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Kronecker Modules and Reductions of a Pair of Bilinear Forms *

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Abstract

We give a short overview on the subject of canonical reduction of a pair of bilinear forms, each being symmetric or alternating, making use of the classification of pairs of linear mappings between vector spaces given by J. Dieudonné.

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The problem of a simultaneous reduction of a pair of symmetric bilinear forms over a given field is classic: this problem has been solved, for fields of characteristic zero, in 1868 by K. Weierstrass, under the assumption that both the forms are not degenerate. Two papers, the first of which by L. Kronecker [4], dated 1890, the second by L. E. Dickson [1], dated 1909, give a complete answer for fields of characteristic zero. Later J. Williamson [9] (1935), [10] (1945) showed that similar results were also valid for any field of characteristic $\neq 2$, but the condition that one of the form is not degenerate is needed again. The case where both the forms are degenerate has been solved by W. Waterhouse [7] (1976), as well as the case of a pair of symmetric bilinear forms (even degenerate) over a field of characteristic 2, [8] (1977).

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Two papers of the 70's by P. Gabriel [3] and R. Scharlau [6] showed that the classification of pairs of linear mappings (or Kronecker modules) by J. Dieudonné [2] (1946), which goes back to the mentioned paper of Kronecker, plays a fundamental role in studying pairs of bilinear forms. More precisely, Scharlau gives a complete answer for a pair of alternating bilinear forms, as pointed out by Waterhouse in [8].

The case where one of the forms is symmetric and the other is alternating has been treated by several authors and can be found in two papers by C. Riehm [5] and Gabriel [3], but the arguments used by Riehm, as well as the ones used by Waterhouse, do not concern any longer the theory of Kronecker modules.

We provide a statement (Theorem 2 and following discussion) which gives an overview on the subject from the point of view of Kronecker modules. This allows us to give an alternative proof of some results which had been given in the mentioned papers.

1. A *Kronecker module over the field* K is a pair

$$\Phi = (\varphi_1 : V' \to V'';\ \varphi_2 : V' \to V'')$$

of linear mappings from a K-vector space V' into a K-vector space V''. We write for short $\Phi = (V', V''; \varphi_1, \varphi_2)$, or simply $\Phi = (V',\ V'')$. An *isomorphism* $\iota : \Phi \to \Psi$ from Φ onto the Kronecker module $\Psi = (W', W''; \psi_1, \psi_2)$ is a pair of bijective linear mappings $\iota = (\iota' : V' \to W'; \iota'' : V'' \to W'')$ such that $\iota''\varphi_h = \psi_h\iota'$ $(h = 1, 2)$.

From the Kronecker module Φ we obtain two further Kronecker modules: the *opposite* of Φ, that is the Kronecker module

$$\Phi^\circ := (V', V''; \varphi_2, \varphi_1),$$

and the *transpose* of Φ, that is the Kronecker module

$${}^t\Phi := (V''^*, V'^*; {}^t\varphi_1, {}^t\varphi_2),$$

where, for a given linear mapping φ, we denote by ${}^t\varphi$ the *transpose* of φ, from the dual V''^* of V'' into the dual V'^* of V', defined by

$${}^{\mathrm{t}}\varphi(x''^*)(x') = x''^*(\varphi(x'))$$

for all $x' \in V'$ and $x''^* \in V''^*$. The Kronecker module Φ is *self-transpose* if there exists an isomorphism $\Phi \to {}^{\mathrm{t}}\Phi$.

Any Kronecker module can be decomposed into the direct sum of indecomposable submodules and, for two such decompositions, the Krull–Remak–Schmidt Theorem applies. This means $\Phi(F) = \Phi_1 \bigoplus \ldots \bigoplus \Phi_t$ for a fixed number t of indecomposable submodules Φ_i, determined up to permutations and isomorphisms. Indecomposable Kronecker modules were classified by Kronecker [4] and Dieudonné [2]: let

$$\begin{aligned} &\Phi_\varphi = (K^n, K^n; id, \varphi) && n > 0, \quad \varphi \in \mathbf{End}_K K^n, \\ &\Phi_n = (K^n, K^{n+1}; \varphi_1, \varphi_2) && n \geq 0, \text{ where} \\ & && \varphi_1 : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0), \\ & && \varphi_2 : (x_1, \dots, x_n) \mapsto (0, x_1, \dots, x_n), \end{aligned} \tag{1}$$

then an indecomposable Kronecker module is isomorphic to one of $\Phi_n, {}^t\Phi_n, \Phi_\varphi$, or Φ_φ°, for a suitable endomorphism $\varphi \in \mathbf{End}_K(K^n)$ which makes K^n into an indecomposable $K[\varphi]$-module. Note that Φ_φ is self-transpose, whereas Φ_n is not, hence Φ is self-transpose precisely if $\dim V' = \dim V''$.

The Krull–Remak–Schmidt Theorem has the following useful corollary (the *exchange theorem*): let $\Phi = \Upsilon_1 \bigoplus \Upsilon_2$ and $\Phi = \bar{\Upsilon}_1 \bigoplus \bar{\Upsilon}_2$ be two decompositions of Φ. Assume that no indecomposable component of Υ_1 (resp. $\bar{\Upsilon}_1$) is isomorphic to any indecomposable component of Υ_2 (resp. $\bar{\Upsilon}_2$), then $\Phi = \Upsilon_1 \bigoplus \bar{\Upsilon}_2 = \bar{\Upsilon}_1 \bigoplus \Upsilon_2$.

2. Let $f_h : V \times V \to K$, $h = 1, 2$, be a pair of bilinear forms, each being symmetric or alternating, defined on a K-vector space V. We can associate to the triple $F = (V; f_1, f_2)$ the self-transpose Kronecker module

$$\Phi(F) := (\bar{f}_1 : V \to V^*, \ \bar{f}_2 : V \to V^*),$$

where, for $x \in V$, $\bar{f}_h(x)$ is the mapping $y \mapsto f_h(x, y)$. For a subspace U of V, we can set

$$U^\perp = \{v \in V : f_1(v, x) = f_2(v, x) = 0 \text{ for any } x \in U\}.$$

We say that F is *decomposable* if $V = U + U^\perp$ for some nontrivial subspace U.

Manifestly, any decomposition of V into the direct sum of two subspaces U_1 and U_2, orthogonal with respect to both f_1 and f_2, provides a decomposition of $\Phi(F)$. The converse is generally not true.

The canonical identification $V = V^{**}$ yields the consequent identification $\Phi(F) = {}^t\Phi(F)$. Hence, the number of components of $\Phi(F)$ isomorphic to Φ_n is the same of the ones isomorphic to ${}^t\Phi_n$. This provides a decomposition of $\Phi(F)$ into self-transpose submodules, having no isomorphic components in common, which gives in turn an orthogonal decomposition of V, as the following lemma claims.

Lemma 1 *Let $\Phi(F) = \Upsilon_1 \bigoplus \Upsilon_2$ with self-transpose Υ_h, $h = 1, 2$. Assume that no component of Υ_1 is isomorphic to any component of Υ_2, then F decomposes.*

Proof Let $\Upsilon_h \equiv (U_h, W_h^*)$, then $V = U_1 \bigoplus U_2$ and $V^* = W_1^* \bigoplus W_2^*$. Consequently $V = W_1 \bigoplus W_2$, corresponding to the decomposition $\Phi(F) = {}^t\Phi(F) = {}^t\Upsilon_1 \bigoplus {}^t\Upsilon_2$. By the exchange theorem, we have the further decompositions $\Phi(F) = {}^t\Upsilon_1 \bigoplus \Upsilon_2 = \Upsilon_1 \bigoplus {}^t\Upsilon_2$, hence $V = W_1 \bigoplus U_2 = U_1 \bigoplus W_2$. As we have $\bar{f}_1(x), \bar{f}_2(x) \in W_i^*$ for any $x \in U_i$, then for any $y \in W_j$, $j \neq i$, it follows $f_h(x, y) = 0$, that is, the latter decompositions of V are orthogonal. □

In view of the above lemma, indecomposable F correspond to Kronecker modules $\Phi(F)$ isomorphic to either $(\Phi_\varphi)^r$ or $(\Phi^\circ_\varphi)^r$, or $(\Phi_n)^s \bigoplus ({}^{\mathrm{t}}\Phi_n)^s$. Moreover, direct computations on the bases show that $s = 1$ and

$r = 1$ for an indecomposable pair of symmetric forms,
$r = 1, 2$ for an indecomposable pair, where one is symmetric and the other is alternating,
$r = 2$ for an indecomposable pair of alternating bilinear forms,

according to [7], [8], [5] and [6]. Therefore we have

Theorem 2 *Let F be indecomposable. Then the Kronecker module $\Phi(F)$ is isomorphic to either Φ_φ or ${\Phi_\varphi}^\circ$, or $\Phi_n \bigoplus {}^{\mathrm{t}}\Phi_n$.*

Let $\mathcal{U}$, $\mathcal{W}$ be bases such that $\Phi(F)$ is represented by matrices (S_1, S_2), the entries of which are given in (1), and A be the matrix of the rowed coordinates of the vectors in $\mathcal{W}^*$ with respect to $\mathcal{U}^*$. A simultaneous reduction to canonical form is now reached through the condition that the product S_hA, $(h = 1, 2)$, as a representation of F, is symmetric or alternating.

In particular, if both the forms are degenerate, i.e. $\Phi(F) = \Phi_n \bigoplus {}^{\mathrm{t}}\Phi_n$, by definition of Φ_n the bases

$$\mathcal{U} = \{u'_1, \ldots, u'_n, u''_1, \ldots, u''_{n+1}\} \quad \text{and} \quad \mathcal{W} = \{w'_1, \ldots, w'_{n+1}, w''_1, \ldots, w''_n\}$$

of V are such that,

$$\begin{aligned} f_1(u'_i, w'_j) = f_1(w''_i, u''_j) &= \delta_{i,j}, \\ f_2(u'_i, w'_j) = f_2(w''_i, u''_j) &= \delta_{i,j-1}, \end{aligned} \tag{2}$$

where $1 \le i \le n$ and $1 \le j \le n + 1$.

Let (J_1, J_2) be the matrix representation of $\Phi(F)$ with respect to $\mathcal{U}$ and $\mathcal{W}^*$, the entries of which are given by (2). Then, the matrix A fulfills the equations $J_hA = S_h$ $(h = 1, 2)$. The partitions $\mathcal{U} = \{u'_1, \ldots, u'_n\} \cup \{u''_1, \ldots, u''_{n+1}\}$ and $\mathcal{W} = \{w'_1, \ldots, w'_{n+1}\} \cup \{w''_1, \ldots, w''_n\}$ allow one to write the equations $J_hA = S_h$ in blocks as

$$\begin{pmatrix} \mathbf{J}_h & \mathbf{0} \\ \mathbf{0} & {}^{\mathrm{t}}\mathbf{J}_h \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} S_h^{11} & S_h^{12} \\ \varepsilon\,{}^{\mathrm{t}}S_h^{12} & S_h^{22} \end{pmatrix} \qquad \varepsilon = \pm 1, \tag{3}$$

where we put

$$J_h = \begin{pmatrix} \mathbf{J}_h & \mathbf{0} \\ \mathbf{0} & {}^{\mathrm{t}}\mathbf{J}_h \end{pmatrix},$$

and

$$(\mathbf{J}_1, \mathbf{J}_2) = \left(\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix} \right)$$

is the current representation of the Kronecker module Φ_n.

One sees that the products ${}^{\mathrm{t}}\mathbf{J}_h A_{22} = S_h^{22}$ are symmetric or alternating just if $A_{22} = \mathbf{0}$, hence $S_h^{22} = \mathbf{0}$. Furthermore, the square matrix A_{12} cannot be singular, so, up to replacing each of the vectors $u_1'', \ldots, u_{n+1}''$ by a suitable linear combination of themselves, we may assume $\mathcal{U}$ such that $A_{12} = \mathbf{I}_{n+1}$. Therefore, the equations (3) turn into

$$\begin{pmatrix} \mathbf{J}_h & \mathbf{0} \\ \mathbf{0} & Y_h \end{pmatrix} \begin{pmatrix} A_{11} & \mathbf{I}_{n+1} \\ A_{21} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} S_h^{11} & \mathbf{J}_h \\ \varepsilon^{\mathrm{t}}\mathbf{J}_h & \mathbf{0} \end{pmatrix}, \tag{4}$$

for suitable matrices Y_h which play no role for our purposes. Hence F has a representation

$$\left(\begin{pmatrix} T_1 & \mathbf{J}_1 \\ \varepsilon^{\mathrm{t}}\mathbf{J}_1 & \mathbf{0} \end{pmatrix}, \begin{pmatrix} T_2 & \mathbf{J}_2 \\ \varepsilon^{\mathrm{t}}\mathbf{J}_2 & \mathbf{0} \end{pmatrix} \right), \tag{5}$$

where we write T_h instead of S_h^{11}.

Replace now each of the vectors u_i' in $\mathcal{U}$ by $u_i' + \sum_{j=1}^{n+1} c_{ij} u_j''$, then

$$f_1\left(u_r' + \sum_{j=1}^{n+1} c_{rj} u_j'', u_s' + \sum_{j=1}^{n+1} c_{sj} u_j'' \right) = f_1(u_r', u_s') + c_{rs} + c_{sr},$$

$$f_2\left(u_r' + \sum_{j=1}^{n+1} c_{rj} u_j'',\ u_s' + \sum_{j=1}^{n+1} c_{sj} u_j'' \right) = f_2(u_r', u_s') + c_{r,\,s+1} + c_{s,\,r+1},$$

for $r, s = 1, \ldots, n$. Let $\mathrm{char} K \neq 2$, then it is possible to find entries c_{ij} which make the above quantities zero. Let $\mathrm{char} K = 2$, then it is still possible to do that, provided $r \neq s$.

The above arguments can be summarized in the following result, which has been proved by Scharlau [6] for pairs of alternating forms, while Waterhouse [7], [8] proved it for pairs of symmetric forms, but there he made use of other tecniques.

Theorem 3 *Let F be an indecomposable pair of degenerate bilinear forms on a K-vector space V, each being symmetric or alternating. Then, V has odd dimension $2n+1$ over K and F has a representation*

$$\left(\begin{pmatrix} D_1 & \mathbf{J}_1 \\ \varepsilon^{\mathrm{t}}\mathbf{J}_1 & \mathbf{0} \end{pmatrix}, \begin{pmatrix} D_2 & \mathbf{J}_2 \\ \varepsilon^{\mathrm{t}}\mathbf{J}_2 & \mathbf{0} \end{pmatrix} \right)$$

for suitable diagonal matrices D_1, D_2. Moreover, if the characteristic of K is not 2, there exists a representation with $D_1 = D_2 = \mathbf{0}$.

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