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A Note on Orthodox Additive Inverse Semirings

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Abstract

We show in an additive inverse regular semiring $(S, +, \cdot)$ with $E^{\bullet}(S)$ as the set of all multiplicative idempotents and $E^{+}(S)$ as the set of all additive idempotents, the following conditions are equivalent:
(i) For all $e, f \in E^{\bullet}(S)$, $ef \in E^{+}(S)$ implies $fe \in E^{+}(S)$.
(ii) $(S, \cdot)$ is orthodox.
(iii) $(S, \cdot)$ is a semilattice of groups.
This result generalizes the corresponding result of regular ring.

Key words: Additive inverse semirings, regular semirings, orthodox semirings.

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1 Introduction

A semiring $(S, +, \cdot)$ is a nonempty set S on which operations of addition, $+$, and multiplication, $\cdot$, have been defined such that the following conditions are satisfied:

(1) $(S, +)$ is a semigroup.
(2) $(S, \cdot)$ is a semigroup.
(3) Multiplication distributes over addition from either side.

A semiring $(S, +, \cdot)$ is called an additive inverse semiring if $(S, +)$ is an inverse semigroup, that is for each $a \in S$ there exists a unique element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$. Additive inverse semirings were first studied by Karvellas [4] in 1974. Karvellas [4] proved the following: (Karvellas (1974), Theorem 3(ii) and Theorem 7) Take any additive inverse semiring $(S, +, \cdot)$.

(i) For all $x, y \in S$, $(x \cdot y)' = x' \cdot y = x \cdot y'$ and $x' \cdot y' = x \cdot y$

(ii) If $a \in aS \cap Sa$ for all $a \in S$ then S is additively commutative.

A semiring $(S, +, \cdot)$ is called regular if for each $a \in S$ there exists $x \in S$ such that $axa = a$. In a regular semiring S, for any element $a \in S$, $V^{\bullet}(a) = \{x \in S : axa = a \text{ and } xax = x\}$. A regular semiring S contains element e such that $e \cdot e = e$. We denote the set of such elements by $E^{\bullet}(S)$. If in a regular semiring S, $E^{\bullet}(S)$ is a subsemigroup of the semigroup of $(S, \cdot)$, then the semiring S is called an orthodox semiring.

Chaptal [1] proved the following result in 1966.

Result 1.1 For a ring $(R, +, \cdot)$ the following conditions are equivalent.

(i) $(R, \cdot)$ is a union of groups.

(ii) $(R, \cdot)$ is an inverse semigroup.

(iii) $(R, \cdot)$ is a semilattice of groups.

Latter J. Zeleznekow [5] proved the following result.

Result 1.2 In a regular ring $(R, +, \cdot)$ the following conditions are equivalent.

(i) $(R, \cdot)$ is orthodox.

(ii) $(R, \cdot)$ is a union of groups.

(iii) $(R, \cdot)$ is an inverse semigroup.

(iv) $(R, \cdot)$ is a semilattice of groups.

These results do not hold in arbitrary semiring [see Example 2.1.]. The aim of this paper is to generalize these results in an additive inverse semiring with some conditions. For notations and terminologies not given in this note, the reader is referred to the monograph of Golan [2] and Howie [3].

2 Orthodox additive inverse semiring

An additive inverse semiring S is called orthodox if $(S, \cdot)$ is an orthodox semigroup.

Example 2.1 [5] Let S be the set of all binary relations on a two element set. Under the operations of union and composition of binary relations, S becomes a semiring in which $(S, \cdot)$ is regular but neither orthodox nor a union of groups.

Example 2.2 Let $(S,+)$ be a semilattice with more than one element. On S, define the multiplication, $\cdot$, by $a \cdot b = a$ for all $a, b \in S$. Then $(S,+,\cdot)$ is a semiring such that $(S,+)$ is an inverse semigroup, $(S,\cdot)$ is orthodox. Hence this semiring is an orthodox additive inverse semiring. In this semiring we find that $(S,\cdot)$ is not an inverse semigroup.

¿From the above example we find that J. Zeleznekow's result is not true in an orthodox additive inverse semiring. Let S be an additive inverse semiring. We say that S satisfies conditions (A) and (B) if for all $a, b \in S$

(A) $a(b+b') = (b+b')a$.

(B) $a + a(b+b') = a$.

Clearly rings, distributive lattices and direct products of distributive lattice and ring are natural examples of such additive inverse semiring. We consider the following example.

Example 2.3 Let $S = \{0, a, b\}$. Define addition and multiplication on S by the following Cayley tables:

$+$	0	a	b
0	0	a	b
a	a	0	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

It is easy to see that $(S,+,\cdot)$ is a semiring such that $(S,+)$ is an additive inverse semiring with conditions (A) and (B).

In the remaining part of this section we assume that S denotes an additive commutative and additive inverse semiring satisfying conditions (A) and (B). Also we assume that $E^+(S) = \{a \in S : a + a = a\}$. Note that $E^+(S)$ is an ideal of S.

We now prove the following Lemma.

Lemma 2.4 *Let $a, b \in S$ be such that $a + b' \in E^+(S)$ and $a + a' = b + b'$. Then $a = b$.*

Proof Since $a + b' \in E^+(S)$ so we have

$$a + b' = (a + b') + (a + b')' = a + b' + b + a' = a + a' + b + b' = b + b'.$$

This leads to, $a + b' + b = b + b' + b$, i.e., $a + a' + a = b$. Hence $a = b$. $\square$

Nest we prove the following important lemma.

Lemma 2.5 *If the semiring S is multiplicatively regular then the following conditions are equivalent.*

(i) For all $e, f \in E^\bullet(S)$, $ef \in E^+(S)$ implies $fe \in E^+(S)$.

(ii) For all $e \in E^{\bullet}(S)$, for all $x \in S$, $ex \in E^{+}(S)$ implies $xe \in E^{+}(S)$.

(iii) For all $n \in \mathbb{N}$, for all $x \in S$, $x^n \in E^{+}(S)$ implies $x \in E^{+}(S)$.

(iv) For all $x \in S$, $x^2 \in E^{+}(S)$ implies $x \in E^{+}(S)$.

(v) For all $x, y \in S$, $xy \in E^{+}(S)$ implies $yx \in E^{+}(S)$.

Furthermore, each is implied by

(vi) $(S, \cdot)$ is orthodox.

Proof (i) $\Rightarrow$ (ii): Let $e \in E^{\bullet}(S)$ and $x \in S$ be such that $ex \in E^{+}(S)$. Then $ex = ex + (ex)' = ex + ex'$. Now,

$$\begin{aligned}
(e + xe')^2 &= e(e + xe') + xe'(e + xe') \\
&= e^2 + exe' + xe'e + xe'xe' \\
&= e^2 + exe' + xe'e + xe'x'e \\
&= e^2 + exe' + xe'e + x(e')'xe \\
&= e^2 + exe' + xe'e + xexe \\
&= e + (ex + ex')e' + xe' + x(ex + ex')e \\
&= e + (exe' + ex'e') + xe' + x(e'x' + e'x)e \\
&= e + e(xe' + xe) + xe' + xe'(x'e + xe) \\
&= e + xe' \text{ (by condition (B))}.
\end{aligned}$$

Thus $e + xe' \in E^{\bullet}(S)$. Let $x^* \in V^{\bullet}(x)$. Now,

$$\begin{aligned}
(e + xe')(xx^*) &= exx^* + xe'xx^* \\
&= exx^* + (xexx^*)' \\
&= exx^* + x'(ex)x^* \in E^{+}(S) \text{ (as } E^{+}(S) \text{ is an ideal of } S).
\end{aligned}$$

But $e + xe', xx^* \in E^{\bullet}(S)$. So by (i), $xx^*(e + xe') \in E^{+}(S)$ and thus $xx^*e + xx^*xe' = xx^*e + xe' \in E^{+}(S)$. Also,

$$\begin{aligned}
xx^*e + xx^*e' &= xx^*e' + xx^*e \\
&= xx^*e' + xx^*e + xx^*e(x + x') \text{ (by condition (B))} \\
&= xx^*(e' + e) + xx^*(x + x')e \text{ (by condition (A))} \\
&= x(e' + e)x^* + xx^*x(e + e') \text{ (by condition (A))} \\
&= x(e'x^* + ex^*) + xe + xe' \\
&= xe(x^{*\prime} + x^*) + xe + xe' \\
&= xe + xe' \text{ (by condition (B))}
\end{aligned}$$

Hence by Lemma 2.4., we have $xx^*e = xe$. Now, $exx^* \in E^{+}(S)$ [as $ex \in E^{+}(S)$] and hence $xe = xx^*e \in E^{+}(S)$.

(ii) $\Rightarrow$ (iii): Take any $x \in S$ with $x^n \in E^{+}(S)$ for some $n > 1$. Let $x^* \in V^{\bullet}(x)$. Then $x^*x^n \in E^{+}(S)$ and so $(x^*x)x^{n-1} \in E^{+}(S)$. But $x^*x \in E^{\bullet}(S)$ and thus $x^{n-1}x^*x \in E^{+}(S)$. This leads to $x^{n-2}xx^*x = x^{n-1} \in E^{+}(S)$. Continuing this process, we have $x \in E^{+}(S)$.

(iii) $\Rightarrow$ (iv): This is obvious.

(iv) $\Rightarrow$ (v): Let $x, y \in S$ be such that $xy \in E^+(S)$. Now $(yx)^2 = y(xy)x \in E^+(S)$. Hence by given condition we have $yx \in E^+(S)$.

(v) $\Rightarrow$ (i): This is obvious.

Thus (i), (ii), (iii), (iv) and (v) are equivalent.

(vi) $\Rightarrow$ (i): Let $e, f \in E^\bullet(S)$ be such that $ef \in E^+(S)$. Because $(S, \cdot)$ is orthodox we have $fe \in E^\bullet(S)$. Then $fe = (fe)^2 = f(ef)e \in E^+(S)$. Thus the proof is completed.

We now generalize Chaptal's Theorem in S.

Theorem 2.6 In a semiring S the following conditions are equivalent.

(i) $(S, \cdot)$ is a union of groups.

(ii) $(S, \cdot)$ is an inverse semigroup.

(iii) $(S, \cdot)$ is a semilattice of groups.

Proof (i) $\Rightarrow$ (ii): Let $(S, \cdot)$ be a union of groups $(G_\alpha, \cdot)(\alpha \in I)$ where I is an index set. Let $e \in E^\bullet(S)$ and $y \in S$. Then,

$$\begin{aligned}(ye + ey'e)^2 &= ye(ye + ey'e) + ey'e(ye + ey'e) \\ &= yeye + yey'e + ey'eye + ey'ey'e \in E^+(S).\end{aligned}$$

Let $(ye + ey'e)^2$ be in the group G_α for some $\alpha \in I$ and let z be the inverse of $(ye + ey'e)$ in G_α. Then $ye + ey'e = (ye + ey'e)(ye + ey'e)z = (ye + ey'e)^2 z \in E^+(S)$, because $E^+(S)$ is an ideal of S. Also, $eye + ey'e = e(ye + y'e) = (ye + y'e)e$ (by condition (A)) $= ye + y'e$. Thus, by Lemma 2.4., we at once have $ye = eye$. Similarly, we have $ey = eye$. Hence $ey = ye$. Thus idempotents in $(S, \cdot)$ are central. Hence, $(S, \cdot)$ is an inverse semigroup.

(ii) $\Rightarrow$ (iii): Let $(S, \cdot)$ be an inverse semigroup. Let $e \in E^\bullet(S)$ and $y \in S$. Now,

$$\begin{aligned}(ye + ey'e)^2 &= ye(ye + ey'e) + ey'e(ye + ey'e) \\ &= yeye + yey'e + ey'eye + ey'ey'e \in E^+(S).\end{aligned}$$

So by (iv) of Lemma 2.5., we have $ye + ey'e \in E^+(S)$. Also, $eye + ey'e = e(ye + y'e) = (ye + y'e)e = ye + y'e$. Hence by Lemma 2.4., we at once have $ye = eye$. Similarly, $ey = eye$. Hence $ey = ye$. Thus idempotents in $(S, \cdot)$ are central. Thus $(S, \cdot)$ is a Clifford semigroup. Hence $(S, \cdot)$ is a semilattice of groups.

(iii) $\Rightarrow$ (i):This is obvious. □

We now prove the following theorem.

Theorem 2.7 If the semiring S is multiplicatively regular then the following conditions are equivalent.

(i) $(S, \cdot)$ is orthodox.

(ii) $(S, \cdot)$ is an inverse semigroup.

Proof (i) $\Rightarrow$ (ii): Let $(S, \cdot)$ be orthodox. Let $e, f \in E^\bullet(S)$. Then $e(f + e'f) = ef + ee'f = ef + eef' = ef + ef' \in E^+(S)$. So by (ii) of Lemma 2.5., we have

$(f+e'f)e \in E^+(S), i.e., fe+e'fe \in E^+(S)$. Also, $efe+e'fe = efe+ef'e = e(fe+f'e) = (fe+f'e)e$ (by condition (A)) $= fe+f'e$. Thus, by Lemma 2.4., we have $efe = fe$. Similarly, we can show that $efe = ef$. Thus, $ef = fe$. So idempotents in $(S,\cdot)$ commutes. Hence, $(S,\cdot)$ is an inverse semigroup.

(ii) $\Rightarrow$ (i): This is obvious. □

We now generalize Zeleznekow's Theorem in a semiring S.

Theorem 2.8 If the semiring S is multiplicatively regular then the following conditions are equivalent.

(i) $(S,\cdot)$ is orthodox.

(ii) $(S,\cdot)$ is a union of groups.

(iii) $(S,\cdot)$ is an inverse semigroup.

(iv) $(S,\cdot)$ is a semilattice of groups.

Proof Follows from Theorem 2.6. and Theorem 2.7. □

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