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LOCALLY BEST QUADRATIC ESTIMATORS

LUBOMÍR KUBÁČEK

Introduction

Let $\boldsymbol{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ be an n-dimensional random vector, $\mathbf{X}$ is a known $n \times k$ matrix and $\boldsymbol{\beta} \in \mathcal{R}^k$ (k-dimensional Euclidean space) is an unknown vector parameter. The mean value of the error vector $\boldsymbol{\varepsilon}$ is $E(\boldsymbol{\varepsilon}) = \boldsymbol{0}$ and its covariance matrix is $E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') = \Sigma = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$. The symmetric matrices $\mathbf{V}_i$, $i = 1, \ldots, p$, are known, $(\vartheta_1, \ldots, \vartheta_p)' = \boldsymbol{\vartheta} \in \vartheta^*$ is unknown vector parameter and ϑ^* is a subset of the space $\mathcal{R}^p$ with nonempty interior. The matrix of the third central moments $E[\boldsymbol{\varepsilon} \otimes (\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')]$ is φ and the matrix of the fourth central moments $E[(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') \otimes (\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')]$ is ψ. The symbol $\otimes$ denotes the tensor product. The parametric space $\mathcal{R}^k \times \vartheta^* \times \varphi^* \times \psi^*$ is denoted by θ^*; φ^* and ψ^* are given subsets of the space of all matrices of the third and the fourth moments, respectively.

The problem is to estimate a function $\gamma(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{c}'\boldsymbol{\beta} + \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\beta} \in \mathcal{R}^k$, $\boldsymbol{\vartheta} \in \vartheta^*$, from a realization $\boldsymbol{y}$ of the random vector $\boldsymbol{Y}$. The vectors $\boldsymbol{c} \in \mathcal{R}^k$, $\boldsymbol{f} \in \mathcal{R}^p$, are given. The unbiased estimator of the function with minimum variance at the given point $\boldsymbol{\theta} = (\boldsymbol{\beta}, \boldsymbol{\vartheta}, \varphi, \psi)$ of the parametric space θ^* is considered in the form $\hat{\gamma} = \boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$, where $\boldsymbol{a} \in \mathcal{R}^n$ and $\mathbf{A}$ is an $n \times n$ matrix.

The aim of this paper is to contribute to the determination of the explicit expression for the vector $\boldsymbol{a}$ and for the matrix $\mathbf{A}$.

The fundamental paper on this problem is [1].

1. Notations and auxiliary statements

Definition 1.1. *Let $\mathcal{S}_n$ and $\mathcal{M}_{m,n}$, respectively, be the space of all symmetric $n \times n$ matrices and the space of all $m \times n$ matrices. Mappings*

$$\mathrm{vec}(\cdot): \mathcal{M}_{m,n} \to \mathcal{R}^{mn},$$
$$\mathrm{vech}(\cdot): \mathcal{S}_n \to \mathcal{R}^{n(n+1)/2}$$
$$(\mathrm{cR})\mathrm{vec}(\cdot): \mathcal{S}_n \to \mathcal{R}^{n(n+1)/2}$$

are given by the following relationships

$$\mathrm{vec}(\mathbf{T}) = (t_{1,1}, t_{2,1}, \ldots, t_{m,1}; t_{1,2}, t_{2,2}, \ldots, t_{m,2}; \ldots; t_{1,n}, t_{2,n}, \ldots, t_{m,n})',$$
$$\mathrm{vech}(\mathbf{S}) = (s_{1,1}, s_{1,2}, \ldots, s_{1,n}; s_{2,2}, s_{2,3}, \ldots, s_{2,n}; \ldots; s_{n-1,n-1}, s_{n-1,n}; s_{n,n})',$$
$$(\mathrm{cR})\mathrm{vec}(\mathbf{S}) = (s_{1,1}, 2s_{1,2}, \ldots, 2s_{1,n}; s_{2,2}, 2s_{2,3}, \ldots, 2s_{2,n}; \ldots; s_{n-1,n-1}, 2s_{n-1,n}; s_{n,n})'.$$

Here $t_{i,j} = \{\mathbf{T}\}_{i,j}$ and $s_{i,j} = \{\mathbf{S}\}_{i,j}$ are the (i, j)-th elements of the matrices $\mathbf{T}$ and $\mathbf{S}$, respectively.

Definition 1.2. *Mapping*

$$(\mathrm{cC})(\cdot): \mathcal{M}_{p,r^2} \to \mathcal{M}_{p,r(r+1)/2}$$

is given by

$$(\mathrm{cC})(\mathbf{M}) = (\boldsymbol{m}_{1,1}, \boldsymbol{m}_{1,2} + \boldsymbol{m}_{2,1}, \ldots, \boldsymbol{m}_{1,r} + \boldsymbol{m}_{r,1}; \boldsymbol{m}_{2,2}, \boldsymbol{m}_{2,3} + \boldsymbol{m}_{3,2}, \ldots, \boldsymbol{m}_{2,r} + \boldsymbol{m}_{r,2}; \ldots; \boldsymbol{m}_{r-1,r-1}, \boldsymbol{m}_{r-1,r} + \boldsymbol{m}_{r,r-1}; \boldsymbol{m}_{r,r}),$$

where

$$\mathbf{M} = (\boldsymbol{m}_{1,1}, \boldsymbol{m}_{1,2}, \ldots, \boldsymbol{m}_{1,r}; \boldsymbol{m}_{2,1}, \boldsymbol{m}_{2,2}, \ldots, \boldsymbol{m}_{2,r}; \ldots; \boldsymbol{m}_{r,1}, \boldsymbol{m}_{r,2}, \ldots, \boldsymbol{m}_{r,r}).$$

Here $\boldsymbol{m}_{i,j} \in \mathcal{R}^p$, $i, j = 1, \ldots, r$, are columns of a matrix $\mathbf{M}$.

An analogous operation on the rows of the matrix $\mathbf{M} \in \mathcal{M}_{p^2,r}$ defines the mapping

$$(\mathrm{cR})(\mathbf{M}): \mathcal{M}_{p^2,r} \to \mathcal{M}_{p(p+1)/2,r}.$$

Lemma 1.1. *For arbitrary matrices $\mathbf{A} \in \mathcal{M}_{p,r}$, $\mathbf{S} \in \mathcal{S}_r$, $\mathbf{B} \in \mathcal{M}_{r,p}$ there holds:* $\mathbf{ASB} = \mathbf{C} \Leftrightarrow (\mathbf{B}' \otimes \mathbf{A})\ \mathrm{vec}(\mathbf{S}) = \mathrm{vec}(\mathbf{C}) \Leftrightarrow (\mathrm{cR})(\mathrm{cC})\ (\mathbf{B}' \otimes \mathbf{A})\ \mathrm{vech}(\mathbf{S}) = (\mathrm{cR})\mathrm{vec}(\mathbf{C})$.

Proof is obvious.

Lemma 1.2. *Let $\mathbf{N} \in \mathcal{S}_n$ be a positive semidefinite* (p.s.d.) *matrix, $\mathbf{A} \in \mathcal{M}_{m,n}$, $\boldsymbol{y} \in \mathcal{R}^m$, $\boldsymbol{y} \in \mathcal{M}(\mathbf{A})$ (column space of the matrix $\mathbf{A}$). Then there exists a matrix $\mathbf{G} \in \mathcal{M}_{n,m}$ with the property:*

$$\forall\{\boldsymbol{x}: \mathbf{A}\boldsymbol{x} = \boldsymbol{y}\}(\mathbf{G}\boldsymbol{y})'\mathbf{N}\mathbf{G}\boldsymbol{y} \leq \boldsymbol{x}'\mathbf{N}\boldsymbol{x}\ \&\ \mathbf{AG}\boldsymbol{y} = \boldsymbol{y};$$

the matrix $\mathbf{G}$ is a solution of the equations $\mathbf{AGA} = \mathbf{A}$, $(\mathbf{GA})'\mathbf{N} = \mathbf{NGA}$. A particular solution is $(\mathbf{N} + \mathbf{A}'\mathbf{A})^-\mathbf{A}'[\mathbf{A}(\mathbf{N} + \mathbf{A}'\mathbf{A})^-\mathbf{A}']^-$. (The symbol $^-$ denotes a g-inversion, i.e. the matrix $\mathbf{S}^-$ fulfils the condition $\mathbf{SS}^-\mathbf{S} = \mathbf{S}$.)

Proof. See [4, p. 44] and [2, Lemma 2.1.12 and Lemma 2.1.15].

The matrix $\mathbf{G}$ from Lemma 1.2 is denoted as $\mathbf{A}^-_{m(\mathbf{N})}$.

Lemma 1.3. *The equation $\mathbf{AXB} = \mathbf{C}$, $\mathbf{A} \in \mathcal{M}_{p,q}$, $\mathbf{X} \in \mathcal{M}_{q,r}$, $\mathbf{B} \in \mathcal{M}_{r,s}$, $\mathbf{C} \in \mathcal{M}_{p,s}$ with an unknown matrix $\mathbf{X}$ has a solution iff $\mathbf{AA}^-\mathbf{CB}^-\mathbf{B} = \mathbf{C}$. If this condition is fulfilled, the general solution is $\mathbf{X} = \mathbf{A}^-\mathbf{CB}^- + \mathbf{T} - \mathbf{A}^-\mathbf{ATBB}^-$, where $\mathbf{T}$ is an arbitrary matrix from $\mathcal{M}_{q,r}$.*

Proof. See [4, p. 24], and Lemma 2.1.5 in [2].

2. Unbiased and invariant estimators

Lemma 2.1. *A function* $\gamma(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{c}'\boldsymbol{\beta} + \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\beta} \in \mathcal{R}^k$, $\boldsymbol{\vartheta} \in \vartheta^*$, *is unbiasedly estimable iff* $\boldsymbol{c} \in \mathcal{M}(\mathbf{X}')$ *and* $\boldsymbol{f} \in \mathcal{M}(\mathbf{K})$, *where* $\mathbf{K} \in \mathcal{S}_p$ *and its* (i, j)*-th element is* $\{\mathbf{K}\}_{i,j} = \mathrm{Tr}(\mathbf{V}_i\mathbf{V}_j - \mathbf{P}\mathbf{V}_i\mathbf{P}\mathbf{V}_j) = \{\tilde{\mathbf{V}}'(\mathbf{I} - \mathbf{P}\otimes\mathbf{P})\tilde{\mathbf{V}}\}_{i,j}$; $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'$, $\tilde{\mathbf{V}} = [\mathrm{vec}(\mathbf{V}_1), \mathrm{vec}(\mathbf{V}_2), \ldots, \mathrm{vec}(\mathbf{V}_p)]$.

Proof. It follows from the definition of unbiasedness and from the assumption on the interior of the set ϑ^*.

Lemma 2.2. *An estimator* $\hat{\gamma} = \boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$ *of the unbiasedly estimable function* $\gamma(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{c}'\boldsymbol{\beta} + \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\beta} \in \mathcal{R}^k$, $\boldsymbol{\vartheta} \in \vartheta^*$ (i.e. $\boldsymbol{c} \in \mathcal{M}(\mathbf{X}')$, $\boldsymbol{f} \in \mathcal{M}(\mathbf{K})$) *is unbiased iff*

$$\mathbf{X}'\boldsymbol{a} = \boldsymbol{c},\ (\mathbf{X}'\otimes\mathbf{X}')\mathrm{vec}(\mathbf{A}) = \boldsymbol{0},\ \tilde{\mathbf{V}}'\mathrm{vec}(\mathbf{A}) = \boldsymbol{f}. \tag{2.1}$$

Proof. The statement is the consequence of the relationships $E(\boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}) = \boldsymbol{a}'E(\boldsymbol{Y}) + [\mathrm{vec}(\mathbf{A})]'E(\boldsymbol{Y}\otimes\boldsymbol{Y})$, $E(\boldsymbol{Y}) = \mathbf{X}\boldsymbol{\beta}$, $E(\boldsymbol{Y}\otimes\boldsymbol{Y}) = (\mathbf{X}\otimes\mathbf{X})(\boldsymbol{\beta}\otimes\boldsymbol{\beta}) + \tilde{\mathbf{V}}\boldsymbol{\vartheta}$ and of the assumption on the interior of the set ϑ^*. (The relationship $E(\boldsymbol{Y}\otimes\boldsymbol{Y}) = (\mathbf{X}\otimes\mathbf{X})(\boldsymbol{\beta}\otimes\boldsymbol{\beta}) + \tilde{\mathbf{V}}\boldsymbol{\vartheta}$ can be obtained in the following way:

$$E(\boldsymbol{Y}\otimes\boldsymbol{Y}) = E[(\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon})\otimes(\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon})] = (\mathbf{X}\boldsymbol{\beta})\otimes(\mathbf{X}\boldsymbol{\beta}) + E[\mathrm{vec}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')] =$$
$$= (\mathbf{X}\otimes\mathbf{X})(\boldsymbol{\beta}\otimes\boldsymbol{\beta}) + \mathrm{vec}\left(\sum_{i=1}^{p}\vartheta_i\mathbf{V}_i\right) = (\mathbf{X}\otimes\mathbf{X})(\boldsymbol{\beta}\otimes\boldsymbol{\beta}) + \tilde{\mathbf{V}}\boldsymbol{\vartheta}.)$$

Note 2.1. As $\forall\{\boldsymbol{y} \in \mathcal{R}^n\}\ \boldsymbol{y}'\mathbf{A}\boldsymbol{y} = (1/2)\boldsymbol{y}'(\mathbf{A} + \mathbf{A}')\boldsymbol{y}$ the symmetric matrix $\mathbf{A}(\in \mathcal{S}_n)$ can be used instead of the matrix $\mathbf{A}(\in \mathcal{M}_{n,n})$ from Lemma 2.2. The condition for the unbiasedness of the estimator $\hat{\gamma} = \boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$ of an unbiasedly estimable function $\gamma(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{c}'\boldsymbol{\beta} + \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\beta} \in \mathcal{R}^k$, $\boldsymbol{\vartheta} \in \vartheta^*$, can be rewritten as

$$\mathbf{X}'\boldsymbol{a} = \boldsymbol{c},\quad (\mathrm{cC})(\mathbf{X}'\otimes\mathbf{X}')\mathrm{vech}(\mathbf{A}) = \boldsymbol{0},\quad (\mathrm{cC})(\tilde{\mathbf{V}}')\mathrm{vech}(\mathbf{A}) = \boldsymbol{f}.$$

Note 2.2. Consider a function $\gamma(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \vartheta^*$. With respect to Lemma 2.2 the unbiased estimator of this function is $\hat{\gamma}_f = \boldsymbol{a}'\boldsymbol{Y} + [\mathrm{vec}(\mathbf{A})]'(\boldsymbol{Y}\otimes\boldsymbol{Y})$, where $\boldsymbol{a}$ and $\mathrm{vec}(\mathbf{A})$ fulfil the conditions $\mathbf{X}'\boldsymbol{a} = \boldsymbol{0}$, $(\mathbf{X}'\otimes\mathbf{X}')\mathrm{vec}(\mathbf{A}) = \boldsymbol{0}$, $\tilde{\mathbf{V}}'\mathrm{vec}(\mathbf{A}) = \boldsymbol{f}$. As $\boldsymbol{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ the estimate is $\boldsymbol{\varepsilon}'\boldsymbol{k}_{\mathbf{X}'} + (\boldsymbol{\varepsilon}'\otimes\boldsymbol{\varepsilon}')\mathrm{vec}(\mathbf{A}) + [(\boldsymbol{\beta}'\mathbf{X}')\otimes\boldsymbol{\varepsilon}' + \boldsymbol{\varepsilon}'\otimes(\boldsymbol{\beta}'\mathbf{X}')]\mathrm{vec}(\mathbf{A})$, where $\boldsymbol{k}_{\mathbf{X}'} \in \mathrm{Ker}(\mathbf{X}') = \{\boldsymbol{u}\colon \boldsymbol{u} \in \mathcal{R}^n, \mathbf{X}'\boldsymbol{u} = \boldsymbol{0}\}$ and thus it contains the member $[(\boldsymbol{\beta}'\mathbf{X}')\otimes\boldsymbol{\varepsilon}' + \boldsymbol{\varepsilon}'\otimes(\boldsymbol{\beta}'\mathbf{X}')]\mathrm{vec}(\mathbf{A})$ that need not be a zero and depends on the parameter $\boldsymbol{\beta}$. In order to remove this dependence, the condition $\mathbf{A}\mathbf{X} = \mathbf{0}$ (in the case of the symmetric matrix $\mathbf{A}$) is used instead of the condition $\mathbf{X}'\mathbf{A}\mathbf{X} = \mathbf{0}$;

$$\mathbf{A}\mathbf{X} = \mathbf{0} \Leftrightarrow (\mathbf{X}'\otimes\mathbf{I})\mathrm{vec}(\mathbf{A}) = \boldsymbol{0} \Leftrightarrow (\mathrm{cC})(\mathbf{X}'\otimes\mathbf{I})\mathrm{vech}(\mathbf{A}) = \boldsymbol{0}.$$

The estimator $\boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$, $\mathbf{A} = \mathbf{A}'$, fulfilling the conditions

$$\mathbf{X}'\boldsymbol{a} = \boldsymbol{c},\quad (\mathbf{X}'\otimes\mathbf{I})\mathrm{vec}(\mathbf{A}) = \boldsymbol{0},\quad \tilde{\mathbf{V}}'\mathrm{vec}(\mathbf{A}) = \boldsymbol{f}$$

is called an invariant estimator. It is clear that the invariant estimator is an unbiased one. (For details see [1] and [3].)

Lemma 2.3. *An invariant estimator of the function* $\gamma(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{c}'\boldsymbol{\beta} + \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\beta} \in \mathcal{R}^k$, $\boldsymbol{\vartheta} \in \vartheta^*$, *exists iff* $\boldsymbol{c} \in \mathcal{M}(\mathbf{X}')$ *and* $\boldsymbol{f} \in \mathcal{M}(\mathbf{K}^{(I)})$, *where* $\{\mathbf{K}^{(I)}\}_{i,j} = \mathrm{Tr}(\mathbf{MV}_i\mathbf{MV}_j) = \{\tilde{\mathbf{V}}'(\mathbf{M}\otimes\mathbf{M})\tilde{\mathbf{V}}\}_{i,j}$, $i, j = 1, \ldots, p$, $\mathbf{M} = \mathbf{I} - \mathbf{P}$.

Proof. See [3, p. 9].

3. Locally best estimators

The following denotations will be used:

$$\mathrm{Var}(\mathbf{Z}_i) = \mathrm{E}\{[\mathbf{Z}_i - \mathrm{E}(\mathbf{Z}_i)][\mathbf{Z}_i - \mathrm{E}(\mathbf{Z}_i)]'\}, \; i = 1, 2,$$
$$\mathrm{cov}(\mathbf{Z}_i, \mathbf{Z}_j) = \mathrm{E}\{[\mathbf{Z}_i - \mathrm{E}(\mathbf{Z}_i)][\mathbf{Z}_j - \mathrm{E}(\mathbf{Z}_j)]'\}, i, j = 1, 2, i \neq j.$$

Lemma 3.1. *For random vectors* $\boldsymbol{Y}$, $\boldsymbol{Y}\otimes\boldsymbol{Y}$, $(\mathrm{cR})(\boldsymbol{Y}\otimes\boldsymbol{Y})$ *the following relationships are true:*

(a) $$\mathrm{cov}\{\boldsymbol{Y}, [(\mathrm{cR})(\boldsymbol{Y}\otimes\boldsymbol{Y})]\} = (\mathrm{cC})\{\mathrm{cov}[\boldsymbol{Y}, (\boldsymbol{Y}\otimes\boldsymbol{Y})]\},$$

(b) $$\mathrm{Var}[(\mathrm{cR})(\boldsymbol{Y}\otimes\boldsymbol{Y})] = (\mathrm{cC})(\mathrm{cR})[\mathrm{Var}(\boldsymbol{Y}\otimes\boldsymbol{Y})] = (\mathrm{cR})(\mathrm{cC})[\mathrm{Var}(\boldsymbol{Y}\otimes\boldsymbol{Y})].$$

Proof. (a) Follows from $[(\mathrm{cR})(\boldsymbol{Y}\otimes\boldsymbol{Y})]' = (\mathrm{cC})[\boldsymbol{Y}\otimes\boldsymbol{Y})']$ and from the bilinearity of $\mathrm{cov}(\cdot, \cdot\cdot)$.

(b) Follows from the relationships:

$$\mathrm{Var}\{[\mathrm{vec}(\mathbf{A})]'(\boldsymbol{Y}\otimes\boldsymbol{Y})\} = \mathrm{Var}\{[\mathrm{vech}(\mathbf{A})]'(\mathrm{cR})(\boldsymbol{Y}\otimes\boldsymbol{Y})\};$$
$$\mathrm{Var}\{[\mathrm{vec}(\mathbf{A})]'(\boldsymbol{Y}\otimes\boldsymbol{Y})\} = [\mathrm{vec}(\mathbf{A})]'\mathrm{Var}(\boldsymbol{Y}\otimes\boldsymbol{Y})\mathrm{vec}(\mathbf{A}) =$$
$$= [\mathrm{vech}(\mathbf{A})]'(\mathrm{cR})[\mathrm{Var}(\boldsymbol{Y}\otimes\boldsymbol{Y})]\mathrm{vec}(\mathbf{A}) =$$
$$= [\mathrm{vech}(\mathbf{A})]'(\mathrm{cC})(\mathrm{cR})\mathrm{Var}(\boldsymbol{Y}\otimes\boldsymbol{Y})\mathrm{vech}(\mathbf{A})$$

and

$$\mathrm{Var}\{[\mathrm{vech}(\mathbf{A})]'(\mathrm{cR})(\boldsymbol{Y}\otimes\boldsymbol{Y})\} =$$
$$= [\mathrm{vech}(\mathbf{A})]'\mathrm{Var}[(\mathrm{cR})(\boldsymbol{Y}\otimes\boldsymbol{Y})]\mathrm{vech}(\mathbf{A}).$$

The assumption $\mathbf{A} = \mathbf{A}'$ is used. The commutation of (cR) and (cC) is obvious.

Lemma 3.2. *Under the assumption from the Introduction the following relationships are valid:*

(a) $$\mathrm{cov}[\boldsymbol{Y}, (\boldsymbol{Y}\otimes\boldsymbol{Y})] = \varphi' + (\boldsymbol{\beta}'\mathbf{X}')\otimes\Sigma + \Sigma\otimes(\boldsymbol{\beta}'\mathbf{X}'),$$

(b) $$\mathrm{Var}(\boldsymbol{Y}\otimes\boldsymbol{Y}) = \psi + [(\mathbf{X}\boldsymbol{\beta})\otimes\mathbf{I}]\varphi' + \varphi[(\boldsymbol{\beta}'\mathbf{X}')\otimes\mathbf{I}] + [\mathbf{I}\otimes(\mathbf{X}\boldsymbol{\beta})]\varphi' +$$
$$+ \varphi[\mathbf{I}\otimes(\boldsymbol{\beta}'\mathbf{X}')] + (\mathbf{X}\boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{X}')\otimes\Sigma + \Sigma\otimes(\mathbf{X}\boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{X}') +$$
$$+ [\mathbf{I}\otimes(\mathbf{X}\boldsymbol{\beta})]\Sigma[(\boldsymbol{\beta}'\mathbf{X}')\otimes\mathbf{I}] + [(\mathbf{X}\boldsymbol{\beta})\otimes\mathbf{I}]\Sigma[\mathbf{I}\otimes(\boldsymbol{\beta}'\mathbf{X}')] - \mathrm{vec}(\Sigma)[\mathrm{vec}(\Sigma)]'.$$

Proof. From the definition of $\mathrm{cov}(\cdot, \cdot\cdot)$ it follows that

$$\mathrm{cov}(\boldsymbol{Y}, \boldsymbol{Y}\otimes\boldsymbol{Y}) = E\{(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta})[\boldsymbol{Y}\otimes\boldsymbol{Y} - E(\boldsymbol{Y}\otimes\boldsymbol{Y})]'\} =$$
$$= E\{\boldsymbol{\varepsilon}[(\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon})\otimes(\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}) - (\mathbf{X}\boldsymbol{\beta})\otimes(\mathbf{X}\boldsymbol{\beta}) - \mathrm{vec}(\Sigma)]'\} =$$
$$= E\{\boldsymbol{\varepsilon}[\boldsymbol{\varepsilon}'\otimes\boldsymbol{\varepsilon}' + \boldsymbol{\varepsilon}'\otimes(\boldsymbol{\beta}'\mathbf{X}') + (\boldsymbol{\beta}'\mathbf{X}')\otimes\boldsymbol{\varepsilon}' - [\mathrm{vec}(\Sigma)]']\},$$

where

$$E[\boldsymbol{\varepsilon}(\boldsymbol{\varepsilon}'\otimes\boldsymbol{\varepsilon}')]=E[(1\otimes\boldsymbol{\varepsilon})(\boldsymbol{\varepsilon}'\otimes\boldsymbol{\varepsilon}')]=E[\boldsymbol{\varepsilon}'\otimes(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')]=\varphi'$$
$$E\{\boldsymbol{\varepsilon}[\boldsymbol{\varepsilon}'\otimes(\boldsymbol{\beta}'\mathbf{X}')]\}=E\{(\boldsymbol{\varepsilon}\otimes 1)[\boldsymbol{\varepsilon}'\otimes(\boldsymbol{\beta}'\mathbf{X}')]\}=$$
$$=[E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')]\otimes(\boldsymbol{\beta}'\mathbf{X}')=\Sigma\otimes(\boldsymbol{\beta}'\mathbf{X}'),$$
$$E\{\boldsymbol{\varepsilon}[(\boldsymbol{\beta}'\mathbf{X}')\otimes\boldsymbol{\varepsilon}']\}=E\{(1\otimes\boldsymbol{\varepsilon})[(\boldsymbol{\beta}'\mathbf{X}')\otimes\boldsymbol{\varepsilon}']\}=$$
$$=(\boldsymbol{\beta}'\mathbf{X}')\otimes[E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')]=(\boldsymbol{\beta}'\mathbf{X}')\otimes\Sigma,$$
$$E\{\boldsymbol{\varepsilon}[\mathrm{vec}(\Sigma)]'\}=\mathbf{0}.$$

If $\boldsymbol{Y}\otimes\boldsymbol{Y}-E(\boldsymbol{Y}\otimes\boldsymbol{Y})=\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}+(\mathbf{X}\boldsymbol{\beta})\otimes\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}\otimes(\mathbf{X}\boldsymbol{\beta})-\mathrm{vec}(\Sigma)$ is taken into account, then

$$\mathrm{Var}(\boldsymbol{Y}\otimes\boldsymbol{Y})=E\{(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\otimes(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')+[(\mathbf{X}\boldsymbol{\beta})\boldsymbol{\varepsilon}']\otimes(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')+$$
$$+(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\otimes[(\mathbf{X}\boldsymbol{\beta})\boldsymbol{\varepsilon}']-\mathrm{vec}(\Sigma)(\boldsymbol{\varepsilon}'\otimes\boldsymbol{\varepsilon}')+[\boldsymbol{\varepsilon}(\boldsymbol{\beta}'\mathbf{X}')]\otimes$$
$$\otimes(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')+(\mathbf{X}\boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{X}')\otimes(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')+[\boldsymbol{\varepsilon}(\boldsymbol{\beta}'\mathbf{X}')]\otimes[(\mathbf{X}\boldsymbol{\beta})\boldsymbol{\varepsilon}']-$$
$$-\mathrm{vec}(\Sigma)[(\boldsymbol{\beta}'\mathbf{X}')\otimes\boldsymbol{\varepsilon}']+(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\otimes[\boldsymbol{\varepsilon}(\boldsymbol{\beta}'\mathbf{X}')]+[(\mathbf{X}\boldsymbol{\beta})\boldsymbol{\varepsilon}']\otimes$$
$$\otimes[\boldsymbol{\varepsilon}(\boldsymbol{\beta}'\mathbf{X}')]+(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\otimes(\mathbf{X}\boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{X}')-\mathrm{vec}(\Sigma)[\boldsymbol{\varepsilon}'\otimes(\boldsymbol{\beta}'\mathbf{X}')]-$$
$$-(\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon})[\mathrm{vec}(\Sigma)]'-[(\mathbf{X}\boldsymbol{\beta})\otimes\boldsymbol{\varepsilon}][\mathrm{vec}(\Sigma)]'-$$
$$-[\boldsymbol{\varepsilon}\otimes(\mathbf{X}\boldsymbol{\beta})][\mathrm{vec}(\Sigma)]'+\mathrm{vec}(\Sigma)[\mathrm{vec}(\Sigma)]'\},$$

where

$$E[(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\otimes(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')]=\psi,$$
$$E\{[(\mathbf{X}\boldsymbol{\beta})\boldsymbol{\varepsilon}']\otimes(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\}=[(\mathbf{X}\boldsymbol{\beta})\otimes\mathbf{I}]E[\boldsymbol{\varepsilon}'\otimes(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')]=[(\mathbf{X}\boldsymbol{\beta})\otimes\mathbf{I}]\varphi',$$
$$E\{(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\otimes[(\mathbf{X}\boldsymbol{\beta})\boldsymbol{\varepsilon}']\}=[\mathbf{I}\otimes(\mathbf{X}\boldsymbol{\beta})]E[(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\otimes\boldsymbol{\varepsilon}]=[\mathbf{I}\otimes(\mathbf{X}\boldsymbol{\beta})]E[(\boldsymbol{\varepsilon}\otimes 1)(\boldsymbol{\varepsilon}'\otimes\boldsymbol{\varepsilon}')]=$$
$$=[\mathbf{I}\otimes(\mathbf{X}\boldsymbol{\beta})]E[(1\otimes\boldsymbol{\varepsilon})(\boldsymbol{\varepsilon}'\otimes\boldsymbol{\varepsilon}')]=$$
$$=[\mathbf{I}\otimes(\mathbf{X}\boldsymbol{\beta})]E[\boldsymbol{\varepsilon}'\otimes(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')]=[\mathbf{I}\otimes(\mathbf{X}\boldsymbol{\beta})]\varphi',$$
$$-\mathrm{vec}(\Sigma)E(\boldsymbol{\varepsilon}'\otimes\boldsymbol{\varepsilon}')=-\mathrm{vec}(\Sigma)[\mathrm{vec}(\Sigma)]';$$
$$E\{[\boldsymbol{\varepsilon}(\boldsymbol{\beta}'\mathbf{X}')]\otimes(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\}=E[\boldsymbol{\varepsilon}\otimes(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')][(\boldsymbol{\beta}'\mathbf{X}')\otimes\mathbf{I}]=\varphi[(\boldsymbol{\beta}'\mathbf{X}')\otimes\mathbf{I}],$$
$$(\mathbf{X}\boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{X}')\otimes E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')=(\mathbf{X}\boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{X}')\otimes\Sigma,$$
$$E\{[\boldsymbol{\varepsilon}(\boldsymbol{\beta}'\mathbf{X}')]\otimes[(\mathbf{X}\boldsymbol{\beta})\boldsymbol{\varepsilon}']\}=[\mathbf{I}\otimes(\mathbf{X}\boldsymbol{\beta})]E(\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}')[(\boldsymbol{\beta}'\mathbf{X}')\otimes\mathbf{I}]=$$
$$=[\mathbf{I}\otimes(\mathbf{X}\boldsymbol{\beta})]\Sigma[(\boldsymbol{\beta}'\mathbf{X}')\otimes\mathbf{I}];$$
$$E\{(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\otimes[\boldsymbol{\varepsilon}(\boldsymbol{\beta}'\mathbf{X}')]\}=E[(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\otimes\boldsymbol{\varepsilon}][\mathbf{I}\otimes(\boldsymbol{\beta}'\mathbf{X}')]=$$
$$=E[(\boldsymbol{\varepsilon}\otimes\mathbf{I})(\boldsymbol{\varepsilon}'\otimes\boldsymbol{\varepsilon})][\mathbf{I}\otimes(\boldsymbol{\beta}'\mathbf{X}')]=$$
$$=E[(\boldsymbol{\varepsilon}\otimes\mathbf{I})\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'][\mathbf{I}\otimes(\boldsymbol{\beta}'\mathbf{X}')]=$$
$$=E[\boldsymbol{\varepsilon}\otimes(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')][\mathbf{I}\otimes(\boldsymbol{\beta}'\mathbf{X}')]=\varphi[\mathbf{I}\otimes(\boldsymbol{\beta}'\mathbf{X}')],$$
$$E\{[(\mathbf{X}\boldsymbol{\beta})\boldsymbol{\varepsilon}']\otimes[\boldsymbol{\varepsilon}(\boldsymbol{\beta}'\mathbf{X}')]\}=[(\mathbf{X}\boldsymbol{\beta})\otimes\mathbf{I}]E(\boldsymbol{\varepsilon}'\otimes\boldsymbol{\varepsilon})[\mathbf{I}\otimes(\boldsymbol{\beta}'\mathbf{X}')]=$$
$$=[(\mathbf{X}\boldsymbol{\beta})\otimes\mathbf{I}]\Sigma[\mathbf{I}\otimes(\boldsymbol{\beta}'\mathbf{X}')].$$

It is obvious how to finish the proof.

Further the denotation

$$\begin{pmatrix}\mathbf{X}'\otimes\mathbf{X}'\\ \tilde{\mathbf{V}}'\end{pmatrix}=\mathbf{Z}'$$

is used. Let

$$\mathscr{G}=\left\{\boldsymbol{a}'\boldsymbol{Y}+\boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}:\mathbf{X}'\boldsymbol{a}=\boldsymbol{c},\ \mathbf{Z}'\mathrm{vec}(\mathbf{A})=\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix}\right\}$$

(i.e. the class of all unbiased linear-quadratic estimators of the unbiasedly estimable function $\gamma(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{c}'\boldsymbol{\beta} + \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\beta} \in \mathcal{R}^k$, $\boldsymbol{\vartheta} \in \vartheta^*$, $\boldsymbol{c} \in \mathcal{M}(\mathbf{X}')$, $\boldsymbol{f} \in \mathcal{M}[\tilde{\mathbf{V}}'(\mathbf{I} - \mathbf{P} \otimes \mathbf{P})\tilde{\mathbf{V}}]$); an estimator $\hat{\gamma}_0 \in \mathcal{G}$ is called the locally best one at the point $(\boldsymbol{\beta}, \boldsymbol{\vartheta}, \varphi, \psi) \in \theta^*$ if

$$\mathrm{Var}_{\boldsymbol{\beta}, \boldsymbol{\vartheta}, \varphi, \psi}(\hat{\gamma}_0) = \min\{\mathrm{Var}_{\boldsymbol{\beta}, \boldsymbol{\vartheta}, \varphi, \psi}(\hat{\gamma}) : \hat{\gamma} \in \mathcal{G}\}.$$

Theorem 3.1. *The locally best unbiased estimator of the function* $\gamma(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{c}'\boldsymbol{\beta} + \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\beta} \in \mathcal{R}^k$, $\boldsymbol{\vartheta} \in \vartheta^*$, $\boldsymbol{c} \in \mathcal{M}(\mathbf{X}')$, $\boldsymbol{f} \in \mathcal{M}[\tilde{\mathbf{V}}'(\mathbf{I} - \mathbf{P} \otimes \mathbf{P})\tilde{\mathbf{V}}]$ *is* $\hat{\gamma} = \boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$, $\boldsymbol{a} \in \mathcal{R}^n$, $\mathbf{A} \in \mathcal{M}_{n,n}$, *where*

$$\boldsymbol{a} = (\mathbf{X}')_{m(*)}\boldsymbol{c} - [\mathbf{I} - (\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\mathbf{D}_{1,2}(\mathbf{Z}')_{m(**)}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix},$$

$$\mathrm{vec}(\mathbf{A}) = (\mathbf{Z}')_{m(**)}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix} - [\mathbf{I} - (\mathbf{Z}')_{m(\mathbf{D}_{2,2})}\mathbf{Z}']\mathbf{D}^-_{2,2}\mathbf{D}_{2,1}(\mathbf{X}')_{m(*)}\boldsymbol{c},$$

$$\mathbf{D}_{1,2} = \mathrm{cov}(\mathbf{Y}, \mathbf{Y} \otimes \mathbf{Y}) = \mathbf{D}'_{2,1},$$

$$\mathbf{D}_{2,2} = \mathrm{Var}(\mathbf{Y} \otimes \mathbf{Y}),$$

$$(*) = \Sigma - \mathbf{D}_{1,2}\mathbf{D}^-_{2,2}[\mathbf{D}_{2,2} - \mathbf{D}_{2,2}(\mathbf{Z}')_{m(\mathbf{D}_{2,2})}\mathbf{Z}']\mathbf{D}^-_{2,2}\mathbf{D}_{2,1},$$

$$(**) = \mathbf{D}_{2,2} - \mathbf{D}_{2,1}\Sigma^-[\Sigma - \Sigma(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\mathbf{D}_{1,2}.$$

Proof. First it is necessary to show that $(*)$ and $(**)$ are p.s.d. matrices and thus the g-inversion $(\mathbf{X}')^-_{m(*)}$ and $(\mathbf{Z}')^-_{m(**)}$ is correctly defined. With respect to the inclusion $\mathcal{M}(\mathbf{D}_{2,1}) \subset \mathcal{M}(\mathbf{D}_{2,2})$ (it is implied by the relation

$$P\{\boldsymbol{Y} \otimes \boldsymbol{Y} - E(\boldsymbol{Y} \otimes \boldsymbol{Y}) \in \mathcal{M}[\mathrm{Var}(\boldsymbol{Y} \otimes \boldsymbol{Y})]\} = 1)$$

it is valid $\mathbf{D}_{2,2}\mathbf{D}^-_{2,2}\mathbf{D}_{2,1} = \mathbf{D}_{2,1}$; thus $(*)$ can be rewritten as

$$(*) = \Sigma - \mathbf{D}_{1,2}\mathbf{D}^-_{2,2}\mathbf{D}_{2,1} + \mathbf{D}_{1,2}\mathbf{D}^-_{2,2}\mathbf{D}_{2,2}(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\mathbf{Z}'\mathbf{D}^-_{2,2}\mathbf{D}_{2,1} =$$
$$= \mathrm{Var}[\boldsymbol{Y} - \mathbf{D}_{1,2}\mathbf{D}^-_{2,2}(\boldsymbol{Y} \otimes \boldsymbol{Y})] + \mathbf{D}_{1,2}\mathbf{D}^-_{2,2}\mathrm{Var}\{\mathbf{Z}[(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}]'(\boldsymbol{Y} \otimes \boldsymbol{Y})\}\mathbf{D}^-_{2,2}\mathbf{D}_{2,1};$$

as $(*)$ is a sum of p.s.d. matrices, $(*)$ is a p.s.d. matrix.

The proof for $(**)$ is analogous.

The necessary and sufficient condition for the statistic

$$(\boldsymbol{a}', [\mathrm{vec}(\mathbf{A})]')\begin{pmatrix}\boldsymbol{Y}\\ \boldsymbol{Y} \otimes \boldsymbol{Y}\end{pmatrix}$$

to be locally the best unbiased estimator of the function $\gamma(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{c}'\boldsymbol{\beta} + \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\beta} \in \mathcal{R}^k$, $\boldsymbol{\vartheta} \in \vartheta^*$, is

$$\begin{pmatrix}\mathbf{X}', & \mathbf{0}\\ \mathbf{0}, & \mathbf{Z}'\end{pmatrix}\begin{pmatrix}\boldsymbol{a}\\ \mathrm{vec}(\mathbf{A})\end{pmatrix} = \begin{pmatrix}\boldsymbol{c}\\ \boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix} \quad \text{(unbiasedness)}$$

and the minimality of the variance

$$\mathrm{Var}\left\{[\boldsymbol{Y}', (\boldsymbol{Y}\otimes\boldsymbol{Y})'] \begin{pmatrix} \boldsymbol{a} \\ \mathrm{vec}(\mathbf{A}) \end{pmatrix}\right\} = (\boldsymbol{a}', [\mathrm{vec}(\mathbf{A})]')\mathbf{D} \begin{pmatrix} \boldsymbol{a} \\ \mathrm{vec}(\mathbf{A}) \end{pmatrix},$$

where

$$\mathbf{D} = \begin{pmatrix} \Sigma, & \mathbf{D}_{1,2} \\ \mathbf{D}_{2,1}, & \mathbf{D}_{2,2} \end{pmatrix}.$$

With respect to Lemma 1.2 it means that

$$\begin{pmatrix} \boldsymbol{a} \\ \mathrm{vec}(\mathbf{A}) \end{pmatrix} = \begin{pmatrix} \mathbf{X}', & \mathbf{0} \\ \mathbf{0}, & \mathbf{Z}' \end{pmatrix}^{-}_{m(\mathbf{D})} \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{0} \\ \boldsymbol{f} \end{pmatrix}.$$

If the denotation

$$\begin{pmatrix} \mathbf{X}', & \mathbf{0} \\ \mathbf{0}, & \mathbf{Z}' \end{pmatrix}^{-}_{m(\mathbf{D})} = \begin{pmatrix} \mathbf{C}_{1,1}, & \mathbf{C}_{1,2} \\ \mathbf{C}_{2,1}, & \mathbf{C}_{2,2} \end{pmatrix}$$

is used, then with respect to Lemma 1.2 two systems of conditions are obtained for the matrices $\mathbf{C}_{i,j}$, $i, j = 1, 2$:

$$\begin{aligned} &\mathbf{X}'\mathbf{C}_{1,1}\mathbf{X}' = \mathbf{X}', \qquad \mathbf{X}'\mathbf{C}_{1,2}\mathbf{Z}' = \mathbf{0}, \\ &\mathbf{Z}'\mathbf{C}_{2,1}\mathbf{X}' = \mathbf{0}, \qquad \mathbf{Z}'\mathbf{C}_{2,2}\mathbf{Z}' = \mathbf{Z}', \end{aligned} \tag{1}$$

$$\begin{aligned} &\left[(\Sigma, \mathbf{D}_{1,2}) \begin{pmatrix} \mathbf{C}_{1,1} \\ \mathbf{C}_{2,1} \end{pmatrix} \mathbf{X}'\right]' = (\Sigma, \mathbf{D}_{1,2}) \begin{pmatrix} \mathbf{C}_{1,1} \\ \mathbf{C}_{2,1} \end{pmatrix} \mathbf{X}', \\ &\left[(\Sigma, \mathbf{D}_{1,2}) \begin{pmatrix} \mathbf{C}_{1,2} \\ \mathbf{C}_{2,2} \end{pmatrix} \mathbf{Z}'\right]' = (\mathbf{D}_{2,1}, \mathbf{D}_{2,2}) \begin{pmatrix} \mathbf{C}_{1,1} \\ \mathbf{C}_{2,1} \end{pmatrix} \mathbf{X}', \\ &\left[(\mathbf{D}_{2,1}, \mathbf{D}_{2,2}) \begin{pmatrix} \mathbf{C}_{1,2} \\ \mathbf{C}_{2,2} \end{pmatrix} \mathbf{Z}'\right]' = (\mathbf{D}_{2,1}, \mathbf{D}_{2,2}) \begin{pmatrix} \mathbf{C}_{1,2} \\ \mathbf{C}_{2,2} \end{pmatrix} \mathbf{Z}'. \end{aligned} \tag{2}$$

With respect to Lemma 1.3 from (1) we obtain:

$$\begin{aligned} &\mathbf{C}_{1,1} = (\mathbf{X}')^{-}, \quad \mathbf{C}_{1,2} = \mathbf{T}_{1,2} - (\mathbf{X}')^{-}\mathbf{X}'\mathbf{T}_{1,2}\mathbf{Z}'(\mathbf{Z}')^{-}, \\ &\mathbf{C}_{2,1} = \mathbf{T}_{2,1} - (\mathbf{Z}')^{-}\mathbf{Z}'\mathbf{T}_{2,1}\mathbf{X}'(\mathbf{X}')^{-}, \quad \mathbf{C}_{2,2} = (\mathbf{Z}')^{-}. \end{aligned}$$

The problem is to determine the matrices $\mathbf{T}_{1,2}$ and $\mathbf{T}_{2,1}$ and to determine the proper types of g-inversion in order to fulfil the condition (2).

First the matrix $\mathbf{T}_{2,1}$ will be determined. When the function $\gamma_c(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{c}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathscr{R}^k$, is estimated, then the matrices $\mathbf{C}_{1,2}$ and $\mathbf{C}_{2,2}$ do not occur in the estimator $\hat{\gamma}_c = \boldsymbol{Y}'(\mathbf{X}')^{-}\boldsymbol{c} + (\boldsymbol{Y}\otimes\boldsymbol{Y})'[\mathbf{T}_{2,1} - (\mathbf{Z}')^{-}\mathbf{Z}'\mathbf{T}_{2,1}\mathbf{X}(\mathbf{X}')^{-}]$. Let $(\mathbf{X}')^{-}$ be fixed; the matrix $\mathbf{T}_{2,1}$ minimizes the dispersion $\mathrm{Var}(\hat{\gamma}_c)$ if it is a solution of the equation $\partial\mathrm{Var}(\hat{\gamma}_c)/\partial\mathbf{T}_{2,1} = \mathbf{0}$. When $(\mathbf{Z}')^{-}_{m(\mathbf{D}_{2,2})}$ is chosen for $(\mathbf{Z}')^{-}$, then the equation $\partial\mathrm{Var}(\hat{\gamma}_c)/\partial\mathbf{T}_{2,1} = \mathbf{0}$ is equivalent to

$$\{\mathbf{I} - [(\mathbf{Z}')^{-}_{m(\mathbf{D}_{2,2})}\mathbf{Z}']'\}\mathbf{D}_{2,2}\mathbf{T}_{2,1}\boldsymbol{c}\boldsymbol{c}' = -\{\mathbf{I} - [(\mathbf{Z}')^{-}_{m(\mathbf{D}_{2,2})}\mathbf{Z}']'\}\mathbf{D}_{2,1}(\mathbf{X}')^{-}\boldsymbol{c}\boldsymbol{c}'$$

(with respect to Lemma 1.3 it can be seen that the last equation can be solved).

As the matrix $\mathbf{T}_{2,1}$ cannot depend on the vector $\boldsymbol{c}$ the particular solution can be chosen in the form

$$\mathbf{T}_{2,1} = -\mathbf{D}_{2,2}^{-}\mathbf{D}_{2,1}(\mathbf{X}')^{-}$$

and thus

$$\mathbf{C}_{2,1}\boldsymbol{c} = -[\mathbf{I} - (\mathbf{Z}')_{m(\mathbf{D}_{2,2})}^{-}\mathbf{Z}']\mathbf{D}_{2,2}^{-}\mathbf{D}_{2,1}(\mathbf{X}')^{-}\boldsymbol{c},$$

because of $\mathbf{X}'(\mathbf{X}')^{-}\boldsymbol{c} = \boldsymbol{c} \in \mathcal{M}(\mathbf{X}')$. The type of the g-inversion $(\mathbf{X}')^{-}$ will be determined later.

We can proceed analogously in the case of the matrix $\mathbf{T}_{1,2}$. When $(\mathbf{Z}')^{-}$ is fixed and the matrix $(\mathbf{X}')_{m(\Sigma)}^{-}$ is chosen for $(\mathbf{X}')^{-}$ in the estimator

$$\boldsymbol{f}'\boldsymbol{\vartheta} = \hat{\gamma}_{\boldsymbol{f}} = (\boldsymbol{Y} \otimes \boldsymbol{Y})'(\mathbf{Z}')^{-} \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{f} \end{pmatrix} +$$

$$+ \boldsymbol{Y}'[\mathbf{T}_{1,2} - (\mathbf{X}')^{-}\mathbf{X}'\mathbf{T}_{1,2}\mathbf{Z}'(\mathbf{Z}')^{-}] \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{f} \end{pmatrix},$$

then the particular solution of the equation $\partial \mathrm{Var}(\hat{\gamma}_{\boldsymbol{f}})/\partial \mathbf{T}_{1,2} = \mathbf{0}$ is

$$\mathbf{T}_{1,2} = -\Sigma^{-}\mathbf{D}_{1,2}(\mathbf{Z}')^{-}.$$

For the linear member of the estimator of the function $\gamma_{\boldsymbol{f}}(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \vartheta^*$,

$$\mathbf{C}_{1,2} \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{f} \end{pmatrix} = -[\mathbf{I} - (\mathbf{X}')_{m(\Sigma)}^{-}\mathbf{X}']\Sigma^{-}\mathbf{D}_{1,2}(\mathbf{Z}')^{-} \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{f} \end{pmatrix}$$

$\left(\text{because of } \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{f} \end{pmatrix} \in \mathcal{M}[\mathbf{Z}'(\mathbf{I} - \mathbf{P} \otimes \mathbf{P})\mathbf{Z}] \subset \mathcal{M}(\mathbf{Z}')\right)$ holds.

It remains to determine the type of the g-inversion $(\mathbf{X}')^{-}$ in the expression for the matrix $\mathbf{C}_{2,1}$ (and thus in the expression for the matrix $\mathbf{C}_{1,1}$) and the type of the g-inversion $(\mathbf{Z}')^{-}$ in the expression for the matrix $\mathbf{C}_{1,2}$ (and thus in the expression for the matrix $\mathbf{C}_{2,2}$).

Consider the first condition from (2). Its right-hand side is

$$\text{r.h.s.} = \Sigma(\mathbf{X}')^{-}\mathbf{X}' - \mathbf{D}_{1,2}[\mathbf{I} - (\mathbf{Z}')_{m(\mathbf{D}_{2,2})}^{-}\mathbf{Z}']\mathbf{D}_{2,2}^{-}\mathbf{D}_{2,1}(\mathbf{X}')^{-}\mathbf{X}' =$$
$$= \{\Sigma - \mathbf{D}_{1,2}\mathbf{D}_{2,2}^{-}[\mathbf{D}_{2,2} - \mathbf{D}_{2,2}(\mathbf{Z}')_{m(\mathbf{D}_{2,2})}^{-}\mathbf{Z}']\mathbf{D}_{2,2}^{-}\mathbf{D}_{2,1}\}(\mathbf{X}')\ \mathbf{X}'.$$

If in the last expression the matrix $(\mathbf{X}')_{m(\bullet)}^{-}$ is substituted for the matrix $(\mathbf{X}')^{-}$, then with respect to Lemma 1.2 the r.h.s. is a symmetric matrix and thus the first condition from (2) is fulfilled. Analogously the choice $(\mathbf{Z}')_{m(\bullet\bullet)}^{-}$ for the matrix $(\mathbf{Z}')^{-}$ fulfils the third condition in (2).

It remains to verify the second condition of (2). Its left-hand side is

$$\text{l.h.s.} = [(\Sigma\mathbf{C}_{1,2} + \mathbf{D}_{1,2}\mathbf{C}_{2,2})\mathbf{Z}']' =$$
$$= \{-\Sigma[\mathbf{I} - (\mathbf{X}')_{m(\Sigma)}^{-}\mathbf{X}']\Sigma^{-}\mathbf{D}_{1,2}(\mathbf{Z}')_{m(\bullet\bullet)}^{-}\mathbf{Z}' + \mathbf{D}_{1,2}(\mathbf{Z}')_{m(\bullet\bullet)}^{-}\mathbf{Z}'\}' =$$
$$= \mathbf{Z}[(\mathbf{Z}')_{m(\bullet\bullet)}^{-}]'\mathbf{D}_{2,1}(\mathbf{X}')_{m(\Sigma)}^{-}\mathbf{X}'.$$

(Lemma 1.2 and $\mathcal{M}(\mathbf{D}_{1,2}) \subset \mathcal{M}(\Sigma) \Rightarrow \Sigma\Sigma^{-}\mathbf{D}_{1,2} = \mathbf{D}_{1,2}$ were utilized.)

The r.h.s. of the second condition from (2) is

$$\text{r.h.s.} = (\mathbf{D}_{2,1}, \mathbf{D}_{2,2}) \begin{pmatrix} \mathbf{C}_{1,1} \\ \mathbf{C}_{2,1} \end{pmatrix} \mathbf{X}' = \mathbf{D}_{2,1}(\mathbf{X}')^-_{m(\cdot)}\mathbf{X}' -$$

$$- \mathbf{D}_{2,2}[\mathbf{I} - (\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\mathbf{Z}']\mathbf{D}^-_{2,2}\mathbf{D}_{2,1}(\mathbf{X}')^-_{m(\cdot)}\mathbf{X}' = \mathbf{Z}[(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}]'\mathbf{D}_{2,1}(\mathbf{X}')^-_{m(\cdot)}\mathbf{X}'.$$

(Lemma 1.2 and $\mathcal{M}(\mathbf{D}_{2,1}) \subset \mathcal{M}(\mathbf{D}_{2,2}) \Rightarrow \mathbf{D}_{2,2}\mathbf{D}^-_{2,2}\mathbf{D}_{2,1} = \mathbf{D}_{2,1}$ were applicated.)

The equality between the r.h.s. and the l.h.s. can be proved in the following way. Let two random vectors

$$\boldsymbol{L}_1 = \boldsymbol{Y} \otimes \boldsymbol{Y} - \mathbf{D}_{2,1}\Sigma^-\boldsymbol{Y} + \mathbf{D}_{2,1}\Sigma^-\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma)}]'\boldsymbol{Y}$$

and

$$\boldsymbol{L}_2 = \boldsymbol{Y} - \mathbf{D}_{1,2}\mathbf{D}^-_{2,2}(\boldsymbol{Y} \otimes \boldsymbol{Y}) + \mathbf{D}_{1,2}\mathbf{D}^-_{2,2}\mathbf{Z}[(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}]'(\boldsymbol{Y} \otimes \boldsymbol{Y})$$

be considered. The expression for the $\text{cov}(\boldsymbol{L}_1, \boldsymbol{L}_2)$:

$$\text{cov}(\boldsymbol{L}_1, \boldsymbol{L}_2) = \mathbf{D}_{2,1} - \mathbf{D}_{2,1}\Sigma^-[\Sigma - \Sigma(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}'] -$$

$$- [\mathbf{D}_{2,2} - \mathbf{D}_{2,2}(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\mathbf{Z}']\mathbf{D}^-_{2,2}\mathbf{D}_{2,1} +$$

$$+ \mathbf{D}_{2,1}\Sigma^-[\Sigma - \Sigma(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\mathbf{D}_{1,2}\mathbf{D}^-_{2,2}[\mathbf{D}_{2,2} - \mathbf{D}_{2,2}(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\mathbf{Z}']\mathbf{D}^-_{2,2}\mathbf{D}_{2,1},$$

can be arranged into two following forms:

$$\text{cov}(\boldsymbol{L}_1, \boldsymbol{L}_2) = \mathbf{D}_{2,1}(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}' -$$

$$- \{\mathbf{I} - \mathbf{D}_{2,1}\Sigma^-[\Sigma - \Sigma(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\mathbf{D}_{1,2}\mathbf{D}^-_{2,2}\}[\mathbf{D}_{2,2} - \mathbf{D}_{2,2}(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\mathbf{Z}']\mathbf{D}^-_{2,2}\mathbf{D}_{2,1} =$$

$$= \mathbf{D}_{2,1}(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}' - (**)[\mathbf{I} - (\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\mathbf{Z}']\mathbf{D}^-_{2,2}\mathbf{D}_{2,1} = F_1;$$

$$\text{cov}(\boldsymbol{L}_1, \boldsymbol{L}_2) = -\mathbf{D}_{2,1}\Sigma^-[\Sigma - \Sigma(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}'] \cdot$$

$$\cdot \{\mathbf{I} - \Sigma^-\mathbf{D}_{1,2}\mathbf{D}^-_{2,2}[\mathbf{D}_{2,2} - \mathbf{D}_{2,2}(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\mathbf{Z}']\mathbf{D}^-_{2,2}\mathbf{D}_{2,1}\} + \mathbf{Z}[(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}]'\mathbf{D}_{2,1} =$$

$$= -\mathbf{D}_{2,1}\Sigma^-\{\mathbf{I} - \mathbf{X}[(\mathbf{X}')^-_{m(\Sigma)}]'\}(*) + \mathbf{Z}[(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}]'\mathbf{D}_{2,1} = F_2.$$

(The relationships

$$\mathbf{D}_{2,2}(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\mathbf{Z}'\mathbf{D}^-_{2,2}\mathbf{D}_{2,1} =$$

$$= \mathbf{Z}[(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}]'\mathbf{D}_{2,2}\mathbf{D}^-_{2,2}\mathbf{D}_{2,1} = \mathbf{Z}[(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}]'\mathbf{D}_{2,1},$$

$$\mathbf{D}_{2,1}\Sigma^-\Sigma(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}'\Sigma^-\mathbf{D}_{1,2} =$$

$$= \mathbf{D}_{2,1}\Sigma^-\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma)}]'\Sigma\Sigma^-\mathbf{D}_{1,2} = \mathbf{D}_{2,1}\mathbf{X}\Sigma^-[(\mathbf{X}')^-_{m(\Sigma)}]'\mathbf{D}_{1,2},$$

etc. resulting from Lemma 1.2 and from $\mathcal{M}(\mathbf{D}_{1,2}) \subset \mathcal{M}(\Sigma)$, $\mathcal{M}(\mathbf{D}_{2,1}) \subset \mathcal{M}(\mathbf{D}_{2,2})$ were utilized.)

Further

$$\mathbf{Z}[(\mathbf{Z}')^-_{m(\cdot\cdot)}]'F_1(\mathbf{X}')^-_{m(\cdot)}\mathbf{X}' = \mathbf{Z}[(\mathbf{Z}')^-_{m(\cdot\cdot)}]'\mathbf{D}_{2,1}(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}'(\mathbf{X}')^-_{m(\cdot)}\mathbf{X}' -$$

$$- (**)(\mathbf{Z}')^-_{m(\cdot\cdot)}\mathbf{Z}'[\mathbf{I} - (\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\mathbf{Z}']\mathbf{D}^-_{2,2}\mathbf{D}_{2,1} = \text{l.h.s.}$$

because of

$$\mathbf{X}'(\mathbf{X}')^-_{m(\cdot)}\mathbf{X}' = \mathbf{X}', \quad \mathbf{Z}[(\mathbf{Z}')^-_{m(\cdot\cdot)}]'(**) = (**)(\mathbf{Z}')^-_{m(\cdot\cdot)}\mathbf{Z}'$$

and

$$(\mathbf{Z}')^-_{m(\cdot\cdot)}\mathbf{Z}'(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\mathbf{Z}' = (\mathbf{Z}')^-_{m(\cdot\cdot)}\mathbf{Z}'.$$

Analogically

$$\mathbf{Z}[(\mathbf{Z}')^-_{m(**)}]'F_2(\mathbf{X}')^-_{m(*)}\mathbf{X}' = \mathbf{Z}[(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}]'\mathbf{D}_{2,1}(\mathbf{X}')^-_{m(*)}\mathbf{X}' = \text{r.h.s.}$$

Lemma 3.3. *The class of all unbiased quadratic estimators of the zero function is*

$$\mathcal{U}_0 = \{\tau_0\colon \tau_0 = \boldsymbol{Y}'[\mathbf{I} - (\mathbf{X}')^-\mathbf{X}']\boldsymbol{u} + (\boldsymbol{Y}\otimes\boldsymbol{Y})'[\mathbf{I} - (\mathbf{Z}')^-\mathbf{Z}']\boldsymbol{t},\ \boldsymbol{u}\in\mathcal{R}^n,\ \boldsymbol{t}\in\mathcal{R}^{n^2}\}.$$

Proof. It is analogous to the proof of Lemma 2.2 considering the relationships $\operatorname{Ker}(\mathbf{X}') = \{\boldsymbol{a}\colon \mathbf{X}'\boldsymbol{a} = \boldsymbol{0}\} = \{[\mathbf{I} - (\mathbf{X}')^-\mathbf{X}']\boldsymbol{u}\colon \boldsymbol{u}\in\mathcal{R}^n\}$ and $\operatorname{Ker}(\mathbf{Z}') = \{[\mathbf{I} - (\mathbf{Z}')^-\mathbf{Z}']\boldsymbol{t}\colon \boldsymbol{t}\in\mathcal{R}^{n^2}\}$.

Note 3.1. Theorem 3.1 can be proved with the help of the relation

$$\operatorname{cov}(\hat{\gamma}, \tau_0) = 0, \quad \tau_0\in\mathcal{U}_0$$

(e.g. [2, p. 55]), which expresses the necessary and sufficient condition for the estimator $\hat{\gamma}$ to be the locally best. Thus it is sufficient to verify

$$\boldsymbol{a}'\Sigma\boldsymbol{b} + \boldsymbol{a}'\mathbf{D}_{1,2}\operatorname{vec}(\mathbf{B}) + [\operatorname{vec}(\mathbf{A})]'\mathbf{D}_{2,1}\boldsymbol{b} + [\operatorname{vec}(\mathbf{A})]'\mathbf{D}_{2,2}\operatorname{vec}(\mathbf{B}) = 0.$$

It is advantageous to choose the zero estimator in the form

$$\tau_0 = \boldsymbol{b}'\boldsymbol{Y} + [\operatorname{vec}(\mathbf{B})]'(\boldsymbol{Y}\otimes\boldsymbol{Y}) = \boldsymbol{Y}'[\mathbf{I} - (\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\boldsymbol{u} + $$
$$+ (\boldsymbol{Y}\otimes\boldsymbol{Y})'[\mathbf{I} - (\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\mathbf{Z}']\boldsymbol{t};\quad \boldsymbol{u}\in\mathcal{R}^n,\ \boldsymbol{t}\in\mathcal{R}^{n^2}.$$

In the following the denotations

$$\mathbf{N}_{1,2} = (\mathrm{cC})(\mathbf{D}_{1,2}), \quad \mathbf{N}_{2,2} = (\mathrm{cC})(\mathrm{cR})(\mathbf{D}_{2,2})$$

are used.

Corollary 3.1. *For the locally best estimator of the function* $\gamma(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{c}'\boldsymbol{\beta} + \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\beta}\in\mathcal{R}^k$, $\boldsymbol{\vartheta}\in\vartheta^*$, $\boldsymbol{c}\in\mathcal{M}(\mathbf{X}')$, $\boldsymbol{f}\in\mathcal{M}[\tilde{\mathbf{V}}'(\mathbf{I} - \mathbf{P}\otimes\mathbf{P})\tilde{\mathbf{V}}]$, *written in the form* $\hat{\gamma} = \boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$, *where* $\mathbf{A}\in\mathcal{S}_n$, *there holds*

$$\boldsymbol{a} = (\mathbf{X}')^-_{m(*)}\boldsymbol{c} - [\mathbf{I} - (\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\mathbf{N}_{1,2}[(\mathrm{cC})(\mathbf{Z}')]^-_{m(**)}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix},$$

$$\operatorname{vech}(\mathbf{A}) = [(\mathrm{cC})(\mathbf{Z}')]^-_{m(**)}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix} - \{\mathbf{I} - [(\mathrm{cC})(\mathbf{Z}')]^-_{m(\mathbf{N}_{2,2})}(\mathrm{cC})(\mathbf{Z}')\}\mathbf{N}_{2,2}^-\mathbf{N}_{2,1}(\mathbf{X}')^-_{m(*)}\boldsymbol{c},$$

where

$$(*) = \Sigma - \mathbf{N}_{1,2}\mathbf{N}_{2,2}^-\{\mathbf{N}_{2,2} - \mathbf{N}_{2,2}[(\mathrm{cC})(\mathbf{Z}')]^-_{m(\mathbf{N}_{2,2})}(\mathrm{cC})(\mathbf{Z}')\}\mathbf{N}_{2,2}^-\mathbf{N}_{2,1},$$
$$(**) = \mathbf{N}_{2,2} - \mathbf{N}_{2,1}\Sigma^-[\Sigma - \Sigma(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\mathbf{N}_{1,2}.$$

If Lemma 3.1 is taken into account, then the statement can be proved analogously to the statement of Theorem 3.1. This form of estimator is more advantageous for numerical computing.

When estimating a function $\gamma_c(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{c}'\boldsymbol{\beta}$, $\boldsymbol{\beta}\in\mathcal{R}^k$, $\boldsymbol{c}\in\mathcal{M}(\mathbf{X}')$, we can restrict ourselves to linear estimators $\boldsymbol{a}'\boldsymbol{Y}$, $\mathbf{X}'\boldsymbol{a} = \boldsymbol{c}$, only. It is obvious that the best estimator from the larger class $\mathcal{U}_\gamma = \{\boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}\colon \mathbf{X}'\boldsymbol{a} = \boldsymbol{c}, \mathbf{Z}'\operatorname{vec}(\mathbf{A}) = \boldsymbol{0}\}$ cannot be worse than the linear one. Analogously when estimating a function $\gamma_f(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \vartheta^*$, $\boldsymbol{f} \in \mathcal{M}[\tilde{\mathbf{V}}'(\mathbf{I} - \mathbf{P} \otimes \mathbf{P})\tilde{\mathbf{V}}]$, we can restrict ourselves to estimators of the form $\boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$. In this case it is also obvious that the best estimator from the larger class $\mathcal{U}_Y$ cannot be worse than the estimator $\boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$. In spite of it the proof of the following theorem can help to gain a deeper insight into the fact.

Theorem 3.2. (a) *Let* $\gamma_c(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{c}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathcal{R}^k$, $\boldsymbol{c} \in \mathcal{M}(\mathbf{X}')$ *be an estimated function. For the estimator*

$$\widehat{\boldsymbol{c}'\boldsymbol{\beta}} = \boldsymbol{c}'[(\mathbf{X}')^-_{m(*)}]'\boldsymbol{Y} - \boldsymbol{c}'\{[\mathbf{I} - (\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\mathbf{Z}']\mathbf{D}^-_{2,2}\mathbf{D}_{2,1}(\mathbf{X}')^-_{m(*)}\}'(\boldsymbol{Y} \otimes \boldsymbol{Y})$$

from Theorem 3.1 it is true that

$$\mathrm{Var}(\widehat{\boldsymbol{c}'\boldsymbol{\beta}}) \leqslant \mathrm{Var}\{\boldsymbol{c}'[(\mathbf{X}')^-_{m(\Sigma)}]'\boldsymbol{Y}\};$$

(on the right-hand side the variance of the best linear unbiased estimator of the function γ_c *is).*

(b) *Let* $\gamma_f(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \vartheta^*$, $\boldsymbol{f} \in \mathcal{M}[\tilde{\mathbf{V}}'(\mathbf{I} - \mathbf{P} \otimes \mathbf{P})\tilde{\mathbf{V}}]$ *be an estimated function. For the estimator*

$$\widehat{\boldsymbol{f}'\boldsymbol{\vartheta}} = (\boldsymbol{0}', \boldsymbol{f}')[(\mathbf{Z}')^-_{m(**)}]'(\boldsymbol{Y} \otimes \boldsymbol{Y}) - (\boldsymbol{0}', \boldsymbol{f}')\{[\mathbf{I} - (\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\mathbf{D}_{1,2}(\mathbf{Z}')^-_{m(**)}\}'\boldsymbol{Y}$$

from Theorem 3.1 it is true that

$$\mathrm{Var}(\widehat{\boldsymbol{f}'\boldsymbol{\vartheta}}) \leqslant \mathrm{Var}\{[\mathrm{vec}(\mathbf{A}_*)]'(\boldsymbol{Y} \otimes \boldsymbol{Y})\},$$

where $\mathrm{vec}(\mathbf{A}_*) = (\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix}$, *which means that* $[\mathrm{vec}(\mathbf{A}_*)]'(\boldsymbol{Y} \otimes \boldsymbol{Y})$ *is the locally best estimator based on the vector* $\boldsymbol{Y} \otimes \boldsymbol{Y}$.

Proof. Consider $\mathbf{A} \in \mathcal{M}_{m,n}$, $\boldsymbol{y} \in \mathcal{M}(\mathbf{A})$, $\mathbf{R} \in \mathcal{S}_n$, $\mathbf{S} \in \mathcal{S}_n$, where the matrices $\mathbf{R}$, $\mathbf{S}$ and $\mathbf{R} - \mathbf{S}$ are p.s.d. Then with respect to Lemma 1.2 there holds:

$$\forall\{\boldsymbol{x}: \mathbf{A}\boldsymbol{x} = \boldsymbol{y}\}\ \|\mathbf{A}^-_{m(\mathbf{S})}\boldsymbol{y}\|_{\mathbf{S}} \leqslant \|\boldsymbol{x}\|_{\mathbf{S}} \leqslant \|\boldsymbol{x}\|_{\mathbf{R}} \Rightarrow$$
$$\Rightarrow \|\mathbf{A}^-_{m(\mathbf{S})}\boldsymbol{y}\|_{\mathbf{S}} \leqslant \min\{\|\boldsymbol{x}\|_{\mathbf{R}}: \mathbf{A}\boldsymbol{x} = \boldsymbol{y}\} = \|\mathbf{A}^-_{m(\mathbf{R})}\boldsymbol{y}\|_{\mathbf{R}}.$$

If we substitute $\boldsymbol{c}$, Σ, $(*)$ for $\boldsymbol{y}$, $\mathbf{S}$, $\mathbf{R}$, then

$$\mathbf{R} - \mathbf{S} = \mathbf{D}_{1,2}\mathbf{D}^-_{2,2}[\mathbf{D}_{2,2} - \mathbf{D}_{2,2}(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\mathbf{Z}']\mathbf{D}^-_{2,2}\mathbf{D}_{2,1}.$$

Further with respect to Lemma 1.2 the matrix $\mathbf{D}_{2,2} - \mathbf{D}_{2,2}(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\mathbf{Z}'$ can be rewritten as

$$\{\mathbf{I} - \mathbf{Z}[(\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}]'\}\mathbf{D}_{2,2}[\mathbf{I} - (\mathbf{Z}')^-_{m(\mathbf{D}_{2,2})}\mathbf{Z}'],$$

which is a p.s.d. matrix; thus $\mathbf{R} - \mathbf{S}$ is p.s.d. and $\boldsymbol{c}$, Σ, $(*)$ fulfil our assumptions. It means that

$$\mathrm{Var}(\widehat{\boldsymbol{c}'\boldsymbol{\beta}}) = \|(\mathbf{X}')^-_{m(*)}\boldsymbol{c}\|^2_{(*)} \leqslant \|(\mathbf{X}')^-_{m(\Sigma)}\boldsymbol{c}\|^2_{\Sigma} = \mathrm{Var}\{\boldsymbol{c}'[(\mathbf{X}')^-_{m(\Sigma)}]'\boldsymbol{Y}\}.$$

The fact that $\boldsymbol{c}'[(\mathbf{X}')^{-}_{m(\Sigma)}]'\boldsymbol{Y}$ is the best unbiased linear estimator of the function γ_c is proved, e.g., in [2, p. 140].

(b) Can be proved analogously.

Lemma 3.4. *For an invariant estimator*

$$\boldsymbol{a}'\boldsymbol{Y}+[\mathrm{vec}(\mathbf{A})]'(\boldsymbol{Y}\otimes\boldsymbol{Y}),\quad \mathbf{A}\in\mathcal{S}_n,$$
$$\mathbf{X}'\boldsymbol{a}=\boldsymbol{c}\in\mathcal{M}(\mathbf{X}'),\quad (\mathbf{X}'\otimes\mathbf{I})\mathrm{vec}(\mathbf{A})=\boldsymbol{0},$$
$$\tilde{\mathbf{V}}'\mathrm{vec}(\mathbf{A})=\boldsymbol{f}\in\mathcal{M}[\tilde{\mathbf{V}}'(\mathbf{M}\otimes\mathbf{M})\tilde{\mathbf{V}}]$$

of the function $\gamma(\boldsymbol{\beta},\boldsymbol{\vartheta})=\boldsymbol{c}'\boldsymbol{\beta}+\boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\beta}\in\mathcal{R}^k$, $\boldsymbol{\vartheta}\in\vartheta^*$, *there holds*

$$\mathrm{Var}\{\boldsymbol{a}'\boldsymbol{Y}+[\mathrm{vec}(\mathbf{A})]'(\boldsymbol{Y}\otimes\boldsymbol{Y})\}=$$
$$=(\boldsymbol{a}',[\mathrm{vec}(\mathbf{A})]')\begin{pmatrix}\Sigma, & \varphi'\\ \varphi, & \psi-\mathrm{vec}(\Sigma)[\mathrm{vec}(\Sigma)]'\end{pmatrix}\begin{pmatrix}\boldsymbol{a}\\ \mathrm{vec}(\mathbf{A})\end{pmatrix}.$$

Proof. If (2.2), Lemma 3.2 and relationships

$$\mathbf{AX}=\mathbf{0}\Leftrightarrow\mathbf{X}'\mathbf{A}=\mathbf{0}\Leftrightarrow$$
$$(\mathbf{X}'\otimes\mathbf{I})\mathrm{vec}(\mathbf{A})=\boldsymbol{0}\Leftrightarrow(\mathbf{I}\otimes\mathbf{X}')\mathrm{vec}(\mathbf{A})=\boldsymbol{0}$$

are taken into account, then

$$\mathrm{cov}(\boldsymbol{Y},\boldsymbol{Y}\otimes\boldsymbol{Y})\mathrm{vec}(\mathbf{A})=$$
$$=\varphi'\mathrm{vec}(\mathbf{A})+(\boldsymbol{\beta}'\otimes\Sigma)(\mathbf{X}'\otimes\mathbf{I})\mathrm{vec}(\mathbf{A})+(\Sigma\otimes\boldsymbol{\beta}')(\mathbf{I}\otimes\mathbf{X}')\mathrm{vec}(\mathbf{A})=\varphi'\mathrm{vec}(\mathbf{A}),$$

and analogously

$$[\mathrm{vec}(\mathbf{A})]'\mathrm{Var}(\boldsymbol{Y}\otimes\boldsymbol{Y})\mathrm{vec}(\mathbf{A})=$$
$$=[\mathrm{vec}(\mathbf{A})]'\psi\mathrm{vec}(\mathbf{A})-[\mathrm{vec}(\mathbf{A})]'\mathrm{vec}(\Sigma)[\mathrm{vec}(\Sigma)]'\mathrm{vec}(\mathbf{A}).$$

Lemma 3.5. *The matrix*

$$\begin{pmatrix}\Sigma, & \varphi'\\ \varphi, & \psi-\mathrm{vec}(\Sigma)[\mathrm{vec}(\Sigma)]'\end{pmatrix}$$

is p.s.d.

Proof. It can be easily proved that the mentioned matrix is $\mathrm{Var}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}\end{pmatrix}$, which is obviously p.s.d.

Further the denotations

$$\mathbf{D}^{(I)}_{2,2}=\psi-\mathrm{vec}(\Sigma)[\mathrm{vec}(\Sigma)]',$$
$$\mathbf{N}^{(I)}_{1,2}=(\mathrm{cC})(\varphi'),$$
$$\mathbf{N}^{(I)}_{2,2}=(\mathrm{cC})(\mathrm{cR})\mathbf{D}^{(I)}_{2,2},$$
$$\mathbf{Z}^{(I)'}=\begin{pmatrix}\mathbf{X}'\otimes\mathbf{I}\\ \tilde{\mathbf{V}}'\end{pmatrix}$$

will be used.

Lemma 3.6. *The class of all invariant estimators of the zero function is*

$$\mathcal{U}_0^{(I)} = \{\tau_0^{(I)} : \tau_0^{(I)} = \boldsymbol{Y}'[\mathbf{I} - (\mathbf{X}')^- \mathbf{X}']\boldsymbol{u} + (\boldsymbol{Y} \otimes \boldsymbol{Y})'[\mathbf{I} - (\mathbf{Z}^{(I)'})^- \mathbf{Z}^{(I)'}]\boldsymbol{t},$$
$$\boldsymbol{u} \in \mathcal{R}^n, \quad \boldsymbol{t} \in \mathcal{R}^{n^2}\}.$$

Proof. For $\tau_0^{(I)} \in \mathcal{U}_0^{(I)}$ the following has to be valiid:

$$\forall\{\boldsymbol{\beta} \in \mathcal{R}^k, \boldsymbol{\vartheta} \in \vartheta^*\} E_{\beta,\vartheta}(\tau_0^{(I)}) = 0 \,\&\, (\mathbf{X}' \otimes \mathbf{I})\text{vec}(\mathbf{A}) = \boldsymbol{0} \Leftrightarrow$$
$$\Leftrightarrow \mathbf{X}'\boldsymbol{a} = \boldsymbol{0} \,\&\, \mathbf{Z}^{(I)'}\text{vec}(\mathbf{A}) = \boldsymbol{0}.$$

Theorem 3.3. *The locally best invariant estimator of the function*

$$\gamma(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{c}'\boldsymbol{\beta} + \boldsymbol{f}'\boldsymbol{\vartheta}, \quad \boldsymbol{\beta} \in \mathcal{R}^k, \quad \boldsymbol{\vartheta} \in \vartheta^*, \quad \boldsymbol{c} \in \mathcal{M}(\mathbf{X}'),$$
$$\boldsymbol{f} \in \mathcal{M}[\tilde{\mathbf{V}}'(\mathbf{M} \otimes \mathbf{M})\tilde{\mathbf{V}}], \text{ is } \hat{\gamma}^{(I)} = \boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}, \quad \boldsymbol{a} \in \mathcal{R}^n, \quad \mathbf{A} \in \mathcal{S}_n,$$

where

$$\boldsymbol{a} = (\mathbf{X}')^-_{m(*)}\boldsymbol{c} - [\mathbf{I} - (\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\varphi(\mathbf{Z}^{(I)'})^-_{m(**)}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix},$$

$$\text{vec}(\mathbf{A}) = (\mathbf{Z}^{(I)'})^-_{m(**)}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix} - [\mathbf{I} - (\mathbf{Z}^{(I)'})^-_{m(\mathbf{D}_{2,2}^{(I)})}\mathbf{Z}^{(I)'}](\mathbf{D}_{2,2}^{(I)})^-\varphi(\mathbf{X}')^-_{m(*)}\boldsymbol{c},$$
$$(*) = \Sigma - \varphi'(\mathbf{D}_{2,2}^{(I)})^-[\mathbf{D}_{2,2}^{(I)} - \mathbf{D}_{2,2}^{(I)}(\mathbf{Z}^{(I)'})^-_{m(\mathbf{D}_{2,2}^{(I)})}\mathbf{Z}^{(I)'}](\mathbf{D}_{2,2}^{(I)})^-\varphi,$$
$$(**) = \mathbf{D}_{2,2}^{(I)} - \varphi\Sigma^-[\Sigma - (\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\varphi'.$$

Proof. The estimator $\tau_0^{(I)}$ from Lemma 3.6 in the form

$$\tau_0^{(I)} = \boldsymbol{Y}'[\mathbf{I} - (\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\boldsymbol{u} + (\boldsymbol{Y} \otimes \boldsymbol{Y})'[\mathbf{I} - (\mathbf{Z}^{(I)'})^-_{m(\mathbf{D}_{2,2}^{(I)})}\mathbf{Z}^{(I)'}]\boldsymbol{t},$$
$$\boldsymbol{u} \in \mathcal{R}^n, \quad \boldsymbol{t} \in \mathcal{R}^{n^2}$$

and Note 3.1 has to be utilized.

Corollary 3.2. *The following form of the locally best invariant estimator is more advantageous for numerical computations:*

$$\boldsymbol{a} = (\mathbf{X}')^-_{m(*)}\boldsymbol{c} - [\mathbf{I} - (\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\mathbf{N}_{1,2}^{(I)}[(\text{cC})(\mathbf{Z}^{(I)'})]^-_{m(**)}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix},$$

$$\text{vech}(\mathbf{A}) = [(\text{cC})(\mathbf{Z}^{(I)'})]^-_{m(**)}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix} -$$

$$- \{\mathbf{I} - [(\text{cC})(\mathbf{Z}^{(I)'})]^-_{m(\mathbf{N}_{2,2}^{(I)})}(\text{cC})(\mathbf{Z}^{(I)'})\}(\mathbf{N}_{2,2}^{(I)})^-\mathbf{N}_{2,1}^{(I)}(\mathbf{X}')^-_{m(*)}\boldsymbol{c},$$

$$(*) = \Sigma - \mathbf{N}_{1,2}^{(I)}(\mathbf{N}_{2,2}^{(I)})^-\{\mathbf{N}_{2,2}^{(I)} - \mathbf{N}_{2,2}^{(I)}[(\text{cC})(\mathbf{Z}^{(I)'})]^-_{m(\mathbf{N}_{2,2}^{(I)})}(\text{cC})(\mathbf{Z}^{(I)'})\}(\mathbf{N}_{2,2}^{(I)})^-\mathbf{N}_{2,1}^{(I)},$$

$$(**) = \mathbf{N}_{2,2}^{(I)} - \mathbf{N}_{2,1}^{(I)}\Sigma^-[\Sigma - \Sigma(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\mathbf{N}_{1,2}^{(I)}.$$

Note 3.2. An analogy of Theorem 3.2 can be proved for invariant estimators from Theorem 3.3.

Theorem 3.4. *If* $\varphi = \mathbf{0}$, *then the locally best invariant estimator of a function* $\gamma(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \boldsymbol{c}'\boldsymbol{\beta} + \boldsymbol{f}'\boldsymbol{\vartheta}$, $\boldsymbol{\beta} \in \mathcal{R}^k$, $\boldsymbol{\vartheta} \in \vartheta^*$, $\boldsymbol{c} \in \mathcal{M}(\mathbf{X}')$, $\boldsymbol{f} \in \mathcal{M}[\tilde{\mathbf{V}}'(\mathbf{M} \otimes \mathbf{M})\tilde{\mathbf{V}}]$ *is*

$$\hat{\gamma} = \boldsymbol{Y}'(\mathbf{X}')^-_{m(\Sigma)}\boldsymbol{c} + (\boldsymbol{Y}\otimes \boldsymbol{Y})'(\mathbf{Z}^{(I)'})^-_{m(\mathbf{D})}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix},$$

where

$$\mathbf{D} = \begin{pmatrix}\Sigma, & \mathbf{0}\\ \mathbf{0}, & \psi - \operatorname{vec}(\Sigma)[\operatorname{vec}(\Sigma)]'\end{pmatrix}.$$

Proof. From Lemma 3.6, Note 3.1, Lemma 1.2 and assumption $\varphi = \mathbf{0}$ it follows that

$$\forall\{\boldsymbol{u}\in\mathcal{R}^n,\ \boldsymbol{t}\in\mathcal{R}^{n^2}\}\boldsymbol{u}'\{\mathbf{I}-\mathbf{X}[(\mathbf{X}')^-]'\}\Sigma(\mathbf{X}')^-_{m(\Sigma)}\boldsymbol{c} +$$

$$+\boldsymbol{t}'\{\mathbf{I}-\mathbf{Z}^{(I)}[(\mathbf{Z}^{(I)'})^-]'\}\mathbf{D}(\mathbf{Z}^{(I)'})^-_{m(\mathbf{D})}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix} = \boldsymbol{0}$$

$\left(\text{because of } \boldsymbol{c}\in\mathcal{M}(\mathbf{X}') \text{ and } \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix}\in\mathcal{M}(\mathbf{Z}^{(I)'})\right)$.

Note 3.3. The vector $\boldsymbol{a}$ and the matrix $\mathbf{A}$ from Theorem 3.3 are the solution of the equations

(a) $$\mathbf{M}\Sigma\boldsymbol{a} + \mathbf{M}\varphi'\operatorname{vec}(\mathbf{A}) = \boldsymbol{0}$$

(b) $$(\mathbf{M}\otimes\mathbf{M})[\psi\operatorname{vec}(\mathbf{A}) + \varphi\boldsymbol{a}] = (\mathbf{M}\otimes\mathbf{M})\tilde{\mathbf{V}}\boldsymbol{\lambda}$$

derived by J. Kleffe (in [1, equations (4.4), (4.5)]) for the locally best invariant estimator. In (b) the vector λ has to exist.

Proof. If the expressions from Theorem 3.3 are substituted for the vector $\boldsymbol{a}$ and the matrix $\mathbf{A}$, then for (a) there holds:

$$\mathbf{M}\Sigma\boldsymbol{a} + \mathbf{M}\varphi'\operatorname{vec}(\mathbf{A}) = \mathbf{M}\Sigma(\mathbf{X}')^-_{m(*)}\boldsymbol{c} - \mathbf{M}\Sigma[\mathbf{I}-(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\cdot$$

$$\cdot\Sigma^-\varphi'(\mathbf{Z}^{(I)'})^-_{m(**)}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix} + \mathbf{M}\varphi'(\mathbf{Z}^{(I)'})^-_{m(**)}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix} -$$

$$-\mathbf{M}\varphi'[\mathbf{I}-(\mathbf{Z}^{(I)'})^-_{m(\mathbf{D}_{2,2}^{(I)})}\mathbf{Z}^{(I)'}](\mathbf{D}^{(I)}_{2,2})^-\varphi(\mathbf{X}')^-_{m(*)}\boldsymbol{c} =$$

$$= \mathbf{M}(*)(\mathbf{X}')^-_{m(*)}\boldsymbol{c} + \mathbf{M}\Sigma(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}'\Sigma^-\varphi'(\mathbf{Z}^{(I)'})^-_{m(**)}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix} = \boldsymbol{0}$$

because of

$$\mathbf{M}(*)(\mathbf{X}')^-_{m(*)}\boldsymbol{c} = \mathbf{M}(*)(\mathbf{X}')^-_{m(*)}\mathbf{X}'\boldsymbol{u} =$$
$$\mathbf{M}\mathbf{X}[(\mathbf{X}')^-_{m(*)}]'(*)\boldsymbol{u} = \boldsymbol{0} \quad (\mathbf{M}\mathbf{X}=\mathbf{0},\ \boldsymbol{c}=\mathbf{X}'\boldsymbol{u})$$

and

$$\mathbf{M}\Sigma(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}' = \mathbf{M}\mathbf{X}[(\mathbf{X}')^-_{m(\Sigma)}]'\Sigma = \mathbf{0}.$$

For (b) the following is valid:

$$(\mathbf{M}\otimes\mathbf{M})\left\{\psi\left\langle(\mathbf{Z}^{(I)'})^-_{m(**)}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{f}\end{pmatrix} - [\mathbf{I}-(\mathbf{Z}^{(I)'})^-_{m(\mathbf{D}_{2,2}^{(I)})}\mathbf{Z}^{(I)'}](\mathbf{D}^{(I)}_{2,2})^-\varphi(\mathbf{X}')^-_{m(*)}\boldsymbol{c}\right\rangle +\right.$$

$$+\varphi\left\langle(\mathbf{X}')^-_{m(*)}\mathbf{c}-[\mathbf{I}-(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\varphi(\mathbf{Z}^{(I)'})^-_{m(**)}\begin{pmatrix}\mathbf{0}\\ \mathbf{f}\end{pmatrix}\right\rangle\Bigg\}=$$

$$=(\mathbf{M}\otimes\mathbf{M})\Bigg\{\langle\psi-\varphi[\mathbf{I}-(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}']\Sigma^-\varphi'\rangle(\mathbf{Z}^{(I)'})^-_{m(**)}\begin{pmatrix}\mathbf{0}\\ f\end{pmatrix}+$$

$$+\langle\mathbf{I}-\{\mathbf{D}^{(I)}_{2,2}+\mathrm{vec}(\Sigma)[\mathrm{vec}(\Sigma)]'\}[\mathbf{I}-(\mathbf{Z}^{(I)'})^-_{m(\mathbf{D}_{2,2}^{(I)})}\mathbf{Z}^{(I)'}](\mathbf{D}^{(I)}_{2,2})^-\rangle\cdot$$

$$\cdot\varphi(\mathbf{X}')^-_{m(*)}\mathbf{c}\}=(\mathbf{M}\otimes\mathbf{M})\Bigg\{\langle(**)+\mathrm{vec}(\Sigma)[\mathrm{vec}(\Sigma)]'\rangle(\mathbf{Z}^{(I)'})^-_{m(**)}\begin{pmatrix}\mathbf{0}\\ \mathbf{f}\end{pmatrix}+$$

$$+\mathbf{D}^{(I)}_{2,2}(\mathbf{Z}^{(I)'})^-_{m(\mathbf{D}_{2,2}^{(I)})}\mathbf{Z}^{(I)'}(\mathbf{D}^{(I)}_{2,2})^-\varphi(\mathbf{X}')^-_{m(*)}\mathbf{c}-\tilde{\mathbf{V}}.$$

$$\cdot\langle\vartheta[\mathrm{vec}(\Sigma)]'[\mathbf{I}-(\mathbf{Z}^{(I)'})^-_{m(\mathbf{D}_{2,2}^{(I)})}\mathbf{Z}^{(I)'}](\mathbf{D}^{(I)}_{2,2})^-\varphi(\mathbf{X}')^-_{m(*)}\mathbf{c}\rangle\Bigg\}=$$

$$=(\mathbf{0},(\mathbf{M}\otimes\mathbf{M})\tilde{\mathbf{V}})[(\mathbf{Z}^{(I)'})^-_{m(**)}]'(**)\mathbf{t}+(\mathbf{M}\otimes\mathbf{M})\tilde{\mathbf{V}}\cdot$$

$$\cdot\left\langle\vartheta[\mathrm{vec}(\Sigma)]'(\mathbf{Z}^{(I)'})^-_{m(**)}\begin{pmatrix}\mathbf{0}\\ \mathbf{f}\end{pmatrix}\right\rangle+$$

$$+(\mathbf{0},(\mathbf{M}\otimes\mathbf{M})\tilde{\mathbf{V}})[(\mathbf{Z}^{(I)'})^-_{m(\Sigma)}]'\mathbf{D}^{(I)}_{2,2}(\mathbf{D}^{(I)}_{2,2})^-\varphi(\mathbf{X}')^-_{m(*)}\mathbf{c}-$$

$$-(\mathbf{M}\otimes\mathbf{M})\tilde{\mathbf{V}}\langle\vartheta[\mathrm{vec}(\Sigma)]'[\mathbf{I}-(\mathbf{Z}^{(I)'})^-_{m(\mathbf{D}_{2,2}^{(I)})}\mathbf{Z}^{(I)'}](\mathbf{D}^{(I)}_{2,2})^-\varphi(\mathbf{X}')^-_{m(*)}\mathbf{c}\rangle=\mathbf{v}.$$

$\left(\begin{pmatrix}\mathbf{0}\\ \mathbf{f}\end{pmatrix}=\mathbf{Z}^{(I)'}\mathbf{t}\right.$ was utilized.$\left.\right)$ It is clear that for the obtained expression $\mathbf{v}$ there exists a vector $\boldsymbol{\lambda}$ such that $(\mathbf{M}\otimes\mathbf{M})\tilde{\mathbf{V}}\boldsymbol{\lambda}$ is equal to the vector $\mathbf{v}$, which was to be proved.

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ЛОКАЛЬНО НАИЛУЧШИЕ КВАДРАТИЧНЫЕ ОЦЕНКИ

Lubomír Kubáček

Резюме

В линейной смешанной модели

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon},\ E(\boldsymbol{\varepsilon}) = \mathbf{0},\quad E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i \quad (\mathbf{X}, \mathbf{V}_1, \ldots, \mathbf{V}_p$$

известные матрицы) нужно определить оценку функции

$$\gamma(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \mathbf{c}'\boldsymbol{\beta} + \mathbf{f}'\boldsymbol{\vartheta},\quad \boldsymbol{\beta} \in \mathscr{R}^k$$

(k-размерное пространство Евклида), $\boldsymbol{\vartheta} \in \vartheta^* \subset \mathscr{R}^p$. Матрицы третьих и четвертых моментов

$$\varphi = E(\boldsymbol{\varepsilon} \otimes (\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')),\quad \psi = E[(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') \otimes (\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')]$$

даны. Несмещенные и инвариантные оценки предполагаются в форме

$$\hat{\gamma} = \mathbf{a}'\mathbf{y} + \mathbf{y}'\mathbf{A}\mathbf{Y}.$$

Приведенны явные выражения для вектора $\mathbf{a}$ и матрицы $\mathbf{A}$ таковы, что результирующие оценки являются в заданной точке $(\boldsymbol{\beta}, \boldsymbol{\vartheta}, \varphi, \psi)$ параметрического пространства наилучшими.