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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 43 (2004), No. 1, 87--104

Persistent URL: http://dml.cz/dmlcz/132939

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **43** (2004) 87–104

On Eliminating Transformations for Nuisance Parameters in Multivariate Linear Model *

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(Received January 27, 2004)

Abstract

The multivariate linear model, in which the matrix of the first order parameters is divided into two matrices: to the matrix of the useful parameters and to the matrix of the nuisance parameters, is considered. We examine eliminating transformations which eliminate the nuisance parameters without loss of information on the useful parameters and on the variance components.

Key words: Multivariate linear regression model, useful and nuisance parameters, LBLUE, eliminating transformation.

2000 Mathematics Subject Classification: 62J05

*Supported by the Council of Czech Government J14/98: 153 100011.

1 Notations, auxiliary statements

The following notations will be used throughout the paper:

R^n	the space of all n-dimensional real vectors;
$\boldsymbol{u}_p$	the real column p-dimensional vector;
$\boldsymbol{A}_{m,n}$, $Tr(\boldsymbol{A})$	the real $m \times n$ matrix, the trace of the matrix $\boldsymbol{A}$;
$\boldsymbol{A}'$, $r(\boldsymbol{A})$	the transpose, the rank of the matrix $\boldsymbol{A}$;
$\boldsymbol{A}^{(j)}$	j-th column of the matrix $\boldsymbol{A}$;
$vec(\boldsymbol{A})$	the column vector $((\boldsymbol{A}^{(1)})', \dots, (\boldsymbol{A}^{(n)})')'$;
$\boldsymbol{A} \otimes \boldsymbol{B}$	the Kronecker (tensor) product of the matrices $\boldsymbol{A}$,$\boldsymbol{B}$;
$\mathscr{M}(\boldsymbol{A})$	the range of the matrix $\boldsymbol{A}$;
$\boldsymbol{A}^-$	a generalized inverse of the matrix $\boldsymbol{A}$ (satisfying $\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{A}$);
$\boldsymbol{A}^+$	the Moore-Penrose generalized inverse of the matrix $\boldsymbol{A}$ (satisfying $\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{A}, \boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+, (\boldsymbol{A}\boldsymbol{A}^+)' = \boldsymbol{A}\boldsymbol{A}^+$, $(\boldsymbol{A}^+\boldsymbol{A})' = \boldsymbol{A}^+\boldsymbol{A}$);
$\boldsymbol{P}_A$	the orthogonal projector onto $\mathscr{M}(\boldsymbol{A})$;
$\boldsymbol{M}_A = \boldsymbol{I} - \boldsymbol{P}_A$	the orthogonal projector onto $\mathscr{M}^{\perp}(\boldsymbol{A}) = Ker(\boldsymbol{A}')$;
$\boldsymbol{I}_k$	the $k \times k$ identity matrix;
$0_{m,n}$	the $m \times n$ null matrix;
$\boldsymbol{o}$	the null element.

If $\mathscr{M}(\boldsymbol{A}) \subset \mathscr{M}(\boldsymbol{V})$, $\boldsymbol{V}$ p.s.d., then the symbol $\boldsymbol{P}_A^V$ denotes the projector on the subspace $\mathscr{M}(\boldsymbol{A})$ in the $\boldsymbol{V}$-seminorm given by the matrix $\boldsymbol{V}$, $||\boldsymbol{x}||_V = \sqrt{\boldsymbol{x}'\boldsymbol{V}\boldsymbol{x}}$; $\boldsymbol{M}_A^V = \boldsymbol{I} - \boldsymbol{P}_A^V = \boldsymbol{I} - \boldsymbol{A}(\boldsymbol{A}'\boldsymbol{V}\boldsymbol{A})^-\boldsymbol{A}'\boldsymbol{V}$. Let $\boldsymbol{N}_{n,n}$ is p.d. (p.s.d.) matrix and $\boldsymbol{A}_{m,n}$ an arbitrary matrix, then the symbol $\boldsymbol{A}^-_{m(N)}$ denotes the matrix satisfying $\boldsymbol{A}\boldsymbol{A}^-_{m(N)}\boldsymbol{A} = \boldsymbol{A}$ and $\boldsymbol{N}\boldsymbol{A}^-_{m(N)}\boldsymbol{A} = [\boldsymbol{N}\boldsymbol{A}^-_{m(N)}\boldsymbol{A}]'$. ($\boldsymbol{A}^-_{m(N)}\boldsymbol{y}$ is a solution of the consistent system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$ whose N-seminorm is minimal, see [4], p. 151). $\boldsymbol{A}^-_{m(N)}$ is called a minimum N-seminorm g-inverse of the matrix $\boldsymbol{A}$. It holds

$$\mathscr{M}(\boldsymbol{A}') \subset \mathscr{M}(\boldsymbol{N}) \quad \Rightarrow \quad \boldsymbol{A}^-_{m(N)}\boldsymbol{N}^-\boldsymbol{A}'(\boldsymbol{A}\boldsymbol{N}^-\boldsymbol{A}')^-.$$

Assertion 1 (see [3], Lemma 16)

$$(\boldsymbol{M}_S\Sigma\boldsymbol{M}_S)^+ = \Sigma^{-1} - \Sigma^{-1}\boldsymbol{S}(\boldsymbol{S}'\Sigma^{-1}\boldsymbol{S})^-\boldsymbol{S}'\Sigma^{-1} = \Sigma^{-1}\boldsymbol{M}_S^{\Sigma^{-1}}, \ \textit{if } \Sigma \textit{ is p.d.},$$

$$(\boldsymbol{M}_S\Sigma\boldsymbol{M}_S)^+ = \Sigma^+ - \Sigma^+\boldsymbol{S}(\boldsymbol{S}'\Sigma^-\boldsymbol{S})^-\boldsymbol{S}'\Sigma^+, \ \textit{if } \Sigma \textit{ is p.s.d. and } \mathscr{M}(\boldsymbol{S}) \subset \mathscr{M}(\Sigma).$$

Assertion 2 *If Σ is p.d. matrix, $\boldsymbol{W}$ p.s.d. and $\boldsymbol{S}$ such matrices, that*

$$\mathscr{M}(\boldsymbol{S}') = \mathscr{M}(\boldsymbol{S}'\boldsymbol{W}\boldsymbol{S}),$$

then (see [6], Lemma 1)

$$(\boldsymbol{M}_S^W)'[\boldsymbol{M}_S^W\Sigma(\boldsymbol{M}_S^W)']^+\boldsymbol{M}_S^W = (\boldsymbol{M}_S\Sigma\boldsymbol{M}_S)^+.$$

2 Multivariate linear model with nuisance parameters

Let

$$\boldsymbol{Y}_{n,m} = \boldsymbol{X}_{n,k}\boldsymbol{B}_{k,l}\boldsymbol{Z}_{l,m} + \varepsilon_{n,m} \tag{1}$$

be a multivariate linear model under consideration. Here $\boldsymbol{Y}$ is an observation matrix, $\boldsymbol{X}$, $\boldsymbol{Z}$, are known nonzero matrices, ε is a random matrix and $\boldsymbol{B}$ is a matrix of unknown parameters

$$\boldsymbol{B} = (\boldsymbol{B}_1, \boldsymbol{B}_2),$$

where $\boldsymbol{B}_1$ is a $k \times r$ matrix of useful parameters which (or their functions) has to be estimated from the observation matrix $\boldsymbol{Y}$ and $\boldsymbol{B}_2$ is a $k \times s$ matrix of nuisance parameters. Thus we consider the model

$$\boldsymbol{Y} = \boldsymbol{X}(\boldsymbol{B}_1, \boldsymbol{B}_2)\begin{pmatrix}\boldsymbol{Z}_1\\ \boldsymbol{Z}_2\end{pmatrix} + \varepsilon. \tag{2}$$

Lemma 1 *The model (2) can be equivalently written in the form*

$$vec(\boldsymbol{Y}) = [\boldsymbol{Z}_1' \otimes \boldsymbol{X}, \boldsymbol{Z}_2' \otimes \boldsymbol{X}]\begin{pmatrix}vec(\boldsymbol{B}_1)\\ vec(\boldsymbol{B}_2)\end{pmatrix} + vec(\varepsilon). \tag{3}$$

where a $r \times m$ matrix $\boldsymbol{Z}_1$ and a $s \times m$ matrix $\boldsymbol{Z}_2$ are known nonzero matrices.

Proof is obvious by virtue of the following statement

$$vec(\boldsymbol{ABC}) = (\boldsymbol{C}' \otimes \boldsymbol{A})vec(\boldsymbol{B}), \tag{4}$$

valid for all matrices of corresponding types. □

Suppose that

1. the observation vector $vec(\boldsymbol{Y})$ has the mean value

$$E[vec(\boldsymbol{Y})] = [\boldsymbol{Z}_1' \otimes \boldsymbol{X}, \boldsymbol{Z}_2' \otimes \boldsymbol{X}]\begin{pmatrix}vec(\boldsymbol{B}_1)\\ vec(\boldsymbol{B}_2)\end{pmatrix},$$

and the covariance matrix

$$var[vec(\boldsymbol{Y})] = \Sigma_\vartheta \otimes \boldsymbol{I}_n,$$

where $m \times m$ matrix Σ_ϑ (the covariance matrix of any column of the matrix $\boldsymbol{Y}$) is such a matrix that

2. $\Sigma_\vartheta = \sum_{i=1}^{p} \vartheta_i V_i$, $\forall \vartheta = (\vartheta_1, \ldots, \vartheta_p)' \in \underline{\vartheta} \subset R^p$, $V_1, \ldots, V_p$ given symmetric matrices,

3. $\underline{\vartheta} \subset R^p$ contains an open sphere in R^p,

4. if $\vartheta \in \underline{\vartheta}$, the matrix Σ_ϑ is positive definite,

5. the matrix Σ_ϑ is not a function of the matrix $\boldsymbol{B} = (\boldsymbol{B}_1, \boldsymbol{B}_2)$,

6. suppose that

$$\mathscr{M}(\boldsymbol{Z}_1' \otimes \boldsymbol{X}, \boldsymbol{Z}_2' \otimes \boldsymbol{X}) \subset \mathscr{M}(\Sigma_\vartheta \otimes \boldsymbol{I}); \tag{5}$$

this condition is warranted by

$$\mathscr{M}(\boldsymbol{Z}_1) \subset \mathscr{M}(\Sigma_\vartheta) \quad \wedge \quad \mathscr{M}(\boldsymbol{Z}_2) \subset \mathscr{M}(\Sigma_\vartheta); \tag{6}$$

and it means that

$$vec(\boldsymbol{Y}) \in \mathscr{M}(\Sigma_\vartheta \otimes \boldsymbol{I}) \ (a.s.).$$

Remark 1 A parametric function $\boldsymbol{p}'vec(\boldsymbol{B}_1), \boldsymbol{p} \in R^{kr}$, is said to be unbiasedly estimable under the model (2) if there exists an estimator $\boldsymbol{L}'vec(\boldsymbol{Y})$, $\boldsymbol{L} \in R^{mn}$, such that $E[\boldsymbol{L}'vec(\boldsymbol{Y})] = \boldsymbol{p}'vec(\boldsymbol{B}_1)$, $\forall vec(\boldsymbol{B}_1), \forall vec(\boldsymbol{B}_2)$.

The equality

$$E[\boldsymbol{L}'vec(\boldsymbol{Y})] = \boldsymbol{L}'(\boldsymbol{Z}_1' \otimes \boldsymbol{X})vec(\boldsymbol{B}_1) + \boldsymbol{L}'(\boldsymbol{Z}_2' \otimes \boldsymbol{X})vec(\boldsymbol{B}_2) = \boldsymbol{p}'vec(\boldsymbol{B}_1),$$

$\forall vec(\boldsymbol{B}_1), \forall vec(\boldsymbol{B}_2)$, is fulfiled if and only if

$$\boldsymbol{p} = (\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{L} \quad \& \quad (\boldsymbol{Z}_2 \otimes \boldsymbol{X}')\boldsymbol{L} = \boldsymbol{o},$$

that is equivalent to

$$\boldsymbol{p} = (\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{M}_{Z_2' \otimes X}\boldsymbol{u}, \quad \boldsymbol{u} \in R^{mn}.$$

Thus the class of all unbiasedly estimable linear functions $\boldsymbol{p}'vec(\boldsymbol{B}_1)$ of the useful parameters in the model (2) is given by

$$\mathscr{E}_1 = \{\boldsymbol{p}'vec(\boldsymbol{B}_1) : \boldsymbol{p} \in \mathscr{M}[(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{M}_{Z_2' \otimes X}] = \mathscr{M}[\boldsymbol{Z}_1\boldsymbol{M}_{Z_2'} \otimes \boldsymbol{X}']\}. \tag{7}$$

Obviously the class of all unbiasedly estimable linear functions $\boldsymbol{q}'vec(\boldsymbol{B}_2)$ of the nuisance parameters in the model (2) is given by

$$\mathscr{E}_2 = \{\boldsymbol{q}'vec(\boldsymbol{B}_2) : \boldsymbol{q} \in \mathscr{M}[(\boldsymbol{Z}_2 \otimes \boldsymbol{X}')\boldsymbol{M}_{Z_1' \otimes X}] = \mathscr{M}[\boldsymbol{Z}_2\boldsymbol{M}_{Z_1'} \otimes \boldsymbol{X}'].$$

Notation 1 Denote $\widehat{vec(\boldsymbol{B}_1)}$ and $\widehat{vec(\boldsymbol{B}_2)}$ an $(\Sigma_\vartheta^{-1} \otimes \boldsymbol{I})$-LS estimator of the vector parameter $vec(\boldsymbol{B}_1)$ and $vec(\boldsymbol{B}_2)$ respectively computed under the linear model (2) (see [1], p. 161). According to the assumption (6) $\boldsymbol{p}'\widehat{vec(\boldsymbol{B}_1)}$, $\boldsymbol{p} \in \mathscr{M}[(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{M}_{Z_2' \otimes X}]$, and $\boldsymbol{q}'\widehat{vec(\boldsymbol{B}_2)}$, $\boldsymbol{q} \in \mathscr{M}[(\boldsymbol{Z}_2 \otimes \boldsymbol{X}')\boldsymbol{M}_{Z_1' \otimes X}]$, are the BLUEs of the function $\boldsymbol{p}'vec(\boldsymbol{B}_1)$ and $\boldsymbol{q}'vec(\boldsymbol{B}_2)$ respectively (see [1], Theorem 5.3.2., p. 162).

Theorem 1

$$\begin{pmatrix} \widehat{vec(\boldsymbol{B}_1)} \\ \widehat{vec(\boldsymbol{B}_2)} \end{pmatrix} =$$

$$= \begin{pmatrix} (\boldsymbol{Z}_1[\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'}]^+\boldsymbol{Z}_1')^-\boldsymbol{Z}_1[\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'}]^+ \otimes (\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}' \\ (\boldsymbol{Z}_2\Sigma_\vartheta^-\boldsymbol{Z}_2')^-\boldsymbol{Z}_2\Sigma_\vartheta^-\boldsymbol{M}_{Z_1'}^{(M_{Z_2'}\Sigma_\vartheta M_{Z_2'})^+} \otimes (\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}' \end{pmatrix} vec(\boldsymbol{Y}).$$

Proof According to [1], Theorem 5.3.1 we have under the model (2)

$$\begin{pmatrix} \widehat{vec(\boldsymbol{B}_1)} \\ \widehat{vec(\boldsymbol{B}_2)} \end{pmatrix} =$$

$$= [(\boldsymbol{Z}_1'\otimes\boldsymbol{X}, \boldsymbol{Z}_2'\otimes\boldsymbol{X})'(\Sigma_\vartheta\otimes\boldsymbol{I})^-(\boldsymbol{Z}_1'\otimes\boldsymbol{X}, \boldsymbol{Z}_2'\otimes\boldsymbol{X})]^- \begin{pmatrix} \boldsymbol{Z}_1\otimes\boldsymbol{X}' \\ \boldsymbol{Z}_2\otimes\boldsymbol{X}' \end{pmatrix} (\Sigma_\vartheta\otimes\boldsymbol{I})^- vec(\boldsymbol{Y})$$

$$= \begin{bmatrix} \boldsymbol{Z}_1\Sigma_\vartheta^-\boldsymbol{Z}_1'\otimes\boldsymbol{X}'\boldsymbol{X}, & \boldsymbol{Z}_1\Sigma_\vartheta^-\boldsymbol{Z}_2'\otimes\boldsymbol{X}'\boldsymbol{X} \\ \boldsymbol{Z}_2\Sigma_\vartheta^-\boldsymbol{Z}_1'\otimes\boldsymbol{X}'\boldsymbol{X}, & \boldsymbol{Z}_2\Sigma_\vartheta^-\boldsymbol{Z}_2'\otimes\boldsymbol{X}'\boldsymbol{X} \end{bmatrix}^- \begin{pmatrix} \boldsymbol{Z}_1\Sigma_\vartheta^{-1}\otimes\boldsymbol{X}' \\ \boldsymbol{Z}_2\Sigma_\vartheta^{-1}\otimes\boldsymbol{X}' \end{pmatrix} vec(\boldsymbol{Y}). \quad (8)$$

Using the following Rohde's formula for generalized inverse of partitioned p.s.d. matrix (see [3], Lemma 13, p.68)

$$\begin{pmatrix} \boldsymbol{A}, & \boldsymbol{B} \\ \boldsymbol{B}', & \boldsymbol{C} \end{pmatrix}^- = \begin{pmatrix} \boldsymbol{A}^- + \boldsymbol{A}^-\boldsymbol{B}(\boldsymbol{C}-\boldsymbol{B}'\boldsymbol{A}^-\boldsymbol{B})^-\boldsymbol{B}'\boldsymbol{A}^-, & -\boldsymbol{A}^-\boldsymbol{B}(\boldsymbol{C}-\boldsymbol{B}'\boldsymbol{A}^-\boldsymbol{B})^- \\ -(\boldsymbol{C}-\boldsymbol{B}'\boldsymbol{A}^-\boldsymbol{B})^-\boldsymbol{B}'\boldsymbol{A}^-, & (\boldsymbol{C}-\boldsymbol{B}'\boldsymbol{A}^-\boldsymbol{B})^- \end{pmatrix}$$

$$= \begin{pmatrix} (\boldsymbol{A}-\boldsymbol{B}\boldsymbol{C}^-\boldsymbol{B}')^-, & -(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{C}^-\boldsymbol{B}')^-\boldsymbol{B}\boldsymbol{C}^- \\ -\boldsymbol{C}^-\boldsymbol{B}'(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{C}^-\boldsymbol{B}')^-, & \boldsymbol{C}^- + \boldsymbol{C}^-\boldsymbol{B}'(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{C}^-\boldsymbol{B}')^-\boldsymbol{B}\boldsymbol{C}^- \end{pmatrix},$$

we get the blocks of the g-inverse matrix in (8):

$$\boldsymbol{A}_{11} = (\boldsymbol{Z}_1[\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'}]^+\boldsymbol{Z}_1')^- \otimes (\boldsymbol{X}'\boldsymbol{X})^-,$$

$$\boldsymbol{A}_{12} = -[(\boldsymbol{Z}_1[\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'}]^+\boldsymbol{Z}_1')^-\boldsymbol{Z}_1\Sigma_\vartheta^-\boldsymbol{Z}_2'(\boldsymbol{Z}_2\Sigma_\vartheta^{=}\boldsymbol{Z}_2)^-$$
$$\otimes(\boldsymbol{X}'\boldsymbol{X})^-(\boldsymbol{X}'\boldsymbol{X})(\boldsymbol{X}'\boldsymbol{X})^-,$$

$$\boldsymbol{A}_{21} = (\boldsymbol{A}_{12})',$$

$$\boldsymbol{A}_{22} = [(\boldsymbol{Z}_2\Sigma_\vartheta^-\boldsymbol{Z}_2')^- \otimes (\boldsymbol{X}'\boldsymbol{X})^-]$$
$$+[(\boldsymbol{Z}_2\Sigma_\vartheta^-\boldsymbol{Z}_2')^-\boldsymbol{Z}_2\Sigma_\vartheta^-\boldsymbol{Z}_1'(\boldsymbol{Z}_1[\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'}]^+\boldsymbol{Z}_1')^-\boldsymbol{Z}_1\Sigma_\vartheta^-\boldsymbol{Z}_2'(\boldsymbol{Z}_2\Sigma_\vartheta^-\boldsymbol{Z}_2')^-$$
$$\otimes(\boldsymbol{X}'\boldsymbol{X})^-(\boldsymbol{X}'\boldsymbol{X})(\boldsymbol{X}'\boldsymbol{X})^-].$$

After some calculations we get

$$\widehat{vec(\boldsymbol{B}_1)} = [(\boldsymbol{Z}_1[\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'}]^+\boldsymbol{Z}_1')^-\boldsymbol{Z}_1[\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'}]^+ \otimes (\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']vec(\boldsymbol{Y}).$$

$$\widehat{vec(\boldsymbol{B}_2)} = [(\boldsymbol{Z}_2\Sigma_\vartheta^-\boldsymbol{Z}_2')^-\boldsymbol{Z}_2\Sigma_\vartheta^-\boldsymbol{M}_{Z_1'}^{[M_{Z_2'}\Sigma_\vartheta M_{Z_2'}]^+} \otimes (\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']vec(\boldsymbol{Y}).$$

The estimates obtained by substitution $\widehat{vec(\boldsymbol{B}_1)}$ into unbiasedly estimable functions $\boldsymbol{p}'vec(\boldsymbol{B}_1)$ are given uniquely. It can be proved if we take the following assertion (see [3], Lemma 8, p.65)

$$\boldsymbol{A}\boldsymbol{B}^-\boldsymbol{C} \text{ is invariant to the choice of the g-inverse } \boldsymbol{B}^-$$
$$\iff \mathscr{M}(\boldsymbol{A}') \subset \mathscr{M}(\boldsymbol{B}') \quad \& \quad \mathscr{M}(\boldsymbol{C}) \subset \mathscr{M}(\boldsymbol{B}), \quad (9)$$

into account. □

Theorem 2 *Let us denote* $\Sigma_0 = \sum_{i=1}^{p} \vartheta_{0,i} \boldsymbol{V}_i$.

a) In model (2) the function $\boldsymbol{g}'\vartheta = \sum_{i=1}^{p} \boldsymbol{g}_i \vartheta_i$, $\vartheta \in \underline{\vartheta}$, *is unbiasedly, quadratically and invariantly estimable (i.e. the estimator has the form* $[vec(\boldsymbol{Y})]'\boldsymbol{A}[vec(\boldsymbol{Y})]$, *where* $\boldsymbol{A}_{mn,mn}$ *is symmetric matrix, the estimator is invariant with respect to the change of the matrix* $\boldsymbol{B}$*) if and only if*

$$\boldsymbol{g} \in \mathscr{M}\left(\boldsymbol{S}_{(M_{(Z_1'\otimes X, Z_2'\otimes X)}(\Sigma_0\otimes I)M_{(Z_1'\otimes X, Z_2'\otimes X)})^+}\right),$$

where

$$\{\boldsymbol{S}_{(M_{(Z_1'\otimes X, Z_2'\otimes X)}(\Sigma_0\otimes I)M_{(Z_1'\otimes X, Z_2'\otimes X)})^+}\}_{i,j} =$$

$$= Tr[(\boldsymbol{M}_{(Z_1'\otimes X, Z_2'\otimes X)}(\Sigma_0\otimes \boldsymbol{I})\boldsymbol{M}_{(Z_1'\otimes X, Z_2'\otimes X)})^+(\boldsymbol{V}_i\otimes \boldsymbol{I})(\boldsymbol{M}_{(Z_1'\otimes X, Z_2'\otimes X)}(\Sigma_0\otimes \boldsymbol{I})$$

$$\times \boldsymbol{M}_{(Z_1'\otimes X, Z_2'\otimes X)})^+(\boldsymbol{V}_j\otimes \boldsymbol{I})],\ \ i,j = 1,\ldots,p.$$

b) If the function $\boldsymbol{g}'\vartheta$ *satisfies the condition from a), then the* ϑ_0*-MINQUE of* $\boldsymbol{g}'\vartheta$ *is given as*

$$\widehat{\boldsymbol{g}'\vartheta} = \sum_{i=1}^{p} \lambda_i (vec(\boldsymbol{Y}))'[\boldsymbol{M}_{(Z_1'\otimes X, Z_2'\otimes X)}(\Sigma_0\otimes \boldsymbol{I})\boldsymbol{M}_{(Z_1'\otimes X, Z_2'\otimes X)}]^+(\boldsymbol{V}_i\otimes \boldsymbol{I})$$

$$\times[\boldsymbol{M}_{(Z_1'\otimes X, Z_2'\otimes X)}(\Sigma_0\otimes \boldsymbol{I})\boldsymbol{M}_{(Z_1'\otimes X, Z_2'\otimes X)}]^+ vec(\boldsymbol{Y}),$$

where the vector $\lambda = (\lambda_1,\ldots,\lambda_p)'$ *is a solution of the system of equations*

$$\boldsymbol{S}_{(M_{(Z_1'\otimes X, Z_2'\otimes X)}(\Sigma_0\otimes I)M_{(Z_1'\otimes X, Z_2'\otimes X)})^+}\lambda = \boldsymbol{g}.$$

Proof see [4], Theorem IV.1.11.

Remark 2 The matrix $S_{(M_{(Z_1'\otimes X, Z_2'\otimes X)}(\Sigma_0\otimes I)M_{(Z_1'\otimes X, Z_2'\otimes X)})^+}$ is called the criterional matrix for the estimability of the function $\boldsymbol{g}'\vartheta$.
As $\boldsymbol{M}_{(Z_1'\otimes X, Z_2'\otimes X)} = \boldsymbol{M}_{Z_2'\otimes X}\boldsymbol{M}_{M_{(Z_2'\otimes X)}(Z_1'\otimes X)} = \boldsymbol{M}_{M_{(Z_2'\otimes X)}(Z_1'\otimes X)}\boldsymbol{M}_{Z_2'\otimes X}$, it holds

$$\{\boldsymbol{S}_{(M_{(Z_1'\otimes X, Z_2'\otimes X)}(\Sigma_0\otimes I)M_{(Z_1'\otimes X, Z_2'\otimes X)})^+}\}_{i,j}$$

$$= Tr[(\boldsymbol{M}_{M_{(Z_2'\otimes X)}(Z_1'\otimes X)}\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes \boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X}\boldsymbol{M}_{M_{(Z_2'\otimes X)}(Z_1'\otimes X)})^+(\boldsymbol{V}_i\otimes \boldsymbol{I})$$

$$\times(\boldsymbol{M}_{M_{(Z_2'\otimes X)}(Z_1'\otimes X)}\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes \boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X}\boldsymbol{M}_{M_{(Z_2'\otimes X)}(Z_1'\otimes X)})^+(\boldsymbol{V}_j\otimes \boldsymbol{I})]$$

$$= Tr[(\boldsymbol{M}_{Z_1'\otimes X}\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes \boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X}\boldsymbol{M}_{Z_1'\otimes X})^+(\boldsymbol{V}_i\otimes \boldsymbol{I})$$

$$\times(\boldsymbol{M}_{Z_1'\otimes X}\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes \boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X}\boldsymbol{M}_{Z_1'\otimes X})^+(\boldsymbol{V}_j\otimes \boldsymbol{I})], i,j = 1,\ldots p,$$

where the equality

$$[\boldsymbol{M}_{M_{(Z_2'\otimes X)}(Z_1'\otimes X)}\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes \boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X}\boldsymbol{M}_{M_{Z_2'\otimes X}(Z_1'\otimes X)}]^+$$

$$= [\boldsymbol{M}_{Z_1'\otimes X}\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes \boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X}\boldsymbol{M}_{Z_1'\otimes X}]^+,$$

was used.

3 Eliminating transformations

There are situation in the practice, that the number of nuisance parameters is much more greater than the number of useful parameters. This fact could cause difficulties in the course of calculations.

There exist two approaches to the problem of nuisance parameters. One of them is to eliminate the nuisance parameters by a transformation of the observation vector provided this transformation is not allowed to cause a loss of information of the useful parameters.

Our task is to eliminate in the model (2) the matrix $\boldsymbol{Z}_2' \otimes \boldsymbol{X}$, belonging to the vector $vec(\boldsymbol{B}_2)$ of nuisance parameters, i.e. we consider the following class of eliminating matrices

$$\mathscr{T} = \{\boldsymbol{T} : \boldsymbol{T}(\boldsymbol{Z}_2' \otimes \boldsymbol{X}) = 0\},$$

that leads us to linear models

$$[\ Tvec(\boldsymbol{Y}), \boldsymbol{T}(\boldsymbol{Z}_1' \otimes \boldsymbol{X})vec(\boldsymbol{B}_1), \boldsymbol{T}(\Sigma_\vartheta \otimes \boldsymbol{I})\boldsymbol{T}'\ ]. \tag{10}$$

The general solution of the matrix equation $\boldsymbol{T}(\boldsymbol{Z}_2' \otimes \boldsymbol{X}) = 0$ is of the form

$$\boldsymbol{T} = \boldsymbol{A}[\boldsymbol{I} - (\boldsymbol{Z}_2' \otimes \boldsymbol{X})(\boldsymbol{Z}_2' \otimes \boldsymbol{X})^-],$$

where $\boldsymbol{A}$ is an arbitrary matrix of the corresponding type, $(\boldsymbol{Z}_2' \otimes \boldsymbol{X})^-$ is some version of generalized inverse of the matrix $\boldsymbol{Z}_2' \otimes \boldsymbol{X}$.

If we choose $(\boldsymbol{Z}_2' \otimes \boldsymbol{X})^- = [(\boldsymbol{Z}_2' \otimes \boldsymbol{X})'\boldsymbol{W}(\boldsymbol{Z}_2' \otimes \boldsymbol{X})]^-(\boldsymbol{Z}_2' \otimes \boldsymbol{X})'\boldsymbol{W}$, where $\boldsymbol{W} = \boldsymbol{W}_1 \otimes \boldsymbol{W}_2$ is an arbitrary p.s.d. matrix such that

$$\mathscr{M}(\boldsymbol{Z}_2 \otimes \boldsymbol{X}') = \mathscr{M}[(\boldsymbol{Z}_2 \otimes \boldsymbol{X}')\boldsymbol{W}(\boldsymbol{Z}_2' \otimes \boldsymbol{X})], \tag{11}$$

then $\boldsymbol{T} = \boldsymbol{A}\boldsymbol{M}^W_{Z_2' \otimes X}$, where $\boldsymbol{M}^W_{Z_2' \otimes X}$ is given uniquely.

First we consider the transformation matrix $\boldsymbol{T} = \boldsymbol{M}^W_{Z_2' \otimes X}$, i.e. we consider linear model

$$[\boldsymbol{M}^W_{Z_2' \otimes X} vec(\boldsymbol{Y}), \boldsymbol{M}^W_{Z_2' \otimes X}(\boldsymbol{Z}_1' \otimes \boldsymbol{X})vec(\boldsymbol{B}_1), \boldsymbol{M}^W_{Z_2' \otimes X}(\Sigma_\vartheta \otimes \boldsymbol{I})(\boldsymbol{M}^W_{Z_2' \otimes X})'],\ \Sigma_\vartheta \text{ p.d.} \tag{12}$$

Remark 3 As $\boldsymbol{M}^{W_1 \otimes W_2}_{Z_2' \otimes X} vec(\boldsymbol{Y}) = (\boldsymbol{I}_m \otimes \boldsymbol{I}_n)vec(\boldsymbol{Y}) - (\boldsymbol{P}^{W_1}_{Z_2'} \otimes \boldsymbol{P}^{W_2}_X)vec(\boldsymbol{Y})$, we can write $\boldsymbol{Y}^{transf} = \boldsymbol{Y} - \boldsymbol{P}^{W_2}_X \boldsymbol{Y}(\boldsymbol{P}^{W_1}_{Z_2'})'$.

Lemma 2 *Let $\boldsymbol{W}$ is p.s.d. matrix such that (11) is valid. Then*

$$\mathscr{M}(\boldsymbol{M}_{Z_2' \otimes X}) = \mathscr{M}([\boldsymbol{M}^W_{Z_2' \otimes X}]').$$

Proof see [7], Lemma 2. □

Thus

$$\mathscr{M}[(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{M}_{Z_2' \otimes X}] = \mathscr{M}[(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')(\boldsymbol{M}^W_{Z_2' \otimes X})'],$$

i.e. the classes of the estimable functions $\boldsymbol{p}'vec(\boldsymbol{B}_1)$ in the model (2) and in the model (12) are identical.

Theorem 3 *The ϑ-LBLUE of the estimable function $\boldsymbol{p}'vec(\boldsymbol{B}_1)$, where $\boldsymbol{p} \in \mathscr{M}[(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{M}_{Z_2'\otimes X}]$ in the model (12) is given as*

$$\boldsymbol{p}'\widehat{vec(\boldsymbol{B}_1)} =$$
$$= \boldsymbol{p}'[(\boldsymbol{Z}_1[\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'}]^+\boldsymbol{Z}_1')^-\boldsymbol{Z}_1[\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'}]^+ \otimes (\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']vec(\boldsymbol{Y}),$$

i.e. it is the same as in the model (2), (see Theorem 1).

Proof According to [2], Theorem 3.1.3 the ϑ-LBLUE in the model (12) is given as

$$\boldsymbol{p}'\widehat{vec(\boldsymbol{B}_1)} =$$
$$= \boldsymbol{p}'\left\{\left[\left(\boldsymbol{M}^W_{Z_2'\otimes X}(\boldsymbol{Z}_1'\otimes\boldsymbol{X})\right)'\right]^-_{m(M^W_{Z_2'\otimes X}(\Sigma_\vartheta\otimes I)(M^W_{Z_2'\otimes X})')}\right\}'\boldsymbol{M}^W_{Z_2'\otimes X}vec(\boldsymbol{Y})$$
$$= \boldsymbol{p}'\{[\boldsymbol{M}^W_{Z_2'\otimes X}(\boldsymbol{Z}_1'\otimes\boldsymbol{X})]'[(\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_\vartheta\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})']^-\boldsymbol{M}^W_{Z_2'\otimes X}(\boldsymbol{Z}_1'\otimes\boldsymbol{X})\}^-$$
$$\times[\boldsymbol{M}^W_{Z_2'\otimes X}(\boldsymbol{Z}_1'\otimes\boldsymbol{X})]'[\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_\vartheta\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})']^-\boldsymbol{M}^W_{Z_2'\otimes X}vec(\boldsymbol{Y})$$
$$= \boldsymbol{p}'\{(\boldsymbol{Z}_1\otimes\boldsymbol{X}')(\boldsymbol{M}^W_{Z_2'\otimes X})'[\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_\vartheta\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})']^+\boldsymbol{M}^W_{Z_2'\otimes X}(\boldsymbol{Z}_1'\otimes\boldsymbol{X})\}^-$$
$$\times(\boldsymbol{Z}_1\otimes\boldsymbol{X}')(\boldsymbol{M}^W_{Z_2'\otimes X})'[\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_\vartheta\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})']^+\boldsymbol{M}^W_{Z_2'\otimes X}vec(\boldsymbol{Y}).$$

Using Assertion 2 and Assertion 1 we get

$$\boldsymbol{p}'\widehat{vec(\boldsymbol{B}_1)} = \boldsymbol{p}'\{(\boldsymbol{Z}_1\otimes\boldsymbol{X}')[\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_\vartheta\otimes\boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X}]^+(\boldsymbol{Z}_1'\otimes\boldsymbol{X})\}^-$$
$$\times(\boldsymbol{Z}_1\otimes\boldsymbol{X}')[\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_\vartheta\otimes\boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X}]^+vec(\boldsymbol{Y})$$
$$= \boldsymbol{p}'\{(\boldsymbol{Z}_1\otimes\boldsymbol{X}')(\Sigma_\vartheta^{-1}\otimes\boldsymbol{I})(\boldsymbol{Z}_1'\otimes\boldsymbol{X}) - (\boldsymbol{Z}_1\otimes\boldsymbol{X}')(\Sigma_\vartheta^{-1}\otimes\boldsymbol{I})(\boldsymbol{Z}_2'\otimes\boldsymbol{X})$$
$$\times[(\boldsymbol{Z}_2\otimes\boldsymbol{X}')(\Sigma_\vartheta^{-1}\otimes\boldsymbol{I})(\boldsymbol{Z}_2'\otimes\boldsymbol{X})]^-(\boldsymbol{Z}_2\otimes\boldsymbol{X}')(\Sigma_\vartheta^{-1}\otimes\boldsymbol{I})(\boldsymbol{Z}_1'\otimes\boldsymbol{X})\}^-$$
$$\times\{(\boldsymbol{Z}_1\otimes\boldsymbol{X}')(\Sigma_\vartheta^{-1}\otimes\boldsymbol{I}) - (\boldsymbol{Z}_1\otimes\boldsymbol{X}')(\Sigma_\vartheta^{-1}\otimes\boldsymbol{I})(\boldsymbol{Z}_2'\otimes\boldsymbol{X})$$
$$\times[(\boldsymbol{Z}_2\otimes\boldsymbol{X}')(\Sigma_\vartheta^{-1}\otimes\boldsymbol{I})(\boldsymbol{Z}_2'\otimes\boldsymbol{X})]^-(\boldsymbol{Z}_2\otimes\boldsymbol{X}')(\Sigma_\vartheta^{-1}\otimes\boldsymbol{I})\}vec(\boldsymbol{Y})$$
$$= \boldsymbol{p}'[(\boldsymbol{Z}_1[\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'}]^+\boldsymbol{Z}_1')^-\boldsymbol{Z}_1[\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'}]^+ \otimes (\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']vec(\boldsymbol{Y}).$$

The validity of

$$\mathscr{M}[\boldsymbol{M}^W_{Z_2'\otimes X}(\boldsymbol{Z}_1'\otimes\boldsymbol{X})] \subset \mathscr{M}[\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_\vartheta\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})']$$

follows from (5) and from regularity of Σ_ϑ. □

Lemma 3

$$(\boldsymbol{M}^W_{Z_2'\otimes X})'[\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})'\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}]^+\boldsymbol{M}^W_{Z_2'\otimes X}$$
$$= [\boldsymbol{M}_{Z_1'\otimes X}\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X}\boldsymbol{M}_{Z_1'\otimes X}]^+.$$

Proof Using Assertions 1,2 we have

$$(\boldsymbol{M}^W_{Z_2'\otimes X})'[\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})'\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}]^+\boldsymbol{M}^W_{Z_2'\otimes X}$$

$$= (\boldsymbol{M}^W_{Z_2'\otimes X})'[\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})']^+\boldsymbol{M}^W_{Z_2'\otimes X}$$

$$-(\boldsymbol{M}^W_{Z_2'\otimes X})'\left[\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})'\right]^+\boldsymbol{M}^W_{Z_2'\otimes X}(\boldsymbol{Z}_1'\otimes\boldsymbol{X})$$

$$\times\{(\boldsymbol{Z}_1\otimes\boldsymbol{X}')(\boldsymbol{M}^W_{Z_2'\otimes X})'\left[\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})'\right]^+\boldsymbol{M}^W_{Z_2'\otimes X}(\boldsymbol{Z}_1'\otimes\boldsymbol{X})\}^-$$

$$\times(\boldsymbol{Z}_1\otimes\boldsymbol{X}')(\boldsymbol{M}^W_{Z_2'\otimes X})'\left[\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})'\right]^+\boldsymbol{M}^W_{Z_2'\otimes X}$$

$$= (\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X})^+ - (\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X})^+$$

$$\times(\boldsymbol{Z}_1'\otimes\boldsymbol{X})[(\boldsymbol{Z}_1\otimes\boldsymbol{X}')(\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X})^+(\boldsymbol{Z}_1'\otimes\boldsymbol{X})]^-$$

$$\times(\boldsymbol{Z}_1\otimes\boldsymbol{X}')(\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X})^+$$

$$= [\boldsymbol{M}_{Z_1'\otimes X}\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X}\boldsymbol{M}_{Z_1'\otimes X}]^+.$$

□

Theorem 4 *A linear function $\boldsymbol{g}'\vartheta$ of the vector parameter $\vartheta\in\underline{\vartheta}\subset R^p$, unbiasedly estimable in the model (2) before eliminating transformation is unbiasedly estimable in the transformed model (12).*

Proof The (i,j)-th element of the criterional matrix in the model (12) is given by

$$\left\{\boldsymbol{S}_{(M_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}M^W_{Z_2'\otimes X}(\Sigma_0\otimes I)(M^W_{Z_2'\otimes X})'M_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)})^+}\right\}_{i,j}$$

$$= Tr\Bigg\{\left[\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})'\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}\right]^+$$

$$\times\boldsymbol{M}^W_{Z_2'\otimes X}(\boldsymbol{V}_i\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})'$$

$$\times\left[\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})'\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}\right]^+$$

$$\times\boldsymbol{M}^W_{Z_2'\otimes X}(\boldsymbol{V}_j\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})'\Bigg\}$$

$$= Tr\Bigg\{\left[\boldsymbol{M}^W_{Z_2'\otimes X})'[\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})'\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}\right]^+$$

$$\times\boldsymbol{M}^W_{Z_2'\otimes X}(\boldsymbol{V}_i\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})'$$

$$\times\left[\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_0\otimes\boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})'\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}\right]^+\boldsymbol{M}^W_{Z_2'\otimes X}(\boldsymbol{V}_j\otimes\boldsymbol{I})\Bigg\}.$$

By Lemma 3 then

$$\left\{ \boldsymbol{S}_{(M_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)} M^W_{Z_2'\otimes X}(\Sigma_0\otimes I)(M^W_{Z_2'\otimes X})' M_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)})^+} \right\}_{i,j} =$$

$$= Tr \left\{ \left[\boldsymbol{M}_{Z_1'\otimes X}\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes \boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X}\boldsymbol{M}_{Z_1'\otimes X}\right]^+ (\boldsymbol{V}_i\otimes \boldsymbol{I}) \right.$$

$$\left. \times \left[\boldsymbol{M}_{Z_1'\otimes X}\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes \boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X}\boldsymbol{M}_{Z_1'\otimes X}\right]^+ (\boldsymbol{V}_j\otimes \boldsymbol{I}) \right\},\ i,j = 1,\ldots,p.$$

Due to the Remark 2 it is evident that the criterional matrices in the model (2) and in the model (12) are identical. □

Theorem 5 *Let $\boldsymbol{g}'\vartheta$, $\vartheta \in \underline{\vartheta}$ be an unbiasedly estimable function. Then the ϑ_0-MINQUE in the model (2) and the ϑ_0-MINQUE in the model (12) after elimination coincide.*

Proof We have seen that each function $\boldsymbol{g}'\vartheta$, that is unbiasedly estimable in the model (2) is unbiasedly estimable in the model (12).

According to Theorem 2 the ϑ_0-MINQUE in the model (12) is given by

$$\widehat{\boldsymbol{g}'\vartheta} = \sum_{i=1}^{p} \lambda_i (vec(\boldsymbol{Y}))'(\boldsymbol{M}^W_{Z_2'\otimes X})'$$

$$\left[\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_0\otimes \boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})'\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}\right]^+$$

$$\times \boldsymbol{M}^W_{Z_2'\otimes X}(\boldsymbol{V}_i\otimes \boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})'$$

$$\times \left[\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}\boldsymbol{M}^W_{Z_2'\otimes X}(\Sigma_0\otimes \boldsymbol{I})(\boldsymbol{M}^W_{Z_2'\otimes X})'\boldsymbol{M}_{M^W_{Z_2'\otimes X}(Z_1'\otimes X)}\right]^+$$

$$\times \boldsymbol{M}^W_{Z_2'\otimes X} vec(\boldsymbol{Y})$$

$$= \sum_{i=1}^{p} \lambda_i (vec(\boldsymbol{Y}))' \left[\boldsymbol{M}_{Z_1'\otimes X}\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes \boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X}\boldsymbol{M}_{Z_1'\otimes X}\right]^+ (\boldsymbol{V}_i\otimes \boldsymbol{I})$$

$$\times \left[\boldsymbol{M}_{Z_1'\otimes X}\boldsymbol{M}_{Z_2'\otimes X}(\Sigma_0\otimes \boldsymbol{I})\boldsymbol{M}_{Z_2'\otimes X}\boldsymbol{M}_{Z_1'\otimes X}\right]^+ vec(\boldsymbol{Y}),$$

i.e. this estimator is identical to the estimator in the model (2)—see Remark 2. Lemma 3 has been taken into account. □

Lemma 4

$$[\boldsymbol{M}_{Z_1'\otimes X}(\Sigma_\vartheta\otimes \boldsymbol{I})\boldsymbol{M}_{Z_1'\otimes X}]^+ = (\Sigma_\vartheta^{-1}\otimes \boldsymbol{I}) - (\boldsymbol{P}^{\Sigma_\vartheta^{-1}}_{Z_1'}\otimes \boldsymbol{P}_X). \tag{13}$$

Proof With respect to Assertion 1

$$[\boldsymbol{M}_{Z_1'\otimes X}(\Sigma_\vartheta \otimes \boldsymbol{I})\boldsymbol{M}_{Z_1'\otimes X}]^+ =$$

$$= (\Sigma_\vartheta^{-1} \otimes \boldsymbol{I}) - (\Sigma_\vartheta^{-1}\boldsymbol{Z}_1'(\boldsymbol{Z}_1\Sigma_\vartheta^{-1}\boldsymbol{Z}_1')^-\boldsymbol{Z}_1\Sigma_\vartheta^{-1} \otimes \boldsymbol{X}[\boldsymbol{X}'\boldsymbol{X}]^-\boldsymbol{X}')$$

$$= (\Sigma_\vartheta^{-1} \otimes \boldsymbol{I}) - (\Sigma_\vartheta^{-1}\boldsymbol{P}_{Z_1'}^{\Sigma_\vartheta^{-1}} \otimes \boldsymbol{P}_X).$$

□

Lemma 5

$$\boldsymbol{M}_{Z_2'\otimes X}^{[M_{Z_1'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_1'\otimes X}]^+} = \boldsymbol{M}_{Z_2'\otimes X}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+\otimes I}. \tag{14}$$

Proof With respect to $\boldsymbol{M}_A^V = \boldsymbol{I} - \boldsymbol{P}_A^V = \boldsymbol{I} - \boldsymbol{A}(\boldsymbol{A}'\boldsymbol{V}\boldsymbol{A})^-\boldsymbol{A}'\boldsymbol{V}$ and using Lemma 4 we get

$$\boldsymbol{M}_{Z_2'\otimes X}^{[M_{Z_1'}\otimes X)(\Sigma_\vartheta\otimes I)M_{Z_1'\otimes X}]^+}$$

$$= (\boldsymbol{I}\otimes\boldsymbol{I}) - (\boldsymbol{Z}_2'\otimes\boldsymbol{X})[(\boldsymbol{Z}_2\otimes\boldsymbol{X}')\{(\Sigma_\vartheta^{-1}\otimes\boldsymbol{I}) - (\Sigma_\vartheta^{-1}\boldsymbol{P}_{Z_1'}^{\Sigma_\vartheta^{-1}}\otimes\boldsymbol{P}_X)\}(\boldsymbol{Z}_2'\otimes\boldsymbol{X})]^-$$

$$\times (\boldsymbol{Z}_2\otimes\boldsymbol{X}')[(\Sigma_\vartheta^{-1}\otimes\boldsymbol{I}) - (\Sigma_\vartheta^{-1}\boldsymbol{P}_{Z_1'}^{\Sigma_\vartheta^{-1}}\otimes\boldsymbol{P}_X)]$$

$$= (\boldsymbol{I}\otimes\boldsymbol{I}) - (\boldsymbol{Z}_2'[\boldsymbol{Z}_2[\boldsymbol{M}_{Z_1'}\Sigma_\vartheta\boldsymbol{M}_{Z_1'}]^+\boldsymbol{Z}_2']^-\boldsymbol{Z}_2[\boldsymbol{M}_{Z_1'}\Sigma_\vartheta\boldsymbol{M}_{Z_1'}]^+\otimes\boldsymbol{X}[\boldsymbol{X}'\boldsymbol{X}]^-\boldsymbol{X}')$$

$$= (\boldsymbol{I}\otimes\boldsymbol{I}) - (\boldsymbol{P}_{Z_2'}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+}\otimes\boldsymbol{P}_X) = \boldsymbol{M}_{Z_2'\otimes X}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+\otimes I}.$$ □

Lemma 6

$$\boldsymbol{P}_{Z_1'\otimes X}^{[M_{Z_2'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_2'\otimes X}]^+}\cdot\boldsymbol{P}_{Z_2'\otimes X}^{[M_{Z_1'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_1'\otimes X}]^+} =$$

$$= \boldsymbol{P}_{Z_2'\otimes X}^{[M_{Z_1'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_1'\otimes X}]^+}\cdot\boldsymbol{P}_{Z_1'\otimes X}^{[M_{Z_2'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_2'\otimes X}]^+} = 0.$$

Proof With respect to Lemma 5

$$\boldsymbol{P}_{Z_2'\otimes X}^{[M_{Z_1'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_1'\otimes X}]^+} = \boldsymbol{P}_{Z_2'\otimes X}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+\otimes I} = \boldsymbol{P}_{Z_2'}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+}\otimes\boldsymbol{P}_X,$$

analogously $\boldsymbol{P}_{Z_1'\otimes X}^{[M_{Z_2'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_2'\otimes X}]^+} = \boldsymbol{P}_{Z_1'}^{[M_{Z_2'}\Sigma_\vartheta M_{Z_2'}]^+}\otimes\boldsymbol{P}_X$. Since

$$\boldsymbol{P}_{Z_1'}^{[M_{Z_2'}\Sigma_\vartheta M_{Z_2'}]^+}\cdot\boldsymbol{P}_{Z_2'}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+} = \boldsymbol{P}_{Z_2'}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+}\cdot\boldsymbol{P}_{Z_1'}^{[M_{Z_2'}\Sigma_\vartheta M_{Z_2'}]^+} = 0,$$

we get the statements. □

Lemma 7

$$\boldsymbol{M}^{\Sigma_\vartheta^{-1}\otimes I}_{(Z_1'\otimes X, Z_2'\otimes X)} = (\boldsymbol{I}\otimes\boldsymbol{I}) - (\boldsymbol{P}^{[M_{Z_2'}\Sigma_\vartheta M_{Z_2'}]^+}_{Z_1'} \otimes \boldsymbol{P}_X) - (\boldsymbol{P}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+}_{Z_2'} \otimes \boldsymbol{P}_X)$$

$$= (\boldsymbol{I}\otimes\boldsymbol{I}) - \boldsymbol{P}^{[M_{Z_2'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_2'\otimes X}]^+}_{Z_1'\otimes X} - \boldsymbol{P}^{[M_{Z_1'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_1'\otimes X}]^+}_{Z_2'\otimes X}$$

$$= \boldsymbol{M}^{[M_{Z_1'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_1'\otimes X}]^+}_{Z_2'\otimes X} \cdot \boldsymbol{M}^{[M_{Z_2'\otimes X}(\Sigma\otimes I)M_{Z_2'\otimes X}]^+}_{Z_1'\otimes X}.$$

Proof

$$\boldsymbol{M}^{\Sigma_\vartheta^{-1}\otimes I}_{(Z_1'\otimes X, Z_2'\otimes X)} = (\boldsymbol{I}\otimes\boldsymbol{I}) - (\boldsymbol{Z}_1'\otimes\boldsymbol{X}, \boldsymbol{Z}_2'\otimes\boldsymbol{X})$$

$$\times\left[\begin{pmatrix}\boldsymbol{Z}_1\otimes\boldsymbol{X}'\\ \boldsymbol{Z}_2\otimes\boldsymbol{X}'\end{pmatrix}(\Sigma_\vartheta^{-1}\otimes\boldsymbol{I})(\boldsymbol{Z}_1'\otimes\boldsymbol{X}, \boldsymbol{Z}_2'\otimes\boldsymbol{X})\right]^{-}\begin{pmatrix}\boldsymbol{Z}_1\Sigma_\vartheta^{-1}\otimes\boldsymbol{X}'\\ \boldsymbol{Z}_2\Sigma_\vartheta^{-1}\otimes\boldsymbol{X}'\end{pmatrix}$$

$$= (\boldsymbol{I}\otimes\boldsymbol{I}) - (\boldsymbol{Z}_1'\otimes\boldsymbol{X}, \boldsymbol{Z}_2'\otimes\boldsymbol{X})$$

$$\times\begin{pmatrix}\boldsymbol{Z}_1\Sigma_\vartheta^{-1}\boldsymbol{Z}_1'\otimes\boldsymbol{X}'\boldsymbol{X}, & \boldsymbol{Z}_1\Sigma_\vartheta^{-1}\boldsymbol{Z}_2'\otimes\boldsymbol{X}'\boldsymbol{X}\\ \boldsymbol{Z}_2\Sigma_\vartheta^{-1}\boldsymbol{Z}_1'\otimes\boldsymbol{X}'\boldsymbol{X}, & \boldsymbol{Z}_2\Sigma_\vartheta^{-1}\boldsymbol{Z}_2'\otimes\boldsymbol{X}'\boldsymbol{X}\end{pmatrix}^{-}\begin{pmatrix}\boldsymbol{Z}_1\Sigma_\vartheta^{-1}\otimes\boldsymbol{X}'\\ \boldsymbol{Z}_2\Sigma_\vartheta^{-1}\otimes\boldsymbol{X}'\end{pmatrix}$$

$$= (\boldsymbol{I}\otimes\boldsymbol{I}) - (\boldsymbol{Z}_1'\otimes\boldsymbol{X}, \boldsymbol{Z}_2'\otimes\boldsymbol{X})\begin{pmatrix}\boldsymbol{A}_{11}, & \boldsymbol{A}_{12}\\ \boldsymbol{A}_{21}, & \boldsymbol{A}_{22}\end{pmatrix}\begin{pmatrix}\boldsymbol{Z}_1\Sigma_\vartheta^{-1}\otimes\boldsymbol{X}'\\ \boldsymbol{Z}_2\Sigma_\vartheta^{-1}\otimes\boldsymbol{X}'\end{pmatrix},$$

where (using the second Rohde's formula)

$$\boldsymbol{A}_{11} = (\boldsymbol{Z}_1[\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'}]^+\boldsymbol{Z}_1')^-\otimes(\boldsymbol{X}'\boldsymbol{X})^-,$$

$$\boldsymbol{A}_{12} = -[(\boldsymbol{Z}_1[\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'}]^+\boldsymbol{Z}_1')^-\boldsymbol{Z}_1\Sigma_\vartheta^{-1}\boldsymbol{Z}_2'(\boldsymbol{Z}_2\Sigma_\vartheta^{-1}\boldsymbol{Z}_2')^-$$
$$\otimes(\boldsymbol{X}'\boldsymbol{X})^-(\boldsymbol{X}'\boldsymbol{X})(\boldsymbol{X}'\boldsymbol{X})^-],$$

and (using the first Rohde's formula)

$$\boldsymbol{A}_{21} = -[(\boldsymbol{Z}_2[\boldsymbol{M}_{Z_1'}\Sigma_\vartheta\boldsymbol{M}_{Z_1'}]^+\boldsymbol{Z}_2')^-\boldsymbol{Z}_2\Sigma_\vartheta^{-1}\boldsymbol{Z}_1'(\boldsymbol{Z}_1\Sigma_\vartheta^{-1}\boldsymbol{Z}_1')^-$$
$$\otimes(\boldsymbol{X}'\boldsymbol{X})^-(\boldsymbol{X}'\boldsymbol{X})(\boldsymbol{X}'\boldsymbol{X})^-],$$

$$\boldsymbol{A}_{22} = (\boldsymbol{Z}_2[\boldsymbol{M}_{Z_1'}\Sigma_\vartheta\boldsymbol{M}_{Z_1'}]^+\boldsymbol{Z}_2')^-\otimes(\boldsymbol{X}'\boldsymbol{X})^-.$$

Substituting these expressions we get the first assertion. The rest of the proof is evident (with respect to Lemma 5 and Lemma 6). □

If we use in the eliminating transformation $\boldsymbol{T} = \boldsymbol{M}^W_{Z_2'\otimes X}$ the following matrix

$$\boldsymbol{W} = \left[\boldsymbol{M}_{Z_1'\otimes X}(\Sigma_\vartheta\otimes\boldsymbol{I})\boldsymbol{M}_{Z_1'\otimes X}\right]^+,$$

we get the transformation matrix (see (14))

$$\boldsymbol{T} = \boldsymbol{M}^{[M_{Z_1'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_1'\otimes X}]^+}_{Z_2'\otimes X} = \boldsymbol{M}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+\otimes I}_{Z_2'\otimes X},$$

that is very useful. It eliminates the nuisance parameters and does not change the design matrix belonging to the vector of useful parameters, i.e. this transformation yields the following model

$$\Big[\boldsymbol{M}_{Z_2'\otimes X}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+\otimes I} vec(\boldsymbol{Y}), (\boldsymbol{Z}_1'\otimes \boldsymbol{X})vec(\boldsymbol{B}_1),$$

$$\boldsymbol{M}_{Z_2'\otimes X}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+\otimes I}(\Sigma_\vartheta\otimes \boldsymbol{I})(\boldsymbol{M}_{Z_2'\otimes X}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+\otimes I})'\Big],\quad \Sigma_\vartheta \text{ p.d.} \tag{15}$$

Remark 4 a) The matrix $\boldsymbol{W} = [\boldsymbol{M}_{Z_1'\otimes X}(\Sigma_\vartheta\otimes \boldsymbol{I})\boldsymbol{M}_{Z_1'\otimes X}]^+$ satisfies the assumption (11), see [2], page 189.

b) Theorem 3, Theorem 4 and Theorem 5 are true in the model (15).

Let us consider the more general model

$$\Big[\boldsymbol{AM}_{Z_2'\otimes X}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+\otimes I} vec(\boldsymbol{Y}), \boldsymbol{A}(\boldsymbol{Z}_1'\otimes \boldsymbol{X})vec(\boldsymbol{B}_1),$$

$$\boldsymbol{AM}_{Z_2'\otimes X}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+\otimes I}(\Sigma_\vartheta\otimes \boldsymbol{I})(\boldsymbol{M}_{Z_2'\otimes X}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+\otimes I})'\boldsymbol{A}'\Big],\quad \Sigma_\vartheta \quad \text{p.d.}, \tag{16}$$

where $\boldsymbol{A}$ is such that

$$\mathscr{M}[(\boldsymbol{Z}_1\otimes \boldsymbol{X}')\boldsymbol{A}'] = \mathscr{M}[(\boldsymbol{Z}_1\otimes \boldsymbol{X}')\boldsymbol{M}_{Z_2'\otimes X}], \tag{17}$$

i.e. the classes of the unbiasedly estimable functions in the model (2) and in the model (16) coincide.

It holds

$$E\left(\boldsymbol{AP}_{Z_1'\otimes X}^{[M_{Z_2'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_2'\otimes X}]^+} vec(\boldsymbol{Y})\right) = E\left(\boldsymbol{AP}_{Z_1'\otimes X}^{[M_{Z_2'}\Sigma_\vartheta M_{Z_2'}]^+\otimes I} vec(\boldsymbol{Y})\right)$$

$$= \boldsymbol{AP}_{Z_1'\otimes X}^{[M_{Z_2'}\Sigma_\vartheta M_{Z_2'}]^+\otimes I}[(\boldsymbol{Z}_1'\otimes \boldsymbol{X})vec(\boldsymbol{B}_1) + (\boldsymbol{Z}_2'\otimes \boldsymbol{X})vec(\boldsymbol{B}_2)]$$

$$= \boldsymbol{A}(\boldsymbol{Z}_1'\otimes \boldsymbol{X})vec(\boldsymbol{B}_1),$$

i.e. $\boldsymbol{AP}_{Z_1'\otimes X}^{[M_{Z_2'}\Sigma_\vartheta M_{Z_2'}]^+\otimes I} vec(\boldsymbol{Y})$ is an unbiased estimator of the vector function $\boldsymbol{A}(\boldsymbol{Z}_1'\otimes \boldsymbol{X})vec(\boldsymbol{B}_1)$ for each matrix $\boldsymbol{A}$.

Lemma 8

$$\boldsymbol{AP}_{Z_1'\otimes X}^{[M_{Z_2'}\Sigma_\vartheta M_{Z_2'}]^+\otimes I} vec(\boldsymbol{Y}) = \boldsymbol{AP}_{Z_1'\otimes X}^{[M_{Z_2'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_2'\otimes X}]^+} vec(\boldsymbol{Y})$$

is the best estimator of its mean value.

Proof We use the basic lemma on the locally best estimators (see [4], p. 84).

The class of the estimators of the null parametric function in the model (2) can be expressed in the form

$$\mathscr{U}_0 = \{\boldsymbol{u}'\boldsymbol{M}^{(\Sigma_\vartheta^{-1}\otimes I)}_{(Z_1'\otimes X, Z_2'\otimes X)} vec(\boldsymbol{Y}), \forall \boldsymbol{u} \in R^{mn}\},$$

as

$$E[\boldsymbol{L}'vec(\boldsymbol{Y})] = \boldsymbol{L}'(\boldsymbol{Z}_1'\otimes \boldsymbol{X}, \boldsymbol{Z}_2'\otimes \boldsymbol{X}) \begin{pmatrix} vec(\boldsymbol{B}_1) \\ vec(\boldsymbol{B}_2) \end{pmatrix} = 0,$$

$$\forall vec(\boldsymbol{B}_1) \in R^{kr},\ \forall vec(\boldsymbol{B}_2) \in R^{ks},$$

$$\iff \boldsymbol{L}'(\boldsymbol{Z}_1'\otimes \boldsymbol{X}, \boldsymbol{Z}_2'\otimes \boldsymbol{X}) = \boldsymbol{o}'$$

$$\iff \boldsymbol{L} \in \mathscr{M}[\boldsymbol{M}_{(Z_1'\otimes X, Z_2'\otimes X)}] = \mathscr{M}[(\boldsymbol{M}^{\Sigma_\vartheta^{-1}\otimes I}_{(Z_1'\otimes X, Z_2'\otimes X)})'].$$

$$cov(\boldsymbol{A}\boldsymbol{P}^{[M_{Z_2'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_2'\otimes X}]^+}_{Z_1'\otimes X} vec(\boldsymbol{Y}), \boldsymbol{u}'\boldsymbol{M}^{\Sigma_\vartheta^{-1}\otimes I}_{(Z_1'\otimes X, Z_2'\otimes X)} vec(\boldsymbol{Y}))$$

$$= \boldsymbol{A}\boldsymbol{P}^{[M_{Z_2'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_2'\otimes X}]^+}_{Z_1'\otimes X} (\Sigma_\vartheta \otimes \boldsymbol{I})(\boldsymbol{M}^{\Sigma_\vartheta^{-1}\otimes I}_{(Z_1'\otimes X, Z_2'\otimes X)})'\boldsymbol{u}$$

$$= \boldsymbol{A}\boldsymbol{P}^{[M_{Z_2'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_2'\otimes X}]^+}_{Z_1'\otimes X} \boldsymbol{M}^{\Sigma_\vartheta^{-1}\otimes I}_{(Z_1'\otimes X, Z_2'\otimes X)}(\Sigma_\vartheta \otimes \boldsymbol{I})\boldsymbol{u} = \boldsymbol{o},\ \forall \boldsymbol{u} \in R^{mn},$$

for each matrix $\boldsymbol{A}$, as according to Lemma 6, Lemma 7

$$\boldsymbol{P}^{[M_{Z_2'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_2'\otimes X}]^+}_{Z_1'\otimes X} \boldsymbol{M}^{\Sigma_\vartheta^{-1}\otimes I}_{(Z_1'\otimes X, Z_2'\otimes X)} = 0. \qquad \square$$

Theorem 6 *In the model (16) the estimators* $\boldsymbol{A}\boldsymbol{P}^{[M_{Z_2'\otimes X}(\Sigma_\vartheta\otimes I)M_{Z_2'\otimes X}]^+}_{Z_1'\otimes X} vec(\boldsymbol{Y})$, *where* $\boldsymbol{A}$ *is an arbitrary matrix such that*

$$\mathscr{M}[(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{A}'] = \mathscr{M}[(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{M}_{Z_2'\otimes X}],$$

create the class of all optimal estimators of the vector function $\boldsymbol{A}(\boldsymbol{Z}_1'\otimes\boldsymbol{X})vec(\boldsymbol{B}_1)$.

Proof Let us denote $\boldsymbol{B} = \boldsymbol{M}^{[M_{Z_1'}\Sigma_\vartheta M_{Z_1'}]^+\otimes I}_{Z_2'\otimes X}$. According to [2], Theorem 3.1.3, the ϑ-LBLUE of the vector function $\boldsymbol{A}(\boldsymbol{Z}_1'\otimes \boldsymbol{X})vec(\boldsymbol{B}_1)$ in the model (16) is

$$\boldsymbol{A}(\widehat{\boldsymbol{Z}_1' \otimes \boldsymbol{X})vec(}\boldsymbol{B}_1) = \boldsymbol{A}(\boldsymbol{Z}_1'\otimes\boldsymbol{X}) \left\{[(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{A}']^-_{m(AB(\Sigma_\vartheta\otimes I)B'A')}\right\}' \boldsymbol{A}\boldsymbol{B}vec(\boldsymbol{Y})$$

$$= \boldsymbol{A}(\boldsymbol{Z}_1'\otimes\boldsymbol{X}) \left\{(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{A}' \left[\boldsymbol{A}\boldsymbol{B}(\Sigma_\vartheta \otimes \boldsymbol{I})\boldsymbol{B}'\boldsymbol{A}'\right]^- \boldsymbol{A}(\boldsymbol{Z}_1'\otimes \boldsymbol{X})\right\}^- (\boldsymbol{Z}_1\otimes\boldsymbol{X}')\boldsymbol{A}'$$

$$\times \left[\boldsymbol{A}\boldsymbol{B}(\Sigma_\vartheta \otimes \boldsymbol{I})\boldsymbol{B}'\boldsymbol{A}'\right]^- \boldsymbol{A}\boldsymbol{B}vec(\boldsymbol{Y})$$

$$= \boldsymbol{A}\boldsymbol{B}(\boldsymbol{Z}_1'\otimes\boldsymbol{X})\{(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{B}'\boldsymbol{A}'[\boldsymbol{A}\boldsymbol{B}(\Sigma_\vartheta \otimes \boldsymbol{I})\boldsymbol{B}'\boldsymbol{A}']^-\boldsymbol{A}\boldsymbol{B}(\boldsymbol{Z}_1'\otimes \boldsymbol{X})\}^-(\boldsymbol{Z}_1\otimes\boldsymbol{X}')$$

$$\times\boldsymbol{B}'\boldsymbol{A}' \left[\boldsymbol{A}\boldsymbol{B}(\Sigma_\vartheta \otimes \boldsymbol{I})\boldsymbol{B}'\boldsymbol{A}'\right]^- \boldsymbol{A}\boldsymbol{B}vec(\boldsymbol{Y}) = \boldsymbol{P}^{[AB(\Sigma_\vartheta\otimes I)B'A']^-}_{AB(Z_1'\otimes X)} \boldsymbol{A}\boldsymbol{B}vec(\boldsymbol{Y}).$$

It is the best unbiased estimator. With respect to the basic lemma on the best estimators

$$cov\left\{\boldsymbol{P}_{AB(Z_1'\otimes X)}^{[AB(\Sigma_\vartheta\otimes I)B'A']^-}\boldsymbol{AB}vec(\boldsymbol{Y}), \boldsymbol{u}'\boldsymbol{M}_{(Z_1'\otimes X, Z_2'\otimes X)}^{\Sigma^{-1}\otimes I}vec(\boldsymbol{Y})\right\} = 0, \ \forall \boldsymbol{u} \in R^{mn},$$

is valid, i.e.

$$\boldsymbol{P}_{AB(Z_1'\otimes X)}^{[AB(\Sigma_\vartheta\otimes I)B'A']^-}\boldsymbol{AB}(\Sigma_\vartheta \otimes \boldsymbol{I})(\boldsymbol{M}_{(Z_1'\otimes X, Z_2'\otimes X)}^{\Sigma_\vartheta^{-1}\otimes I})'\boldsymbol{u}'$$

$$= \boldsymbol{P}_{AB(Z_1'\otimes X)}^{[AB(\Sigma_\vartheta\otimes I)B'A']^-}\boldsymbol{ABM}_{(Z_1'\otimes X, Z_2'\otimes X)}^{\Sigma_\vartheta^{-1}\otimes I}(\Sigma_\vartheta \otimes \boldsymbol{I})\boldsymbol{u}' = 0, \quad \forall u \ \in R^{mn}.$$

Thus

$$\boldsymbol{P}_{AB(Z_1'\otimes X)}^{[AB(\Sigma_\vartheta\otimes I)B'A']^-}\boldsymbol{ABM}_{Z_1'\otimes X}^{[M_{Z_2'}\Sigma_\vartheta M_{Z_2'}]^+\otimes I} = 0,$$

where Lemma 7 and Lemma 5 have been utilized. From this equality it follows

$$\boldsymbol{P}_{AB(Z_1'\otimes X)}^{[AB(\Sigma_\vartheta\otimes I)B'A']^-}\boldsymbol{AB}vec(\boldsymbol{Y}) = \boldsymbol{P}_{AB(Z_1'\otimes X)}^{[AB(\Sigma_\vartheta\otimes I)B'A']^-}\boldsymbol{ABP}_{Z_1'\otimes X}^{[M_{Z_2'}\Sigma_\vartheta M_{Z_2'}]^+\otimes I}vec(\boldsymbol{Y}).$$

Let us denote

$$\boldsymbol{C} = (\boldsymbol{Z}_1 \otimes \boldsymbol{X}')\boldsymbol{B}'\boldsymbol{A}'[\boldsymbol{AB}(\Sigma_\vartheta \otimes \boldsymbol{I})\boldsymbol{B}'\boldsymbol{A}']^-\boldsymbol{AB}(\boldsymbol{Z}_1' \otimes \boldsymbol{X}).$$

Then

$$\boldsymbol{A}(\widehat{\boldsymbol{Z}_1' \otimes \boldsymbol{X})vec(}\boldsymbol{B}_1) = \boldsymbol{P}_{AB(Z_1'\otimes X)}^{[AB(\Sigma_\vartheta\otimes I)B'A']^-}\boldsymbol{AB}vec(\boldsymbol{Y})$$

$$= \boldsymbol{AB}(\boldsymbol{Z}_1' \otimes \boldsymbol{X})\boldsymbol{C}^-\boldsymbol{C}[(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')([\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'}]^+ \otimes \boldsymbol{I})(\boldsymbol{Z}_1' \otimes \boldsymbol{X})]^-$$

$$\times(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')[(\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'})^+ \otimes \boldsymbol{I}]vec(\boldsymbol{Y})$$

$$= \boldsymbol{A}(\boldsymbol{Z}_1' \otimes \boldsymbol{X})[(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')([\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'}]^+ \otimes \boldsymbol{I})(\boldsymbol{Z}_1' \otimes \boldsymbol{X})]^-$$

$$\times(\boldsymbol{Z}_1 \otimes \boldsymbol{X}')[(\boldsymbol{M}_{Z_2'}\Sigma_\vartheta\boldsymbol{M}_{Z_2'})^+ \otimes \boldsymbol{I}]vec(\boldsymbol{Y})$$

$$= \boldsymbol{AP}_{Z_1'\otimes X}^{(M_{Z_2'}\Sigma_\vartheta M_{Z_2'})^+\otimes I}vec(\boldsymbol{Y}),$$

(the best estimator of its mean value $\boldsymbol{A}(\boldsymbol{Z}_1'\otimes\boldsymbol{X})vec(\boldsymbol{B}_1)$ according to Lemma 8).

The following equivalence has been taken into account

$$\boldsymbol{AM}_{Z_2'\otimes X}^{(M_{Z_2'}\Sigma_\vartheta M_{Z_2'})^+\otimes I}(\boldsymbol{Z}_1'\otimes\boldsymbol{X})\boldsymbol{C}^-\boldsymbol{C} = \boldsymbol{AM}_{Z_2'\otimes X}^{(M_{Z_2'}\Sigma_\vartheta M_{Z_2'})^+\otimes I}(\boldsymbol{Z}_1'\otimes\boldsymbol{X}) = \boldsymbol{AB}(\boldsymbol{Z}_1'\otimes\boldsymbol{X})$$

$$\iff \mathscr{M}\left[\left(\boldsymbol{AM}_{Z_2'\otimes X}^{(M_{Z_2'}\Sigma_\vartheta M_{Z_2'})^+\otimes I}(\boldsymbol{Z}_1' \otimes \boldsymbol{X})\right)'\right] \subset \mathscr{M}(\boldsymbol{C}').$$

The g-inverse matrix in the matrix $\boldsymbol{C}$ can be chosen arbitrarily. If we chose it positive definite, the condition on the right side of the equivalence is obvious. □

Example 1 Let us consider following situation (see [5]). When laying the foundations for a large building it is necessary to determine the moment at which the subsoil (after large landscaping has been done) stabilizes to the point that it is possible to continue construction without risk of following damage.

There are n points chosen at the building site and their heights are repeatedly measured at the moments $t_1, \ldots, t_m$. It is necessary to create a model describing the subsidence of the subsoil at the chosen points and to estimate the unknown parameters of this model on the basis of the results of the repeated measurements.

The result of the measurement at the i-th point in the j-th epoch could be described as follows:

$$\eta_i(t_j) = \kappa_i - \beta_1(1 - e^{-\beta_2 t_j}) + \varepsilon_{ij},\ i = 1, \ldots, n,\ j = 1, \ldots, m, \tag{18}$$

where κ_i is the height of the i-th point at time t_0, the function $\beta_1(1 - e^{-\beta_2 t})$ describes the movement of the earth-strata at each point. The parameters $\beta_1 > 0, \beta_2 > 0$ are the same at the different points, i.e. we suppose that the geological composition of the subsoil is homogenous. The aim is to estimate the unknown parameters β_1, β_2 and κ_i, $i = 1, \ldots, n$.

The civil engineer needs to know when it is possible to continue the construction, i.e. when the subsidence of the subsoil at the points is insignificant. It means that it is necessary to determine such τ that

$$\beta_1(1 - e^{-\beta_2 \tau)}\ \geq\ C\beta_1,$$

where $0 < C < 1$ is a suitable constant which is sufficiently close to 1. It is possible to continue the construction at the time $t \geq \tau$.

The model (18) is not linear in parameters; we linearize it by using the first two members of the Taylor expansion of the function $\beta_1(1 - e^{-\beta_2 t})$ at the suitable point $(\beta_{1,0}, \beta_{2,0})$, $\beta_{1,0} > 0, \beta_{2,0} > 0$.

We get the model

$$\eta_i(t_j) =$$
$$= \kappa_i - [\beta_{1,0}(1 - e^{-\beta_{2,0} t_j}) + (1 - e^{-\beta_{2,0} t_j})(\beta_1 - \beta_{1,0}) + \beta_{1,0} t_j e^{-\beta_{2,0} t_j}(\beta_2 - \beta_{2,0})] + \varepsilon_{ij},$$
$$i = 1, \ldots, n,\ j = 1, \ldots, m.$$

Denote

$$Y_i^{(j)} = \eta_i(t_j) + \beta_{1,0}(1 - e^{-\beta_{2,0} t_j}),\ \varphi_1(t) = -(1 - e^{-\beta_{2,0} t}),\ \varphi_2(t) = -\beta_{1,0} t e^{-\beta_{2,0} t},$$

$$\delta\beta_1 = \beta_1 - \beta_{1,0}, \quad \delta\beta_2 = \beta_2 - \beta_{2,0},\ i = 1, \ldots, n,\ j = 1, \ldots, m.$$

Thus

$$Y_i^{(j)} = \kappa_i + \varphi_1(t_j)\delta\beta_1 + \varphi_2(t_j)\delta\beta_2 + \varepsilon_{ij},\ i = 1, \ldots, n,\ j = 1, \ldots, m.$$

Let us consider the observation vector

$$\boldsymbol{Y} = (Y^{(1)}, \ldots, Y^{(m)}), \qquad \boldsymbol{Y}^{(j)} = (Y_1^{(j)}, \ldots, Y_n^{(j)}).$$

The model described above could be rewritten in the form

$$\boldsymbol{Y} = \boldsymbol{X}(\boldsymbol{B}_1, \boldsymbol{B}_2)\begin{pmatrix}\boldsymbol{Z}_1\\ \boldsymbol{Z}_2\end{pmatrix} + \varepsilon,$$

where

$$\boldsymbol{X} = \boldsymbol{I}_k, \quad \boldsymbol{B}_1 = \begin{pmatrix}\delta\beta_1, & \delta\beta_2\\ \delta\beta_1, & \delta\beta_2\\ \vdots & \vdots\\ \delta\beta_1, & \delta\beta_2\end{pmatrix} \quad \boldsymbol{B}_2 = \begin{pmatrix}\kappa_1\\ \kappa_2\\ \vdots\\ \kappa_n\end{pmatrix},$$

$$\boldsymbol{Z}_1 = \begin{pmatrix}\varphi_1(t_1), & \varphi_1(t_2), & \dots & \varphi_1(t_m)\\ \varphi_2(t_1), & \varphi_2(t_2), & \dots & \varphi_2(t_m)\end{pmatrix} \quad \boldsymbol{Z}_2 = (1, 1, \dots, 1).$$

The $n \times 2$ matrix $\boldsymbol{B}_1$ is a matrix of useful parameters, the $n \times 1$ matrix $\boldsymbol{B}_2$ is a matrix of nuisance parameters.
Let us choose $n = 2, \quad m = 2, \quad t_1 = 1, \quad t_2 = 6, \quad \beta_{1,0} = 1, \quad \beta_{2,0} = 1,$

$$\boldsymbol{Z}_1 = \begin{pmatrix}-0,6321 & -0,9975\\ -0,3679 & -0,0149\end{pmatrix}, \quad \boldsymbol{Z}_2 = (1, 1).$$

$$\boldsymbol{B}_1 = \begin{pmatrix}\delta\beta_1 & \delta\beta_2\\ \delta\beta_1 & \delta\beta_2\end{pmatrix}, \quad \boldsymbol{B}_2 = \begin{pmatrix}\kappa_1\\ \kappa_2\end{pmatrix},$$

For the sake of simplicity let us choose $\boldsymbol{W} = \boldsymbol{I}, \quad \Sigma = \sigma^2\boldsymbol{I}$, then we have for $\boldsymbol{X} = \boldsymbol{I}$

$$\boldsymbol{M}^{W}_{Z_2'\otimes X} = \boldsymbol{I} - [\boldsymbol{Z}_2'(\boldsymbol{Z}_2\boldsymbol{Z}_2')^-\boldsymbol{Z}_2 \otimes \boldsymbol{I}] = \begin{pmatrix}0.5 & 0 & -0.5 & 0\\ 0 & 0.5 & 0 & -0.5\\ -0.5 & 0 & 0.5 & 0\\ 0 & -0.5 & 0 & 0.5\end{pmatrix},$$

$$\boldsymbol{M}^{[M_{Z_1'}\Sigma M_{Z_1'}]^+\otimes I}_{Z_2'\otimes X} = \boldsymbol{I} - [\boldsymbol{Z}_2'(\boldsymbol{Z}_2\boldsymbol{M}_{Z_1'}\boldsymbol{Z}_2')^-\boldsymbol{Z}_2\boldsymbol{M}_{Z_1'} \otimes \boldsymbol{I}] = \begin{pmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\boldsymbol{P}^{[M_{Z_2'}\Sigma M_{Z_2'}]^+\otimes I}_{Z_1'} = \boldsymbol{Z}_1'(\boldsymbol{Z}_1\boldsymbol{M}_{Z_2'}\boldsymbol{Z}_1')^-\boldsymbol{Z}_1\boldsymbol{M}_{Z_2'} \otimes \boldsymbol{I}$$

$$= \begin{pmatrix}-0.3917 & 0 & 0.3917 & 0\\ 0 & -0.3917 & 0 & 0.3917\\ -1.3917 & 0 & 1.3917 & 0\\ 0 & -1.3917 & 0 & 1.3917\end{pmatrix}.$$

All these matrices eliminate the nuisance parameters.

Remark 5 Papers [3], [6] deal with univariate model, in [7] there is the multivariate linear model (2) with $var[vec(\boldsymbol{Y})] = \boldsymbol{I} \otimes \Sigma_\vartheta$ considered.

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