface with boundary. We extend the ideas of map homomorphisms, regularity, and voltage assignments to holey maps. We show that a holey map M is regular if and only if every map homomorphism from M is regular.

Gross [3] first used voltage assignments to describe covering maps (see [4] for a detailed exposition). Voltage assignments have been used to construct covering maps rich in automorphisms (e.g., [6]), or more generally, to guarantee the existence of certain map homomorphisms (e.g., [1]). Our research continues along this line.

We give a special extended voltage assignment which gives a large automorphism group in the covering map. Using this we will give voltage-based proofs of MacBeath's Theorem (constructing oriented maps with the largest possible

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automorphism groups), of Grünbaum's Conjecture (on the existence of maps of each possible type), and of the fact that every map has a regular cover.

We also use extended voltage assignments to forbid the existence of certain automorphisms. More precisely, we describe a covering map which has no automorphisms besides those lifting from the base map; moreover, every automorphism of the base map lifts to one of the covering map. Using this we construct an infinite sequence of orientably regular maps with trivial exponent groups and increasing valencies. These maps settle a conjecture of Nedela and Škoviera [10].

In summary, describing covering spaces using extended voltage assignments on holey maps gives us a great deal of control over automorphisms in the lift.

The paper is organized as follows. In Section 2, we define our holey maps. Section 3 discusses map homomorphisms and regular maps. Section 4 introduces extended voltage assignments and the lift of a map. We then examine automorphisms in the lift in Section 5. We give two special voltage assignments in Section 6. Finally, Section 7 gives three applications of our results.

2. Holey maps

We are interested in symmetries of maps; so it is convenient to define maps algebraically. Our definition allows us to extend maps to "holey maps". We first give our definition, then relate holey maps to the more common concept of a graph embedded on a surface.

Let F be a finite set of *flags*, and let R, L, and T be three involutions on F. We say that the quadruple $M = (F, R, L, T)$ is a *holey map* if

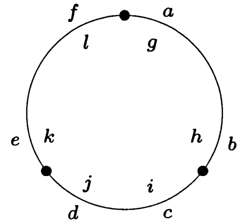
(i) $LT = TL$,

(ii) the group generated by R, L, and T acts transitively on F.

The second condition is a type of connectivity. If we drop this condition, we call M a *holey premap.* A holey premap can be written as the disjoint union of holey maps based on the flag orbits under the action of the group generated by R, L, and T.

We now describe the holey map corresponding to a cellular graph embedding. Consider the embedding of the triangle K_3 in the plane as shown in the left half of Figure 1. The flag set is $F = \{a, b, \dots, l\}$. Each edge of K_3 corresponds to four flags, one for each side of each end. The transversal involution T swaps the two flags on the same end of an edge and is given on the right of the Figure 1. The longitudinal involution L swaps the two flags on the same side of an edge. The rotational involution R records which flags appear together in corners around a vertex of the embedded graph. Observe that $LT = TL$, and the group generated by R, L, and T acts transitively on F. Hence $M = (F, R, L, T)$ is a holey map.

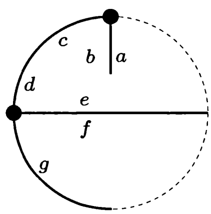
We make several observations about this example. First, the vertices of the graph correspond to the orbits under the action of T and R on the flags. Second, the faces correspond to the orbits under the action of L and R. Third, the edges correspond to the orbits under the action of L and T. Each such edge-orbit has exactly four flags.



$$F = \{a,b,c,d,e,f,g,h,i,j,k,l\}$$
$$R = (af)(bc)(de)(gl)(hi)(jk)$$
$$L = (ab)(cd)(ef)(gh)(ij)(kl)$$
$$T = (ag)(bh)(ci)(dj)(ek)(fl)$$

FIGURE 1. A triangle on the sphere.

Figure 1 illustrates an important subclass of holey maps. A *map* is a holey map $M = (F, R, L, T)$, where R, L, T, and LT are fixed-point-free. These maps are equivalent to those of [12] and are also equivalent to cellular graph embeddings. In a map, the underlying surface has no holes, and the underlying graph has no semiedges. Similarly, a *premap* is a holey premap with R, L, T, and LT fixed-point-free. What distinguishes a holey map from a map is the presence of fixed points in R, L, T, or LT. Each type of fixed-point can be interpreted topologically as either a hole in the surface or a semiedge. The unifying concept is that a flag is fixed by a permutation if moving along an edge (via L), across an edge (via T), or around a vertex (via R) crosses the boundary of the hole.



$$F = \{a,b,c,d,e,f,g\}$$
$$R = (a)(bc)(de)(fg)$$
$$L = (ab)(cd)(e)(f)(g)$$
$$T = (ab)(c)(d)(ef)(g)$$

FIGURE 2. A holey map with semiedges.

We illustrate some of the possibilities in Figure 2. This holey map has flag set F, transverse involution T, longitudinal involution L, and rotational involution R as shown. There are two faces: the upper and lower half inside the circle. The hole is the outside of the circle.

We examine the fixed points of these involutions. First, c and d are not fixed by L, but are by T. This corresponds to an edge lying on the boundary of the hole. The flag g is fixed under both L and T and corresponds to a semiedge lying on the boundary of the hole. Flags a and b are swapped by both T and L and hence are fixed points of LT. Crossing the edge transversally or longitudinally lead to the same flag. This corresponds to a semiedge projecting into the middle of a face and it is not on the boundary of the hole. Flags e and f are fixed by L but swapped by T. This corresponds to a semiedge which disappears into the hole. Finally, the rotation R fixes the flag a and corresponds to a corner of a map on the boundary of the hole.

Observe that the vertices of the graph correspond to the orbits of flags under the action of T and R. Similarly, the faces correspond to the orbits under the action of L and R. The edges correspond to the orbits under the action of L and T. Semiedges arise from either fixed points of L or of LT. For a more detailed explanation of the relation between fixed points and holey maps, we refer the reader to [2] – there holey maps and semiedges are called algebraic maps and free edges respectively – or to [11].

Another description of graph embeddings and its relation with holey maps is as follows. Let $M = (F, R, L, T)$ be a holey premap with fixed-point-free R, L, and T. Let $\mathcal{O}_{RT}(a)$ denote the orbit of flag a under the action of the permutation RT (we compose permutations from right to left and consider them acting on the left). Similarly, $\mathcal{O}_{RT}(Ta)$ is the orbit of flag Ta. We claim that these orbits are disjoint for every flag a. To see this, suppose there is an a and a nonnegative m such that $(RT)^m a = Ta$. Choose a and m such that m is as small as possible. Then

$$(RT)^{m-2}(RTa) = (RT)^{m-1}a = TR(RT)^m a = TRTa = T(RTa).$$

This contradicts the minimality of m unless m is equal to 0 or 1. In the former case, we get $a = Ta$, while in the latter case, we have $R(Ta) = Ta$. Both cases are impossible because T and R are fixed-point-free. Similarly, the orbits $\mathcal{O}_{RL}(a)$, and $\mathcal{O}_{RL}(La)$ are also disjoint for every $a \in F$.

The pairs of orbits $\{\mathcal{O}_{RT}(a), \mathcal{O}_{RT}(Ta)\}$ correspond to vertices in the topological realization of M. The cardinality $|\mathcal{O}_{RT}(a)| = |\mathcal{O}_{RT}(Ta)|$ is the valency of the vertex corresponding to $\{\mathcal{O}_{RT}(a), \mathcal{O}_{RT}(Ta)\}$. The analogous statement holds for faces using pairs of orbits $\{\mathcal{O}_{RL}(a), \mathcal{O}_{RL}(La)\}$.

A holey map is of *type* $\{q, r\}$ if q and r are respectively the least common multiples of all face lengths and all vertex valencies.

A map M is *orientable* if the group generated by the permutations RT and LT has two orbits in F. If D is one of these orbits, then we say that the pair (M, D) is an *oriented map*. Topologically, M is orientable if its underlying surface is orientable. A particular choice of D corresponds to giving the surface one of the two possible orientations. Observe that if a is in one orientation, then Ra, La, and Ta are in the other orientation. Bryant and Singerman [2] have extended orientability to holey maps, but the extension is not needed here.

An oriented map M is equivalent to a triple (D, P, K), where D is a set of *arcs*, P and K are permutations of D such that K is a fixed-point-free involution, and the group generated by P and K acts transitively on D. Under this equivalence, the triple (D, P, K) corresponds to the triple (D, RT, LT). Topologically, the arcs are directed edges in the graph, P is a rotation scheme, and K reverses the direction on an edge. For a more detailed topological interpretation, we refer the reader to [8].

When dealing with an oriented map M, we will usually prefer to work with the triple (D, P, K) rather than the quadruple (F, R, L, T) and will denote both of them by M. Also, if an oriented map $M = (D, P, K)$ is given, we will always assume that it comes from a quadruple (F, R, L, T) to which it is related in the way just described.

A *word* X in R, L, T is a sequence $X = S_n, S_{n-1}, \dots, S_1$, with all $S_i \in \{R, L, T\}$. The word X^{-1} is the sequence $S_1, S_2, \dots, S_n$. Given words X_1 and X_2, the word X_1X_2 is obtained by juxtaposition. The word X^k juxtaposes k copies of X.

If (F, R, L, T) is a holey premap, then the word X corresponds in a natural way to a permutation of F (in this permutation, the S_i are read from right to left). This permutation will also be denoted by X. Thus, given a flag $a \in F$, $Xa = S_nS_{n-1}\dots S_1a$. A *flag-walk* is a sequence $W = a_1, a_2, \dots, a_n$ of flags, such that for i, $1 \le i \le n-1$, $a_{i+1} = Sa_i$ for some $S \in \{R, L, T\}$. We say that W is *closed* if $a_1 = a_n$. If $X = S_n, S_{n-1}, \dots, S_1$ is a word and $a \in F$, then we say that the flag-walk

$$a, S_1a, S_2S_1a, S_3S_2S_1a, \dots, Xa$$

is the *walk corresponding to* X *and* a. We denote this walk by $W(X, a)$.

If M is a holey premap and $X = S_n, S_{n-1}, \dots, S_1$ is a word in R, L, and T, then X_M is the word $U_n, U_{n-1}, \dots, U_1$ in R_M, L_M, and T_M, where $U_i = (S_i)_M$ for each i. To simplify the notation, we will usually omit the subscript.

All these notions can naturally be adapted – with appropriate adjustments – to oriented maps. Thus we speak of *words* in P and K, or of *arc-walks*, without fear of confusing the reader.

3. Map homomorphisms

A *map homomorphism* $f\colon M \to N$ between two holey maps is a function $f\colon F_M \to F_N$ which commutes with R, L, and T; specifically, for every flag a we have $f(R_M a) = R_N f(a)$, $f(L_M a) = L_N f(a)$, and $f(T_M a) = T_N f(a)$. It follows that f is always onto, and that it is one-to-one if and only if $|F_M| = |F_N|$. Such a bijection is a *map isomorphism.* If $M = N$, then f is a *map automorphism.* The map automorphisms of M form a group $\mathrm{Aut}(M)$ under composition. It is an easy but important observation that for fixed $a \in F_M$ and $b \in F_N$, there is at most one map homomorphism $M \to N$ mapping a to b. (This is false if we extend homomorphisms to holey premaps.) In particular, if $\mathrm{Aut}(M)$ acts transitively on F_M, then the action is regular. In this case, we say that M is *regular.*

For example, consider the flag set $\{a_1, a_2, \dots, a_{2m}\}$ where the subscripts are read cyclicly. Let T be the fixed-point-free involution swapping a_{2j-1} and a_{2j} as j ranges from 1 to m. Similarly, let R be the fixed-point-free involution swapping a_{2j} and a_{2j+1}. Finally, let L be the identity. Then (F, R, L, T) is a regular holey map. The corresponding graph embedding has one vertex and m semiedges with each semiedge disappearing into a hole.

We define the concepts of oriented map homomorphisms, isomorphisms, automorphisms etc. for oriented maps in the obvious way. Thus, e.g., an *oriented homomorphism* $f\colon M \to N$ is a mapping $f\colon D_M \to D_N$ satisfying $f(P_M a) = P_N f(a)$ and $f(K_M a) = K_N f(a)$ for every $a \in D_M$. An oriented map M which possesses an oriented automorphism mapping a to b for all pairs $a, b \in D_M$ is called *orientably regular.* An example is the tetrahedron embedded on an oriented sphere. We will write $\mathrm{OAut}(M)$ for the set of all oriented automorphisms of M. It can be shown that if M is an oriented map, then every oriented map automorphism of M is (or more precisely corresponds to) a map automorphism of M. However, the converse is in general not true, since reflections of M are map automorphisms of M which are not oriented map automorphisms. An example is again the tetrahedron, which has twice as many map automorphisms as oriented map automorphisms. In general, we may only conclude that $\mathrm{OAut}(M) \subseteq \mathrm{Aut}(M)$. Similar relations hold between oriented and general map homomorphisms and isomorphisms.

If $f\colon M \to N$ is a map homomorphism, and $b \in F_N$, then the *fiber* $f^{-1}(b)$ is the set $\{a \in F_M : f(a) = b\}$. A *deck transformation* of f is any map automorphism A of M such that $fA = f$. Equivalently, a deck transformation is a map automorphism which preserves fibers. The set of all deck transformations of f forms a group under function composition. Deck transformations are sometimes called automorphisms of coverings, so we denote this group by $\mathrm{Aut}(f)$. A map homomorphism f is *regular* if the group $\mathrm{Aut}(f)$ acts transitively on a fiber $f^{-1}(b)$. It is well known that in such case the group acts transitively on each fiber, so the concept is well defined. Recall that there is at most one automorphism mapping one flag to another, so $|\mathrm{Aut}(f)| \le |f^{-1}(b)|$. If $\mathrm{Aut}(f)$ acts transitively on a fiber, then the action of $\mathrm{Aut}(f)$ on a fiber is regular.

Proposition 1. *Let M and N be a pair of holey maps, and let $a \in F_M$, $b \in F_N$. There is a map homomorphism $f\colon M \to N$ with $f(a) = b$ if and only if for every word X*

$$Xa = a \implies Xb = b. \tag{1}$$

P r o o f. We first prove the sufficiency. Suppose that $f\colon M \to N$ is a map homomorphism with $f(a) = b$, and let X be a word. It follows from the definition of a map homomorphism that $fX = Xf$. Assuming that $Xa = a$, we get

$$Xb = Xf(a) = f(Xa) = f(a) = b.$$

To prove the necessity, let M, N, a and b be as above and assume that they satisfy Condition (1). Let c be an arbitrary flag in F_M. There exists a word X with $Xa = c$ because the group generated by R_M, L_M and T_M acts transitively on F_M. Define the desired mapping $f\colon F_M \to F_N$ by setting $f(c) = Xb$ for this word X. We need to show that f is well defined. If $X_1a = X_2a = c$, then $X_2^{-1}X_1a = a$. Hence by (1), $X_2^{-1}X_1b = b$. This implies $X_1b = X_2b$ as desired.

It remains to show that f is a map homomorphism, i.e., for every $c \in F_M$ we have $f(R_Mc) = R_Nf(c)$, $f(L_Mc) = L_Nf(c)$, and $f(T_Mc) = T_Nf(c)$. We will prove the first of these equalities; the other two follow in a similar fashion. As before, first find a word X with $Xa = c$. Then

$$\begin{aligned} f(R_Mc) &= f\big(R_M(Xa)\big) = f\big((R_MX)a\big) = (R_NX)b \\ &= R_N(Xb) = R_Nf(Xa) = R_Nf(c), \end{aligned}$$

and so the proposition is demonstrated. □

The following is the oriented version of Proposition 1.

Proposition 2. *Let $M = (D_M, P_M, K_M)$ and $N = (D_N, P_N, K_N)$ be oriented maps, and let $a \in D_M$ and $b \in D_n$. There is an oriented map homomorphism $f\colon M \to N$ with $f(a) = b$ if and only if for every word X in P and K, $Xa = a \implies Xb = b$.*

Let Λ denote the free group $\mathrm{Fr}\langle R, L, T\rangle$ generated by symbols R, L, and T. Given an arbitrary map M, let $\lambda_M\colon \Lambda \to \langle R_M, L_M, T_M\rangle$ be the group homomorphism determined by $\lambda_M(R) = R_M$, $\lambda_M(L) = L_M$ and $\lambda_M(T) = T_M$. The group Λ acts on the set F_M with the action given by $Xa = \lambda_M(X)a$ for all $a \in F_M$. Note that this action depends not only on Λ and F_M, but also on the structure of M. Let $\mathrm{Stab}_\Lambda(a)$ denote the set $\{X \in \Lambda :\ Xa = a\}$. We can rephrase Proposition 1 as follows.

COROLLARY 3. *Let M and N be holey maps, and let $a \in F_M$, $b \in F_N$. There is a map homomorphism $f\colon M \to N$ satisfying $f(a) = b$ if and only if*

$$\mathrm{Stab}_\Lambda(a) \subseteq \mathrm{Stab}_\Lambda(b)\,.$$

Now we prove a statement characterizing regular maps in terms of the homomorphisms they admit. We first need a definition and a lemma. If Γ is a group and Ψ its subgroup, then $N_\Gamma(\Psi)$ is the normalizer of Ψ in Γ, that is, $N_\Gamma(\Psi) = \{g \in \Gamma :\ g^{-1}\Psi g = \Psi\}$. It is easy to show that if Ψ has finite index in Γ, then $N_\Gamma(\Psi) = \{g \in \Gamma :\ g^{-1}\Psi g \subset \Psi\}$. We use this fact in the proofs of Lemma 4 and Theorem 5.

LEMMA 4. *Let M and N be holey maps, let $a \in F_M$ and $b \in F_N$. Suppose that $f\colon M \to N$ is a map homomorphism with $f(a) = b$. Then f is regular if and only if*

$$\mathrm{Stab}_\Lambda(b) \subseteq N_\Lambda\bigl(\mathrm{Stab}_\Lambda(a)\bigr)\,.$$

P r o o f. First note that f satisfies $f(Xa) = Xb$ for every $X \in \Lambda$. Hence

$$\begin{aligned} f^{-1}(b) &= \bigl\{c \in F_M :\ f(c) = b\bigr\} \\ &= \{Xa :\ X \in \Lambda,\ \ Xb = b\} = \bigl\{Xa :\ X \in \mathrm{Stab}_\Lambda(b)\bigr\}\,. \end{aligned}$$

Also, by Proposition 1,

$$\begin{aligned} &\bigl\{c \in F_M :\ \exists A \in \mathrm{Aut}(M)\ \text{s.t.}\ A(a) = c\bigr\} \\ =&\bigl\{c \in F_M :\ (\forall X \in \Lambda)(Xa = a \implies Xc = c)\bigr\} \\ =&\bigl\{Ya :\ Y \in \Lambda,\ (\forall X \in \Lambda)\bigl(Xa = a \implies X(Ya) = Ya\bigr)\bigr\} \\ =&\bigl\{Ya :\ Y \in \Lambda,\ (\forall X \in \Lambda)\bigl(X \in \mathrm{Stab}_\Lambda(a) \implies Y^{-1}XY \in \mathrm{Stab}_\Lambda(a)\bigr)\bigr\} \\ =&\bigl\{Ya :\ Y \in N_\Lambda\bigl(\mathrm{Stab}_\Lambda(a)\bigr)\bigr\}\,. \end{aligned}$$

These two set equalities show that $\mathrm{Aut}(M)$ acts transitively on $f^{-1}(b)$ (and thus, f is regular) if and only if

$$\bigl\{Xa :\ X \in \mathrm{Stab}_\Lambda(b)\bigr\} \subseteq \bigl\{Ya :\ Y \in N_\Lambda\bigl(\mathrm{Stab}_\Lambda(a)\bigr)\bigr\}\,. \tag{2}$$

Equation (2) is satisfied when $\mathrm{Stab}_\Lambda(b) \subseteq N_\Lambda\bigl(\mathrm{Stab}_\Lambda(a)\bigr)$.

Conversely, assume that (2) holds true, and let $X \in \mathrm{Stab}_\Lambda(b)$. There is $Y \in N_\Lambda\bigl(\mathrm{Stab}_\Lambda(a)\bigr)$ for which $Xa = Ya$. Then $Y^{-1}X \in \mathrm{Stab}_\Lambda(a) \subseteq N_\Lambda\bigl(\mathrm{Stab}_\Lambda(a)\bigr)$ and $X = Y(Y^{-1}X) \in N_\Lambda\bigl(\mathrm{Stab}_\Lambda(a)\bigr)$. □

THEOREM 5. *A holey map M is regular if and only if every map homomorphism f from M to another holey map N is regular.*

P r o o f. We will first prove the "only if" part. We will do this by showing that for every $a \in F_M$, $N_\Lambda\bigl(\mathrm{Stab}_\Lambda(a)\bigr) = \Lambda$. The result will then be an immediate consequence of Lemma 4.

As in the proof of Lemma 4, we have

$$\{c \in F_M : \exists A \in \mathrm{Aut}(M) \text{ s.t. } A(a) = c\} = \{Ya : Y \in N_\Lambda(\mathrm{Stab}_\Lambda(a))\}.$$

The regularity of M implies that these two sets must equal F_M. If now $X \in \Lambda$ is arbitrary, then there is $Y \in N_\Lambda(\mathrm{Stab}_\Lambda(a))$ with $Xa = Ya$, and the same reasoning as in the proof of Lemma 4 gives $X \in N_\Lambda(\mathrm{Stab}_\Lambda(a))$.

For the "if" part, let N be the holey map containing a single flag b. Define a mapping $f\colon F_M \to F_N$ by $f(a) = b$ for every $a \in F_M$. Then f is a map homomorphism and hence is regular by hypothesis. Therefore, the group $\mathrm{Aut}(f)$ acts transitively on the fiber $f^{-1}(b) = F_M$, so M is regular. □

Theorem 5 is an important reason for working with holey maps instead of the special class of maps. The theorem is false if one only considers homomorphisms onto maps. For example, let M be the connected spherical map with two vertices, one link, and one loop. Any map homomorphism is an automorphism and is regular, yet M is not regular.

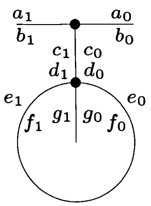
4. Voltages and lifts

Let $M = (F, R, L, T)$ be a holey map, and let Γ be a finite group. Suppose that α, β, and γ are functions from F to Γ. The triple $\sigma = (\alpha, \beta, \gamma)$ is an *extended voltage assignment* if for every $a \in F$:

(i) $\alpha(Ra) = \alpha(a)^{-1}$,
(ii) $\beta(La) = \beta(a)^{-1}$,
(iii) $\gamma(Ta) = \gamma(a)^{-1}$,
(iv) $\beta(a)\gamma(La) = \gamma(a)\beta(Ta)$.

Given an extended voltage assignment σ on M, define the *lift* M^σ as follows. Set $M^\sigma = (F^\sigma, R^\sigma, L^\sigma, T^\sigma)$, where $F^\sigma = F \times \Gamma$; we will use a_g to denote the pair (a, g). Define $R^\sigma(a_g) = (Ra)_{g\alpha(a)}$, $L^\sigma(a_g) = (La)_{g\beta(a)}$ and $T^\sigma(a_g) = (Ta)_{g\gamma(a)}$. The lift is in general a holey premap – that is, a finite collection of disjoint holey maps. However, it can be shown that the components of M^σ are pairwise isomorphic.

In [1], it is shown that if M is a map and σ is an extended voltage assignment on M, then there is a second extended voltage assignment $\nu = (\alpha', \beta', \gamma')$ which gives an isomorphic lift and has β' and γ' identically equal to 1_Γ. However, this is not true for holey maps, so we need here the full generality.



$$R = (a_0a_1)(b_0c_0)(b_1c_1)(d_0e_0)(d_1e_1)(f_0g_0)(f_1g_1)$$
$$L = (a_0b_0)(a_1b_1)(c_0d_0)(c_1d_1)(e_0e_1)(f_0f_1)(g_0g_1)$$
$$T = (a_0b_0)(a_1b_1)(c_0c_1)(d_0d_1)(e_0f_0)(e_1f_1)(g_0g_1)$$

FIGURE 3. A lifted map.

An interesting simple extended voltage assignment can be obtained as follows. Start with a holey map M and a voltage group $\Gamma = Z_2$ with two elements. Define $\alpha(a) = 1$ if and only if $Ra = a$, $\beta(a) = 1$ if and only if $La = a$, and $\gamma(a) = 1$ if and only if $Ta = a$. It can be checked that Conditions (i) – (iv) are satisfied; Condition (iv) is the only one which is not immediate. This extended voltage assignment has the nice geometric interpretation of taking two copies of the map and matching up the holes in their underlying surfaces. Figure 3 shows the map M^σ thus constructed from Figure 2. The surface is the sphere without boundary; the underlying embedded graph has two vertices, one link, one loop, and three semiedges.

Let M be a holey map, and let σ be an extended voltage assignment. Suppose that $X = S_n, S_{n-1}, \dots, S_1$ is a word, and $W = W(X, a_0) = a_0, a_1, a_2, \dots, a_n$ is the walk corresponding to X and a_0. The *voltage* of W, $\sigma(W)$, is the group product $g_1 g_2 \cdots g_n$, where $g_i = \alpha(a_{i-1})$ if $S_i = R_M$, $g_i = \beta(a_{i-1})$ if $S_i = L_M$, and $g_i = \gamma(a_{i-1})$ if $S_i = T_M$. Note that $X\big((a_0)_g\big) = (a_n)_{g\sigma(W)}$ in M^σ. In particular, $W\big(X, (a_0)_g\big)$ is closed if and only if W is closed in M and $\sigma(W) = 1_\Gamma$.

5. Automorphisms in lifts

In this section, we examine the relationship between automorphisms in a lifted map and automorphisms in the base map. The following proposition is a modification of Theorem 9 from [1]. Its proof is similar and is omitted.

PROPOSITION 6. *Let M be a holey map, and let σ be an extended voltage assignment with values in a finite group Γ. Further, let a and b be flags in M such that $b = Aa$ for some $A \in \mathrm{Aut}(M)$. Suppose that the flags a_g and b_h belong to the same component M_C^σ of M^σ. Then there exists an automorphism $\tilde{A} \in \mathrm{Aut}(M_C^\sigma)$ with $b_h = \tilde{A}a_g$ if and only if for every closed flag-walk W rooted at a:*

$$\sigma(W) = 1_\Gamma \implies \sigma(AW) = 1_\Gamma . \tag{3}$$

If M is an oriented map, then an equivalent definition of an extended voltage assignment can be formulated using P_M and K_M. Under this definition, an *oriented voltage assignment* is an arbitrary mapping from D_M to Γ. The corresponding lift is then a premap $M^\nu = (D_M^\nu, P_M^\nu, K_M^\nu)$ defined by $D_M^\nu = D_M \times \Gamma$, $P_M^\nu(a_g) = (P_M a)_{g\nu(a)}$, and $K_M^\nu(a_g) = (Ka)_g$. As we remarked earlier, if M is a map and σ an extended voltage assignment on M, then there exists an equivalent extended voltage assignment $\sigma' = (\alpha', 1, 1)$. The corresponding oriented voltage assignment ν is given by

$$\nu(a) = \alpha'(T_M a) \tag{4}$$

for every $a \in D_M$. Using this equivalence of σ and ν, we obtain the following oriented version of Proposition 6.

PROPOSITION 7. *Let M be an oriented map, and let ν be an oriented voltage assignment with values in the finite group Γ. Further, let a and b be arcs in M such that $b = Aa$ for some $A \in \mathrm{OAut}(M)$. Suppose that the flags a_g and b_h belong to the same component M_C^ν of M^ν. Then there exists an automorphism $\tilde{A} \in \mathrm{OAut}(M_C^\nu)$ with $\tilde{A}a_g = b_h$ if and only if for every closed arc-walk W in M rooted at a*

$$\nu(W) = 1_\Gamma \implies \nu(AW) = 1_\Gamma .$$

6. Two special lifts

In this section, we give two special extended voltage assignments. The first guarantees that the lifted map has a large automorphism group. The second guarantees that the lifted map has no other automorphisms.

It is known that each map M is covered by an orientably regular map; a proof based on the idea of factoring a universal covering of an orientable surface is given in [8]. Here, we give a different argument valid for holey maps as well using a natural extended voltage assignment in the group $\langle R_M, L_M, T_M \rangle$.

THEOREM 8. *For every holey map M there exists a regular holey map $\tilde{M}$ and a map homomorphism $\tilde{M} \to M$.*

P r o o f. Let M be a holey map. Take the group $\Gamma = \langle R_M, L_M, T_M \rangle$ as the voltage group and consider the following canonical extended voltage assignment $\sigma = (\alpha, \beta, \gamma)$: $\alpha(a) = R_M$, $\beta(a) = L_M$, and $\gamma(a) = T_M$ for each $a \in F_M$. Let $\tilde{M}$ be a component of the lift M^σ, let $a, b \in F_M$, and $g, h \in \Gamma$ be such that $a_g, b_h \in F_{\tilde{M}}$. We show that there is an automorphism of $\tilde{M}$ sending a_g to b_h.

Let X be a word where the flag-walk $W(X, a_g)$ is closed in $\tilde{M}$. Then $W(X, a)$ is closed in M and $\sigma\big(W(X, a)\big) = 1_\Gamma$. However, by the definition of σ, we have

$\sigma(W(X,a)) = (X^{-1})_M$, and therefore $(X^{-1})_M = 1_\Gamma$. We also have $X_M = 1_\Gamma$ and $X_M b = b$, which shows that the flag-walk $W(X,b)$ in M is closed and has voltage 1_Γ. Thus $W(X,b_h)$ is also closed, and by Proposition 1, there is an automorphism of $\tilde{M}$ mapping a_g to b_h. We conclude that $\tilde{M}$ is a regular holey map. The map homomorphism $\tilde{M} \to M$ is simply the covering projection $a_g \mapsto a$. □

A routine calculation shows that for a given holey map $M = (F,R,L,T)$ such that all of R, L, T and LT are non-identity permutations, the canonical voltage assignment σ described in the preceding proof yields a regular map $\tilde{M}$ as a covering space of M – that is, the permutations R^σ, L^σ, T^σ, and $L^\sigma T^\sigma$ of the flag set of $\tilde{M}$ are fixed-point-free. Moreover, if R, L, and T are fixed-point-free, then both M and $\tilde{M}$ are of the same type. We have:

COROLLARY 9. *Let $M = (F,R,L,T)$ be a holey map such that R, L, and T are fixed-point-free, and such that LT is not the identity permutation of F. Then there exists a regular map $\tilde{M}$ of the same type as M with a map homomorphism $\tilde{M} \to M$.*

We next want a construction of a nontrivial lift (i.e., the components of M^σ are not isomorphic to M) of an arbitrary map M such that the automorphisms of M^σ are exactly the lifts of automorphisms of M. For this we need a special extended voltage assignment.

Given a holey map $M = (F,R,L,T)$ with a fixed-point-free R, let the group Γ_p be the direct product of $|F|/2$ copies of the cyclic group $\mathbb{Z}_p$, where p is any positive integer. Let e_i $(1 \le i \le |F|/2)$ denote the "ith unit vector" in Γ_p, i.e., the vector whose ith coordinate is 1 and whose remaining coordinates are all 0. Let $a_i \in F$, $1 \le i \le |F|/2$, be such that $\{a_i : \ 1 \le i \le |F|/2\}$ contains exactly one flag from each orbit of R. Define a mapping $\alpha_p \colon F \to \Gamma_p$ by setting $\alpha_p(a_i) = e_i$ and $\alpha_p(Ra_i) = -e_i$. Note that for this definition to be valid, we need R to be fixed-point-free. Let $\beta_p \colon F \to \Gamma_p$ and $\gamma_p \colon F \to \Gamma_p$ be both identically equal to 1_{Γ_p}. Then $\sigma_p = (\alpha_p, \beta_p, \gamma_p)$ satisfies (i) – (iv) and is an extended voltage assignment on M. We call this a *corner voltage assignment.*

When $p = 2$, the α_p above is well-defined even if R has fixed-points. We extend the definition of a corner voltage assignment to this special case.

We next show how to lift automorphisms using a corner voltage assignment.

LEMMA 10. *Let M be a holey map with R fixed-point-free, and let $a, b \in F_M$. Assume further that there is an automorphism A of M which maps a to b. Then whenever a_g and b_h are in the same component $M_C^{\sigma_p}$ of M^{σ_p}, there is an automorphism $\tilde{A}$ in $\mathrm{Aut}(M_C^{\sigma_p})$ mapping a_g to b_h.*

P r o o f. It follows from the definition of α_p that every automorphism A of M induces a mapping given by $\alpha_p(a) \mapsto \alpha_p(Aa)$ which extends to an automorphism of Γ_p. This, together with the fact that β_p and γ_p are identically equal to 1_Γ, implies that Condition (3) is satisfied. Therefore, Lemma 6 applies and the conclusion follows. □

Similarly, we can forbid automorphisms in the lift of a corner voltage assignment.

LEMMA 11. *Let M be a holey map with a fixed-point-free R_M, and let $a, b \in F_M$. Assume that there is no automorphism of M mapping a to b. If p is a sufficiently large prime and $g, h \in \Gamma_p$ are such that a_g and b_h are in the same component $M_C^{\sigma_p}$ of M^{σ_p}, then there is no automorphism in $\mathrm{Aut}(M_C^{\sigma_p})$ mapping a_g to b_h.*

P r o o f. Since there is no automorphism of M mapping a to b, it follows from Proposition 1 that there is a word X such that $Xa = a$ and $Xb \neq b$. Let s be the smallest positive integer for which the permutation X^s fixes b. Note that, if t is an integer, then $X^t b = b$ if and only if t is divisible by s. Let p be a prime larger than s. Then X^p fixes a_g in M^{σ_p} (since $X^p a = a$ and $\sigma_p(W(X^p, a)) = p\sigma_p(W(X, a)) = 1_{\Gamma_p}$), but not b_h (as s does not divide p and thus $X^p b \neq b$). □

Now we are ready to prove the desired result.

THEOREM 12. *Let M be a holey map with a fixed-point-free R_M. Then there is a group Γ and a nontrivial extended voltage assignment σ in Γ such that whenever a_g and b_h are in the same component M_C^σ of M^σ, there is $\tilde{A} \in \mathrm{Aut}(M_C^\sigma)$ with $\tilde{A} a_g = b_h$ if and only if there is $A \in \mathrm{Aut}(M)$ with $Aa = b$.*

P r o o f. For every pair $a, b \in F_M$ for which there is no $A \in \mathrm{Aut}(M)$ such that $Aa = b$, let $p_{a,b}$ be a prime satisfying the conclusion of Lemma 11. If p is a prime greater than all $p_{a,b}$, then we may set $\Gamma = \Gamma_p$ and $\sigma = \sigma_p$. The components of M^{σ_p} are clearly non-isomorphic to M. The rest of the statement follows by Lemmas 10 and 11. □

We give the oriented version of Theorem 12.

THEOREM 13. *Let M be an oriented map. Then there is a group Γ and a nontrivial oriented voltage assignment ν in Γ such that for all $a, b \in D_M$ and $g, h \in \Gamma$ with a_g and b_h in the same component M_c^ν of M^ν, there is $\tilde{A} \in \mathrm{OAut}(M_C^\nu)$ with $\tilde{A} a_g = b_h$ if and only if there is $A \in \mathrm{OAut}(M)$ with $Aa = b$.*

S k e t c h o f t h e p r o o f. Let the oriented map $M = (D_M, P_M, K_M)$ correspond to a general map $M = (F_M, R_M, L_M, T_M)$. Let σ_p be defined for this latter map as above. Define ν_p as in (4), that is, $\nu_p(a) = \alpha_p(T_M a)$ for all $a \in D_M$. Lemmas analogous to Lemmas 10 and 11 hold for ν_p, and the result follows in the same manner as Theorem 12. □

7. Three applications

Recall that an oriented map $M = (D_M, P_M, K_M)$ is *orientably regular* if for every $a, b \in D_M$ there is an oriented automorphism A of M such that $Aa = b$. A generalization of orientable regularity was considered in [10], where the term exponent of a map was introduced. In [10], the exponent was defined for an arbitrary oriented map; we will present here a simplified definition which applies only to orientably regular maps.

An integer e is an *exponent* of an orientably regular map M if M and $M^e = \big(D_M, (P_M)^e, K_M\big)$ are isomorphic. We make several observations about exponents. Let M be an orientably regular map of valence r. First, if e is an exponent of M, then $\gcd(r,e) = 1$. Second, if e is an integer and k is a multiple of r, then $M^e = M^{e+k}$. Third, if e_1 and e_2 are exponents, then so is e_1e_2. It follows that the exponents form a subgroup of Z_r^*, the multiplicative group of the ring of integers modulo r. We call this group the *exponent group* of M and denote it by $\mathrm{Ex}(M)$. We say that M has *trivial exponent group* if $|\,\mathrm{Ex}(M)| = 1$.

The exponent group of an orientably regular map gives us information on the degree of symmetry of the map. For example, -1 is in the exponent group if and only if the map is reflexible. Maps with large exponent groups have higher symmetry than maps with small exponent groups. Nedela and Škoviera asked [10] for a construction of maps which would be orientably regular, and thus highly symmetric, but would not have too much symmetry. More precisely, they were asking for constructions of infinite sequences of orientably regular maps with trivial exponent group and increasing valencies of their underlying graphs. We now use the ideas of this paper to produce such sequences. The result is an immediate consequence of the next theorem. The voltage ν_p is defined as in the proof of Theorem 13.

THEOREM 14. *Let M be an orientably regular map with a trivial exponent group. Let r be the valency of the graph underlying M, and let l be the length of the faces. If p is a sufficiently large prime, then each component of M^{ν_p} is an orientably regular map with trivial exponent group. Moreover, the valency r' of its underlying graph and the length l' of its faces are greater than r and l respectively.*

Proof. Suppose that p is given, and let N be a component of M^{ν_p}. Then N is orientably regular by the orientable version of Lemma 10. Also, it is immediate that $r' = pr > r$ and $l' = pl > l$.

It remains to show that $\mathrm{Ex}(N) = 1$. Suppose that e is coprime with r'. We will show that the maps N and N^e are non-isomorphic. Since N is orientably regular, it suffices to show that for a fixed flag $a_g \in D_N$ there is no oriented map isomorphism f from N to N^e such that $f(a_g) = a_g$. Because e is coprime with r', it is also coprime with r. Therefore, M and M^e are not orientably isomorphic since $\mathrm{Ex}(M) = 1$. In particular, there is no oriented isomorphism $f'\colon M \to M^e$ such that $f'(a) = a$. By Proposition 1, there is a word X such that $X_M a = a$ and $X_{M^e} a \neq a$. As in Lemma 11, we conclude that if p is a large enough prime, there is no f satisfying the above requirements. Call this prime p_e. Next, let p be a prime greater than every p_e as e ranges over all integers coprime with r which are between 1 and r. It follows that $\mathrm{Ex}(M^{\nu_p}) = 1$. □

It is well-known (see, e.g., [13; Theorems 14–21]) that if M is an oriented map whose genus g is at least 2, then the number of oriented automorphisms of M is at most $84(g-1)$. MacBeath's Theorem ([9]) says that there are infinitely many extremal maps, i.e., maps for which $|\mathrm{OAut}(M)| = 84(g-1)$. As an application of Theorem 8, we will give a constructive proof of MacBeath's Theorem using extended voltage assignments.

THEOREM 15. *There are infinitely many oriented maps M whose genus is at least 2 and for which $|\mathrm{OAut}(M)| = 84(g-1)$.*

Proof. Euler's formula shows that every orientably regular map M whose faces have length 7 and whose valency is 3 satisfies $|\mathrm{OAut}(M)| = 84(g-1)$ (see [13; Theorems 14–21]). We construct infinitely many such maps with genus larger than 1. First, start with an infinite sequence $N_1, N_2, N_3, \dots$ of toroidal holey maps. For every k, the underlying graph of N_k contains $14k$ vertices, $20k$ edges, and $2k$ semiedges. The map N_5 is depicted in Figure 4 below; all other maps N_k follow the same pattern and are obtained by horizontal extension or truncation of N_5. The torus in Figure 4 has no boundary.

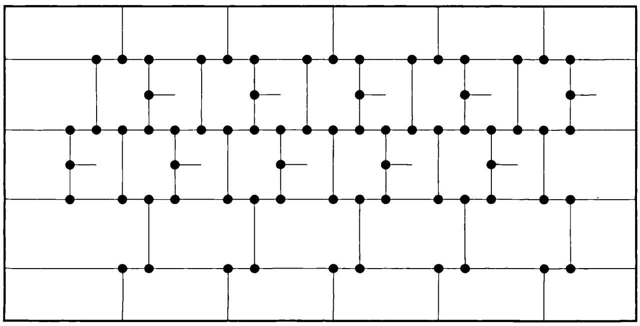


FIGURE 4. The map N_5.

It is easily checked that for every k the holey map $N_k = (F, R, L, T)$ has the permutations R, L, and T fixed-point-free, and has LT not the identity permutation. Also, the type of N_k is $\{7, 3\}$. If σ is the canonical voltage assignment from Theorem 8 on N_k in the group $\Gamma = \langle R, L, T\rangle$, then, by the proof of

Theorem 8 and by Corollary 9, every component $M_k = M$ of N_k^σ is a regular trivalent map with face length 7. Moreover, M is orientable, because the group generated by RT and LT in N_k has two orbits.

Finally, since $|F_{M_k}| \geq |F_{N_k}|$ for every k and since $\{|F_{N_k}| : k = 1, 2, 3, \dots\}$ is infinite, $\{|F_{M_k}| : k = 1, 2, 3 \dots\}$ is infinite as well. In particular, there are infinitely many nonisomorphic maps among the M_k. □

As a final application of techniques presented in this paper, we give a short proof of Grünbaum's Conjecture [5] on the existence of orientably regular map of any given type.

THEOREM 16. *For any $r \geq 2$ and $l \geq 2$ there exists an orientably regular map with vertex valency r and face length l.*

Proof. We may assume that $l \geq r$ since the dual of an orientably regular map is again orientably regular. In view of Theorem 8 and Corollary 9, it is sufficient to construct a holey map $M = M_{l,r}$ of type $\{l, r\}$ with a fixed-point-free R_M, L_M, T_M and a non-trivial $L_M T_M$ for each $2 \leq r \leq l$. We consider three cases.

First suppose that $l = r$. Maps $M_{2,2}$ and $M_{3,3}$ are trivial to obtain. To *construct $M_{r,r}$ for $r \geq 4$, it suffices to take a single-face toroidal embedding of a one-vertex graph with two loops and* $r - 4$ semiedges.

Second suppose that $r + 1 \leq l \leq 2r$. Then $M = M_{l,r}$ can be chosen as a spherical embedding of a star with $l - r + 1$ vertices $u, v_1, \dots, v_{l-r}$, with $l - r$ edges $uv_1, \dots, uv_{l-r}$, and $2r - l$ semiedges incident with u.

Third suppose that $l = sr + t$, where $r + 1 \leq t \leq 2r$. Replace the edge uv_1 in $M_{t,r}$ by a path $uw_1 \dots w_s v_1$ and attach $r - 2$ semiedges to each of the vertices $w_1, \dots, w_s$ to obtain the map $M_{l,r}$.

It is easy to check that the maps $M_{l,r}$ are indeed of type $\{l, r\}$, that is, the least common multiple of their face lengths and vertex valencies is l and r, respectively. The Theorem now follows from Corollary 9 and from the fact that the corresponding regular covers are orientable. □

We note that similar ideas for short proofs of MacBeath's Theorem and Grünbaum's Conjecture have been given by Jendroľ, Nedela, and Škoviera [7]. We acknowledge their priority.

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