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OSCILLATION OF SOLUTIONS FOR NONLINEAR SECOND ORDER NEUTRAL EQUATIONS WITH DEVIATING ARGUMENTS

WANG PEIGUANG* — YU YUANHONG**

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ABSTRACT. This paper discusses the oscillatory conditions of second order neutral differential equations with continuous distributed deviating arguments.

1. Introduction

The study of the oscillatory behavior of the solutions of neutral delay differential equations, besides its theoretical interest, is important in application. Examples of their applications can be found in [1]. A few results on the oscillatory behavior of the solutions of second order neutral delay differential equations are recently obtained in [2], [3], [4], [5], [6], [7], [8] and their references. However, it is noticeable that most of the cases are with discrete delay. The aim of this paper is to extend some results in [2] – [6], [8] to the following nonlinear equation with continuous deviating arguments

$$\left[a(t)\left[y(t)+\sum_{i=1}^{m} c_i(t)y\big(\tau_i(t)\big)\right]'\right]' + \int_a^b f\big(t,\xi,y[g(t,\xi)]\big)\,\mathrm{d}\sigma(\xi) = 0\,, \qquad t \ge t_0\,, \tag{1}$$

and to establish some new oscillatory criteria.

It is easy to see that (1) includes the following equation

$$\left[a(t)\left[y(t)+\sum_{i=1}^{m} c_i(t)y\big(\tau_i(t)\big)\right]'\right]' + \sum_{j=1}^{n} f_j\big(t,y[g_j(t)]\big) = 0\,, \qquad t \ge t_0\,, \tag{2}$$

then the obtained oscillatory criteria improve and generalize some known results.

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Suppose that the following conditions hold.

(H_1) $a(t), c_i(t) \in C([t_0,+\infty),\mathbb{R}_+)$; $f(t,\xi,y) \in C([t_0,+\infty) \times [a,b] \times \mathbb{R},\mathbb{R})$; $\mathbb{R}_+ = [0,+\infty)$;

(H_2) $\tau_i(t) \in C([t_0,+\infty),\mathbb{R})$; $\tau_i(t) \le t$, and $\lim\limits_{t\to+\infty} \tau_i(t) = +\infty$, $i \in I_m = \{1,2,\dots,m\}$;

(H_3) $g(t,\xi) \in C([t_0,+\infty) \times [a,b],\mathbb{R})$; $g(t,\xi) \le t$ for $\xi \in [a,b]$, $g(t,\xi)$ is nondecreasing with respect to t and ξ, respectively; and $\lim\limits_{t\to+\infty} \min\limits_{\xi\in[a,b]} \{g(t,\xi)\} = +\infty$;

(H_4) $\sigma(\xi) \in ([a,b],\mathbb{R})$ is nondecreasing, integral of equation (1) is a Stieltjes integral.

Let $u \in C([t_{-1},+\infty);\mathbb{R})$, where

$$t_{-1} = \min\Big\{ \min_{i\in I_m}\Big\{ \inf_{t\ge t_0} \tau_i(t)\Big\}, \min_{\xi\in[a,b]} g\{(t,\xi)\}, t_0\Big\} ,$$

be a given function and let y_0 be a given constant. Using the method of steps, equation (1) has a unique solution $y(t) \in C([t_{-1},+\infty);\mathbb{R})$ in the sense that both $y(t) + \sum\limits_{i=1}^{m} c_i(t)y(\tau_i(t))$ and $a(t)\Big[y(t) + \sum\limits_{i=1}^{m} c_i(t)y(\tau_i(t))\Big]'$ are continuously differentiable for $t \ge t_0$, $y(t)$ satisfies equation (1) and

$$y(s) = u(s) \quad \text{for} \quad s \in [t_{-1},t_0] , \qquad \Big[y(t) + \sum_{i=1}^{m} c_i(t)y(\tau_i(t))\Big]'_{t=t_0} = y_0 .$$

For further questions concerning existence and uniqueness of solutions of neutral delay differential equations, see [1].

Definition 1. A function $y(t)$ is called *eventually positive* (*negative*) if there exists a number $t_1 \ge t_0$ such that $y(t) > 0$ (< 0) holds for all $t > t_1$.

Definition 2. A solution $y(t)$ of equation (1) is said *oscillatory* if it is not eventually zero solution and it has an unbounded set of zeros. Otherwise, it is called *nonoscillatory.*

For the sake of convenience, we assume that every inequality about functional values is true for all sufficiently large t.

2. Main results

Theorem 1. *Suppose that the following conditions hold*

$$\sum_{i=1}^{m} c_i(t) \le 1 \qquad \textit{and} \qquad \int_{t_0}^{+\infty} \frac{1}{a(s)}\, \mathrm{d}s = +\infty , \tag{3}$$

there exist function $Q(t,\xi) \in C\big([t_0,+\infty) \times [a,b], \mathbb{R}_+\big)$ *and* $F(y) \in C(\mathbb{R},\mathbb{R})$ *such that*

$$f(t,\xi,y)\operatorname{sgn} y \ge Q(t,\xi)F(y)\operatorname{sgn} y \tag{4}$$

and

$$-F(-y) \ge F(y) \ge \lambda y > 0\,, \qquad (\,y>0\,,\ \ \lambda>0 \ \textit{ is a constant}). \tag{5}$$

If there exists a $\frac{\mathrm{d}}{\mathrm{d}t}g(t,a)$ *and function* $\varphi \in C'\big([t_0,+\infty),\mathbb{R}_+\big)$ *such that*

$$\int_{t_0}^{+\infty} \left[\lambda\varphi(s)\int_a^b Q(s,\xi)\left\{1-\sum_{i=1}^m c_i\big[g(s,\xi)\big]\right\}\,\mathrm{d}\sigma(\xi) - \frac{a[g(s,a)]\varphi'^2(s)}{4\varphi(s)g'(s,a)}\right]\,\mathrm{d}s = +\infty\,, \tag{6}$$

then all solutions of equation (1) *are oscillatory.*

P r o o f . Suppose to the contrary that there exists a nonoscillatory solution $y(t)$ of equation (1). We may assume that $y(t)$ is an eventually positive solution. Let

$$z(t) = y(t) + \sum_{i=1}^m c_i(t)y\big(\tau_i(t)\big)\,. \tag{7}$$

Then equation (1) can be written as

$$\big[a(t)z'(t)\big]' + \int_a^b f\big(t,\xi,y[g(t,\xi)]\big)\,\mathrm{d}\xi = 0\,. \tag{8}$$

It follows from (7) that

$$y\big[g(t,\xi)\big] = z\big[g(t,\xi)\big] - \sum_{i=1}^m c_i\big[g(t,\xi)\big]y\big(\tau_i[g(t,\xi)]\big)\,.$$

It follows from (H_1) and (H_2) that

$$z(t) \ge y(t) \qquad \text{and} \qquad z(t) \ge 0\,. \tag{9}$$

Moreover, we have

$$\big[a(t)z'(t)\big]' \le 0\,, \tag{10}$$

thus $a(t)z'(t)$ is decreasing with respect to t, and we can prove that $z'(t) \ge 0$. In fact, let there exist a $t_1 \ge t_0$ such that $z'(t) < 0$, $t \ge t_1$. Integrating both sides of (10) from $[t_1,t]$, we have $a(t)z'(t) \le a(t_1)z'(t_1)$, furthermore, for $t_2 \ge t_1$, we have

$$z(t) \le z(t_2) + a(t_1)z(t_1)\int_{t_2}^t \frac{1}{a(s)}\,\mathrm{d}s\,.$$

Let $t \to +\infty$; using (3), it follows that $\lim\limits_{t\to+\infty} z(t) = -\infty$, which is in contradiction with $z(t) > 0$.

From (9), we have $F\big(y[g(t,\xi)]\big) \geq \lambda y\big[g(t,\xi)\big] > 0$, thus

$$0 \geq \big[a(t)z'(t)\big]' + \lambda \int\limits_a^b Q(t,\xi)\Big\{z[g(t,\xi)] - \sum_{i=1}^m c_i[g(t,\xi)]y\big(\tau_i[g(t,\xi)]\big)\Big\}\, \mathrm{d}\sigma(\xi)\,. \tag{11}$$

It follows from (H_1), (9) and (11) that

$$\big[a(t)z'(t)\big]' + \lambda \int\limits_a^b Q(t,\xi)\Big\{1 - \sum_{i=1}^m c_i\big[g(t,\xi)\big]\Big\} z\big[g(t,\xi)\big]\, \mathrm{d}\sigma(\xi) \leq 0\,. \tag{12}$$

Noticing that $g(t,\xi)$ is nondecreasing with respect to ξ, we have $g(t,a) \leq g(t,\xi)$ for $\xi \in [a,b]$, thus

$$\big[a(t)z'(t)\big]' + \lambda z\big[g(t,a)\big] \int\limits_a^b Q(t,\xi)\Big\{1 - \sum_{i=1}^m c_i\big[g(t,\xi)\big]\Big\}\, \mathrm{d}\sigma(\xi) \leq 0\,. \tag{13}$$

Set

$$w(t) = \varphi(t)\frac{a(t)z'(t)}{z\big[g(t,a)\big]}\,; \tag{14}$$

then $w(t) \geq 0$. From the condition of Theorem 1, we have $z'\big[g(t,a)\big] = \frac{\mathrm{d}z}{\mathrm{d}g}\frac{\mathrm{d}}{\mathrm{d}t}g(t,a)$. Since $a(t)z'(t)$ is decreasing, and noticing that $g(t,\xi) \leq t$ for $\xi \in [a,b]$, we have

$$a(t)z'(t) \leq a\big[g(t,a)\big]z'\big[g(t,a)\big]\,.$$

Thus

$$\begin{aligned} w'(t) &= \frac{\varphi'(t)\big[a(t)z'(t)\big]}{z\big[g(t,a)\big]} + \varphi(t)\frac{\big[a(t)z'(t)\big]'}{z\big[g(t,a)\big]} - \frac{\varphi(t)\big[a(t)z'(t)\big]z'\big[g(t,a)\big]g'(t,a)}{z^2\big[g(t,a)\big]} \\ &\leq \frac{\varphi(t)\big[a(t)z'(t)\big]'}{z\big[g(t,a)\big]} + \frac{a\big[g(t,\xi)\big]\varphi'^2(t)}{4\varphi(t)g'(t,a)} \\ &\quad - \left[\sqrt{\frac{\varphi(t)g'(t,a)}{a\big[g(t,a)\big]}}\frac{a(t)z'(t)}{z\big[g(t,a)\big]} - \frac{\varphi'(t)}{2}\sqrt{\frac{a\big[g(t,a)\big]}{\varphi(t)g'(t,a)}}\right]^2 \\ &\leq \varphi(t)\frac{\big[a(t)z'(t)\big]'}{z\big[g(t,a)\big]} + \frac{a\big[g(t,a)\big]\varphi'^2(t)}{4\varphi(t)g'(t,a)}\,. \end{aligned}$$

It follows from (13) that

$$w'(t) \le -\left[\lambda\varphi(t)\int_a^b Q(t,\xi)\left\{1-\sum_{i=1}^m c_i[g(t,\xi)]\right\} \mathrm{d}\sigma(\xi) - \frac{a[g(t,a)]\varphi'^2(t)}{4\varphi(t)g'(t,a)}\right].$$

Integrating both sides of the above last inequality from t_1 to t ($t > t_1$), we have

$$w(t) \le w(t_1) - \int_{t_1}^t \left[\lambda\varphi(s)\int_a^b Q(s,\xi)\left\{1-\sum_{i=1}^m c_i[g(s,\xi)]\right\} \mathrm{d}\sigma(\xi) - \frac{a[g(s,a)]\varphi'^2(s)}{4\varphi(s)g'(s,a)}\right] \mathrm{d}s. \quad (15)$$

Let $t \to +\infty$, then by (6) and (15), we have $w(t) \to -\infty$, which leads to a contradiction with $w(t) > 0$.

Let $y(t)$ be an eventually negative solution of equation (1). Let $x(t) = -y(t)$, then equation (1) will change to the following equation

$$\left[a(t)\left[x(t)+\sum_{i=1}^m c_i(t)x(\tau_i(t))\right]'\right]' + \int_a^b f^*(t,\xi,x[g(t,\xi)]) \,\mathrm{d}\xi = 0, \qquad t \ge t_0, \tag{1*}$$

where $f^*(t,\xi,x[g(t,\xi)]) \equiv -f(t,\xi,-x[g(t,\xi)])$.

Conditions (4) and (5) imply that

$$\begin{aligned} f^*(t,\xi,x[g(t,\xi)]) &\equiv -f(t,\xi,-x[g(t,\xi)]) \\ &\ge Q(t,\xi)\{-F(-x[g(t,\xi)])\} \ge Q(t,\xi)F(x[g(t,\xi)]), \end{aligned}$$

therefore, we can use the same method to prove the result. This completes the proof of Theorem 1. □

Remark 1. Theorem 1 generalizes Theorem 1 in [2], [3], [5], [6] and [8; Theorem 2].

Remark 2. If function $\varphi(t) \equiv 1$, we have the following corollary.

COROLLARY. *Suppose that* (3) -(5) *hold. If*

$$\int_{t_0}^{+\infty}\int_a^b Q(s,\xi)\left\{1-\sum_{i=1}^m c_i[g(s,\xi)]\right\} \mathrm{d}\sigma(\xi)\,\mathrm{d}s = +\infty,$$

then all solutions of equation (1) *are oscillatory.*

Remark 3. Corollary generalizes Theorem 1 in [2], [4], [5], [6], [8].

THEOREM 2. *Suppose that* (3) – (5) *hold, and*

(i) $c_i(t) \equiv c_i \geq 0$, *there exist* $a'(t)$ *and* $\tau_i'(t) > 0$, $i \in I_m$,
(ii) *there exists a function* $g_t'(t,\xi) \in C([t_0, +\infty) \times [a,b], \mathbb{R}_+)$,
(iii) *there exists a function* $\eta(t) \in C([t_0, +\infty), \mathbb{R}_+)$ *such that*

$$Q(t,\xi) \geq \eta(t), \qquad t \geq t_0, \quad \xi \in [a,b]. \tag{16}$$

If

$$\int_{t_0}^{+\infty} \eta(s) \, \mathrm{d}s = +\infty, \tag{17}$$

then the derivatives of all differentiable solutions of equation (1) *are oscillatory.*

Proof. Suppose that there exists a differentiable solution $y(t)$ of equation (1) such that we eventually have

$$y(t) > 0 \qquad \text{and} \qquad y'(t) > 0, \tag{18}$$

$$y(t) > 0 \qquad \text{and} \qquad y'(t) < 0. \tag{19}$$

First, suppose that (18) holds. Let

$$v(t) = \frac{a(t)z'(t)}{\int_a^b y[g(t,\xi)] \, \mathrm{d}\sigma(\xi)}. \tag{20}$$

Then it follows that $v(t) > 0$, and from (i), we know that there exists y'', therefore y' is continuous, and it follows from (ii) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b y[g(t,\xi)] \, \mathrm{d}\sigma(\xi) = \int_a^b \frac{\mathrm{d}y}{\mathrm{d}g} g_t'(t,\xi) \, \mathrm{d}\sigma(\xi) \geq 0.$$

Then

$$\begin{aligned} v'(t) &= \frac{[a(t)z'(t)]'}{\int_a^b y[g(t,\xi)] \, \mathrm{d}\sigma(\xi)} - \frac{[a(t)z'(t)] \frac{\mathrm{d}}{\mathrm{d}t} \int_a^b y[g(t,\xi)] \, \mathrm{d}\sigma(\xi)}{\left[\int_a^b y[g(t,\xi)] \, \mathrm{d}\sigma(\xi)\right]^2} \\ &\leq \frac{[a(t)z'(t)]'}{\int_a^b y[g(t,\xi)] \, \mathrm{d}\sigma(\xi)} \leq -\lambda \eta(t). \end{aligned} \tag{21}$$

Integrating both sides of above inequality from t_2 to t $(t > t_2)$, we have

$$v(t) \le v(t_2) - \lambda \int_{t_2}^{t} \eta(s)\, \mathrm{d}s\,. \tag{22}$$

Let $t \to +\infty$; from (17), we have $v(t) \to -\infty$, which leads to a contradiction with $v(t) > 0$.

Next, suppose that (19) holds. By (17), there exists a $T \ge t_0$ such that

$$\int_{T}^{t} \eta(s)\, \mathrm{d}s > 0\,, \qquad t > T\,. \tag{23}$$

Using $y'(t) < 0$ and $g(t,\xi)$ is nondecreasing with respect to t, we have

$$y\big[g(s,\xi)\big] \ge y\big[g(t,\xi)\big] > 0\,, \qquad\qquad s \le t\,,$$

$$\int_{a}^{b} y\big[g(s,\xi)\big]\, \mathrm{d}\sigma(\xi) \ge \int_{a}^{b} y\big[g(t,\xi)\big]\, \mathrm{d}\sigma(\xi) > 0\,, \qquad s \le t\,.$$

Thus using (4), (5), (17) and (23), we have

$$\begin{aligned} \int_{T}^{t}\int_{a}^{b} f\big(s,\xi,y[g(s,\xi)]\big)\, \mathrm{d}\sigma(\xi)\, \mathrm{d}s &\ge \int_{T}^{t}\int_{a}^{b} Q(s,\xi)F\big(y[g(s,\xi)]\big)\, \mathrm{d}\sigma(\xi)\, \mathrm{d}s \\ &\ge \lambda \left(\int_{T}^{t} \eta(s)\, \mathrm{d}s \right) \left(\int_{a}^{b} y\big[g(t,\xi)\big]\, \mathrm{d}\sigma(\xi) \right) > 0\,. \end{aligned} \tag{24}$$

Integrating both sides of equation (1) from T to t $(t > T)$, and using (24), we have

$$a(t)z'(t) - a(T)z'(T) = -\int_{T}^{t}\int_{a}^{b} f\big(s,\xi,y[g(s,\xi)]\big)\, \mathrm{d}\sigma(\xi)\, \mathrm{d}s < 0\,,$$

thus

$$a(t)z'(t) < a(T)z'(T)\,. \tag{25}$$

Integrating both sides of above inequality from T_1 to t $(t > T_1)$, we have

$$z(t) < z(T_1) + a(T)z'(T) \int_{T_1}^{t} \frac{1}{a(s)}\, \mathrm{d}s\,. \tag{26}$$

Noticing $z'(t) = y'(t) + \sum_{i=1}^{m} c_i y'(\tau_i(t))\tau_i'(t) < 0$, we have $\lim_{t\to+\infty} z(t) = -\infty$, which contradicts $z(t) > 0$.

For the case of a differentiable solution $y(t)$ of equation (1) that eventually have

$$y(t) < 0 \qquad \text{and} \qquad y'(t) > 0\,,$$

or

$$y(t) < 0 \qquad \text{and} \qquad y'(t) < 0\,,$$

we can also prove the result by the same argument. This completes the proof of Theorem 2. □

Remark 4. Theorem 2 generalizes Theorem 2 in [2], [4], [5], [6] and [8; Theorem 3].

Now, we give some examples.

EXAMPLE 1.

$$\Big[t\big[y(t) + \big(1 - \tfrac{1}{t}y(t-\tau)\big)\big]'\Big]' + \int_1^2 ty\big(\tfrac{1}{2}t\xi\big)\sqrt{1 + y^2\big(\tfrac{1}{2}t\xi\big)}\,\operatorname{arctg}\xi \,\mathrm{d}\xi = 0\,, \qquad t \ge 1\,, \tag{27}$$

in which

$$\tau > 0\,, \quad a(t) = t\,, \quad c(t) = 1 - \frac{1}{t}\,,$$
$$g(t,\xi) = \frac{1}{2}t\xi\,, \quad f(t,\xi,y) = tx\sqrt{1+y^2}\,\operatorname{arctg}\xi\,.$$

Choosing $Q(t,\xi) = \frac{\pi}{4}$, $F(y) = y\sqrt{1+y^2}$, the conditions of Corollary are satisfied. Therefore all the solutions of equation (27) are oscillatory.

EXAMPLE 2.

$$\begin{aligned} &\Big[\mathrm{e}^{-t}\big[y(t) + \big(\tfrac{2}{3} - \tfrac{1}{3}\,\mathrm{e}^{-2t}\,y(t-\tau_1) + \tfrac{1}{3}y(t-\tau_2)\big)\big]'\Big]' \\ &\qquad + \int_{-2}^{-1} \mathrm{e}^{t+2\xi}\,y(t+\xi)\big[1 + y^{\frac{2}{3}}(t+\xi)\big]\,\mathrm{d}\xi = 0\,, \qquad t \ge 1\,, \end{aligned} \tag{28}$$

in which

$$\tau_1, \tau_2 > 0\,, \quad a(t) = \mathrm{e}^{-t}\,, \quad c_1(t) = \frac{2}{3} - \frac{1}{3}\,\mathrm{e}^{-2t}\,, \quad c_2(t) = \frac{1}{3}\,,$$
$$g(t,\xi) = t + \xi\,, \quad f(t,\xi,y) = \mathrm{e}^{(t+2\xi)}\,y\big(1 + y^{\frac{2}{3}}\big)\,.$$

Choosing $Q(t,\xi) = \mathrm{e}^{t+2\xi}$, $F(y) = y\big(1 + y^{\frac{2}{3}}\big)$, $\varphi(t) = \sqrt{t}$, the conditions of Theorem 1 are satisfied. Therefore all solutions of equation (28) are oscillatory.

EXAMPLE 3.

$$\Big[\tfrac{1}{t}\big[y(t)+\tfrac{1}{2}y(t-\tau_1)+\tfrac{1}{2}y(t-\tau_2)\big]'\Big]' + \int\limits_0^1 \frac{t+\xi}{\sin\xi} y\big(\tfrac{t}{2}+\xi\big)\, \mathrm{e}^{y^2(\frac{t}{2}+\xi)}\, \mathrm{d}\xi = 0\,, \qquad t \ge 2\sqrt{2}\,, \tag{29}$$

in which $\tau_1, \tau_2 > 0$, $g(t,\xi) = \frac{t}{2} + \xi$, $f(t,\xi,y) = \frac{t+\xi}{\sin\xi} y\, \mathrm{e}^{y^2}$.

Choosing $Q(t,\xi) = t+\xi$, $F(y) = y\,\mathrm{e}^{y^2}$, $\eta(t) = t$, the conditions of Theorem 2 are satisfied. Therefore the derivatives of differentiable solution of equation (29) are oscillatory.

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