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WEAK NONLINEARITY OF GROWTH CURVE MODELS

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ABSTRACT. In a deformation measurement a link between the mean value of an observation vector and coordinates characterizing an investigated object need not be linear. Also functions characterizing the time course of coordinate changes of points need not be linear in parameters. Thus the problem arises whether estimators can be obtained by linear methods or not. Some criteria for a linearization are given in the paper.

1. Introduction

Deformation measurement can be characterized as follows. Several points characterizing a state of an investigated object (dams, bridges, gas holders, etc.) are located on it. Their positions are given by a k-dimensional coordinate vector $\boldsymbol{\beta} \in \mathbb{R}^k$ (k-dimensional Euclidean space). The value of $\boldsymbol{\beta}$ is changing during the investigation and it can be expressed in the form $\{\boldsymbol{\beta}\}_i = \beta_i = \beta_i(t, \gamma_i)$, $i = 1, \dots, k$, where $\beta_i(\cdot, \cdot\cdot)$ is a known function and γ_i is an unknown vector parameter linked with the time varying coordinate β_i and t is the time. The positions of the vector $\boldsymbol{\beta}$ are measured at several time moments $t_1, \dots, t_m$ (epochs), and differences among vectors $\boldsymbol{\beta}(t_1), \dots, \boldsymbol{\beta}(t_m)$ characterized by a time course of the functions $\beta_1(\cdot, \cdot\cdot), \dots, \beta_k(\cdot, \cdot\cdot)$ are a basis for studying a behaviour of the investigated object deformations. (Another model of deformation measurement is investigated in [3].)

Positions of characterizing points, i.e. values $\boldsymbol{\beta}(t_1), \dots, \boldsymbol{\beta}(t_m)$ of the vector $\boldsymbol{\beta}$ at the times $t_1, \dots, t_m$, respectively, are determined in m experiments, each of them characterized by a nonlinear regression model. The models are linearized

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and, by a standard method, estimators $\hat{\boldsymbol{\beta}}(t_1),\dots,\hat{\boldsymbol{\beta}}(t_m)$ from these experiments are determined. Then a problem arises how to estimate parameters γ_i, $i = 1,\dots,k$, occurring in the functions $\beta_i(\cdot,\cdot\cdot)$, $i = 1,\dots,k$.

A standard procedure is the following one. At time points $t_1,\dots,t_m$, approximate values $\boldsymbol{\beta}_{0,i}$, $i = 1,\dots,m$, of the vector $\boldsymbol{\beta}$ (i.e. a position of the characterizing points) are known and the nonlinear regression models are linearized at them, i.e. the linear term in the Taylor series of the model is used only. Thus the estimators $\hat{\boldsymbol{\beta}}(t_i)$, $i = 1,\dots,m$, are given in the form $\hat{\boldsymbol{\beta}}(t_i) = \boldsymbol{\beta}_{0,i} + \delta\hat{\boldsymbol{\beta}}(t_i)$, $i = 1,\dots,m$. The vectors $\hat{\boldsymbol{\beta}}(t_i)$, $i = 1,\dots,m$, are a basis for a determination of the parameters γ_i, $i = 1,\dots,k$. In general the functions $\beta_i(\cdot,\gamma_i)$ are nonlinear in the parameter γ_i and thus again a problem of linearization arises.

Since a utilization of a nonlinear estimation theory in the investigated case is complicated, the linearization of the mentioned regression models and the functions $\beta_i(\cdot,\cdot\cdot)$ seems to be the only possible way. Nevertheless it can lead to nonadmissible biases in estimators of the vectors $\boldsymbol{\beta}(t_i)$, γ_i, $i = 1,\dots,k$, and their variances, i.e. to a nonadequate interpretation of the deformations. In practice it can have far-reaching consequences.

The aim of the paper is to contribute to a solution of the problem in such a way that there are given sufficient conditions under which estimators of the vectors $\boldsymbol{\beta}(t_1),\dots,\boldsymbol{\beta}(t_m)$, and the parameters $\gamma_1,\dots,\gamma_k$, can be obtained in the framework of linear models. The conditions are based on measures of nonlinearity inspired by Bates and Watts [1].

2. Notations and auxiliary statements

Let $\boldsymbol{Y}_i$ be an observation vector at the time t_i, $i = 1,\dots,m$, and let $\boldsymbol{Y}_i \sim N_n\{\boldsymbol{f}[\boldsymbol{\beta}(t_i,\gamma)],\boldsymbol{\Sigma}\}$, i.e. $\boldsymbol{Y}_i$ is normally distributed with the mean value $\boldsymbol{f}[\boldsymbol{\beta}(t_i,\gamma)]$ and the covariance matrix $\boldsymbol{\Sigma}$. The known function $\boldsymbol{f}[\boldsymbol{\beta}(t_i,\gamma)]$ is of the form

$$\{\boldsymbol{f}[\boldsymbol{\beta}(t_i,\gamma)]\}_s = f_s[\beta_1(t_i,\gamma_1),\dots,\beta_k(t_i,\gamma_k)]\,, \quad s = 1,\dots,n\,.$$

The vectors $\boldsymbol{Y}_1,\dots,\boldsymbol{Y}_m$ are stochastically independent.

In practice, in horizontal deformations the functions $f_i(\boldsymbol{\beta})$ are of two kinds. When a distance between points $P(\beta_r,\beta_{r+1})$ and $P(\beta_s,\beta_{s+1})$ is measured,

$$f_i(\boldsymbol{\beta}) = \sqrt{(\beta_s-\beta_r)^2+(\beta_{s+1}-\beta_{r+1})^2}\,.$$

When an angle at a point $P(\beta_r,\beta_{r+1})$ between directions on points $P(\beta_s,\beta_{s+1})$ and $P(\beta_t,\beta_{t+1})$ is measured, then

$$f_j(\boldsymbol{\beta}) = \operatorname{arctg}\left(\frac{\beta_{t+1}-\beta_{r+1}}{\beta_t-\beta_r}\right) - \operatorname{arctg}\left(\frac{\beta_{s+1}-\beta_{r+1}}{\beta_s-\beta_r}\right)\,.$$

The functions $\beta_i(\cdot,\cdot\cdot)$ are either chosen after several experiments, when a time course of values $\hat{\beta}_i(t_1),\dots,\hat{\beta}_i(t_m)$ hint a type of the function, or it is known from a theory of investigated deformations what kind of function have to be chosen.

In some cases, e.g. in vertical deformation measurements, experiments for a determination of the parameters $\boldsymbol{\beta}(t_1),\dots,\boldsymbol{\beta}(t_m)$ can be characterized by a linear regression model, i.e.

$$\mathbf{Y}_i = \mathbf{F}\boldsymbol{\beta}(t_i) + \boldsymbol{\varepsilon}_i\,, \qquad \mathrm{Var}(\mathbf{Y}_i) = \boldsymbol{\Sigma}$$

and the functions can be given in the form $\beta_i(t) = \sum\limits_{r=1}^{s} \gamma_{i,r}\phi_r(t)$, where $\gamma_{i,r}$ are unknown parameters and $\phi_r(\cdot)$ are known, e.g. polynomials. When the notation $\gamma_i' = (\gamma_{i,1},\dots,\gamma_{i,s})$, $i = 1,\dots,k$, is used, then we obtain well-known linear growth curve model

$$(\mathbf{Y}_1,\dots,\mathbf{Y}_m) = \mathbf{F}\begin{pmatrix}\gamma_1' \\ \vdots \\ \gamma_k'\end{pmatrix}\begin{pmatrix}\phi_1(t_1), & \phi_1(t_2), & \dots, & \phi_1(t_m) \\ \dots & \dots & \dots & \dots \\ \phi_s(t_1), & \phi_s(t_2), & \dots, & \phi_s(t_m)\end{pmatrix} + \boldsymbol{\varepsilon}$$

where $\boldsymbol{\varepsilon}$ is an error matrix.

In the following let $\mathrm{vec}(\mathbf{A}_{m,n}) = \begin{pmatrix}\mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n\end{pmatrix}$, where $\mathbf{A}_{m,n}$ is an $m \times n$ matrix with the columns $\mathbf{a}_1,\dots,\mathbf{a}_n$.

The symbol $\otimes$ denotes the Kronecker multiplication, i.e

$$\begin{pmatrix}a_{1,1}, & a_{1,2} \\ a_{2,1}, & a_{2,2}\end{pmatrix} \otimes \mathbf{B} = \begin{pmatrix}a_{1,1}\mathbf{B}, & a_{1,2}\mathbf{B} \\ a_{2,1}\mathbf{B}, & a_{2,2}\mathbf{B}\end{pmatrix}.$$

The model which is in our focus can be written now in the form

$$\underline{\mathbf{Y}} = (\mathbf{Y}_1,\dots,\mathbf{Y}_m) = (\boldsymbol{\beta}(t_1),\dots,\boldsymbol{\beta}(t_m)) + \boldsymbol{\varepsilon}\,, \qquad \mathrm{Var}[\mathrm{vec}(\underline{\mathbf{Y}})] = \mathbf{I}\otimes\boldsymbol{\Sigma}\,,$$
$$\boldsymbol{\beta}(t_i) = \left(\beta_1(t_i,\gamma_1),\dots,\beta_k(t_i,\gamma_k)\right)'\,, \qquad i = 1,\dots,m$$

($\mathbf{I}_{m,m}$ is an $m \times m$ identity matrix).

The symbol $\mathbf{P}_{\mathbf{F}}^{\boldsymbol{\Sigma}^{-1}}$ denotes the projection matrix on the subspace $\mathcal{M}(\mathbf{F}) = \{\mathbf{F}\mathbf{u} : \mathbf{u} \in \mathbb{R}^k\}$ in the norm $\|\mathbf{x}\|_{\boldsymbol{\Sigma}^{-1}} = \sqrt{\mathbf{x}'\boldsymbol{\Sigma}^{-1}\mathbf{x}}$ given by the positive definite (p.d.) matrix $\boldsymbol{\Sigma}^{-1}$ and $\mathbf{M}_{\mathbf{F}}^{\boldsymbol{\Sigma}^{-1}} = \mathbf{I} - \mathbf{P}_{\mathbf{F}}^{\boldsymbol{\Sigma}^{-1}}$.

In the following the Taylor series of the second order will be assumed to be sufficient to express the functions $f_s(\cdot)$, $s = 1,\dots,n$, and $\beta_j(t,\cdot\cdot)$, $j = 1,\dots,k$.

It means

$$\begin{aligned}
\boldsymbol{f}[\boldsymbol{\beta}(t,\boldsymbol{\gamma})] &= \boldsymbol{f}[\boldsymbol{\beta}(t,\boldsymbol{\gamma}^{(0)})] + \mathbf{F}(t)\mathbf{G}(t)\delta\boldsymbol{\gamma} + \frac{1}{2}\boldsymbol{\kappa}(t) + \frac{1}{2}\mathbf{F}(t)\boldsymbol{\tau}(t),\\
\mathbf{F}(t) &= \partial\boldsymbol{f}(\boldsymbol{u})/\partial\boldsymbol{u}'\big|_{\boldsymbol{u}=\boldsymbol{\beta}(t,\boldsymbol{\gamma}^{(0)})},\\
\mathbf{G}(t) &= \partial\boldsymbol{\beta}(\boldsymbol{v})/\partial\boldsymbol{v}'\big|_{\boldsymbol{v}=\boldsymbol{\gamma}^{(0)}},\\
\boldsymbol{\kappa}(t) &= \big(\kappa_1(t),\dots,\kappa_n(t)\big)',\\
\kappa_i(t) &= \delta\boldsymbol{\gamma}'\mathbf{G}'(t)\partial^2 f_i(\boldsymbol{u})/\partial\boldsymbol{u}\partial\boldsymbol{u}'\big|_{\boldsymbol{u}=\boldsymbol{\beta}(t,\boldsymbol{\gamma}^{(0)})}\mathbf{G}(t)\delta\boldsymbol{\gamma}, \qquad i=1,\dots,n,\\
\boldsymbol{\tau}(t) &= \big(\tau_1(t),\dots,\tau_k(t)\big)',\\
\tau_j(t) &= \delta\boldsymbol{\gamma}'\partial^2\beta_j(t,\boldsymbol{v})/\partial\boldsymbol{v}\partial\boldsymbol{v}'\big|_{\boldsymbol{v}=\boldsymbol{\gamma}^{(0)}}\delta\boldsymbol{\gamma}, \qquad j=1,\dots,k.
\end{aligned}$$

After m epochs the model can be written in the form

$$E[\operatorname{vec}(\boldsymbol{Y}_1,\dots,\boldsymbol{Y}_m)] = \boldsymbol{f}_0 + \mathbf{FG}\delta\boldsymbol{\gamma} + \frac{1}{2}\boldsymbol{\kappa} + \frac{1}{2}\mathbf{F}\boldsymbol{\tau}, \tag{1}$$

$$\begin{aligned}
\boldsymbol{f}_0 &= \Big[\boldsymbol{f}'[\boldsymbol{\beta}(t_1,\boldsymbol{\gamma}^{(0)})],\dots,\boldsymbol{f}'[\boldsymbol{\beta}(t_m,\boldsymbol{\gamma}^{(0)})]\Big]',\\
\mathbf{F} &= \begin{pmatrix} \mathbf{F}(t_1), & \mathbf{0}, & \dots, & \mathbf{0}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{F}(t_2), & \dots, & \mathbf{0}, & \mathbf{0}\\ \dots & \dots & \dots & \dots & \dots\\ \mathbf{0}, & \mathbf{0}, & \dots, & \mathbf{0}, & \mathbf{F}(t_m)\end{pmatrix},\\
\mathbf{G} &= \big(\mathbf{G}'(t_1),\dots,\mathbf{G}'(t_m)\big)',\\
\mathbf{G}(t_i) &= \begin{pmatrix} \partial\boldsymbol{\beta}_1(t_i,\boldsymbol{\gamma}_1)/\partial\boldsymbol{\gamma}_1', & \mathbf{0}, & \mathbf{0}, & \dots, & \mathbf{0}\\ \mathbf{0}, & \partial\boldsymbol{\beta}_2(t_i,\boldsymbol{\gamma}_2)/\partial\boldsymbol{\gamma}_2', & \mathbf{0}, & \dots, & \mathbf{0}\\ \dots & \dots & \dots & \dots & \dots\\ \mathbf{0}, & \mathbf{0}, & \mathbf{0}, & \dots, & \partial\boldsymbol{\beta}_k(t_i,\boldsymbol{\gamma}_k)/\partial\boldsymbol{\gamma}_k'\end{pmatrix},
\end{aligned}$$

$$\delta\boldsymbol{\gamma} = (\delta\boldsymbol{\gamma}_1',\dots,\delta\boldsymbol{\gamma}_k')', \quad \boldsymbol{\kappa} = \big(\boldsymbol{\kappa}'(t_1),\dots,\boldsymbol{\kappa}'(t_m)\big)',$$

$$\boldsymbol{\tau} = \big(\boldsymbol{\tau}'(t_1),\dots,\boldsymbol{\tau}'(t_m)\big)',$$

$$\operatorname{Var}[\operatorname{vec}(\boldsymbol{Y}_1,\dots,\boldsymbol{Y}_m)] = \mathbf{I}\otimes\boldsymbol{\Sigma}.$$

The model (1) is given by its mean value

$$\boldsymbol{f}(\boldsymbol{\beta}) = \boldsymbol{f}_0 + \mathbf{FG}\delta\boldsymbol{\gamma} + \frac{1}{2}\boldsymbol{\kappa} + \frac{1}{2}\mathbf{F}\boldsymbol{\tau}$$

and its linear version is

$$\boldsymbol{f}(\boldsymbol{\beta}) = \boldsymbol{f}_0 + \mathbf{FG}\delta\boldsymbol{\gamma}. \tag{2}$$

A link between the mean value of the observation vectors $\boldsymbol{Y}_1,\dots,\boldsymbol{Y}_m$ and the parameter $\boldsymbol{\gamma}$ is expressed here. The parameter $\boldsymbol{\gamma}$ is unbiasedly estimable in this model if and only if $\mathrm{r}(\mathbf{FG}) = s$ and the matrix Σ is p.d. Thus the model cannot be used until the number of measured epochs multiplied by the dimension k of the parameter $\boldsymbol{\beta}$ is larger than the dimension of the parameter $\boldsymbol{\gamma}$. The model of first m $(mk > s)$ epochs must be considered in the form

$$E[(\mathrm{vec}(\underline{\boldsymbol{Y}})] = \boldsymbol{f} = \boldsymbol{f}_0 + \mathbf{F}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\beta}). \tag{3}$$

This model (usually in its linear version) is used for the determination of the vectors $\hat{\boldsymbol{\beta}}(t_1), \dots, \hat{\boldsymbol{\beta}}(t_m)$.

When a determination of the parameters $\gamma_1, \dots, \gamma_k$ comes into consideration, then the relation between $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ must be taken into account, i.e.

$$\boldsymbol{\beta} = \boldsymbol{\beta}_0 + \mathbf{G}\delta\boldsymbol{\gamma} + \frac{1}{2}\boldsymbol{\tau}(\delta\boldsymbol{\gamma}). \tag{4}$$

The linear versions of the two last models are

$$\boldsymbol{f} = \boldsymbol{f}_0 + \mathbf{F}\delta\boldsymbol{\beta} \tag{5}$$

and

$$\boldsymbol{\beta} = \boldsymbol{\beta}_0 + \mathbf{G}\delta\boldsymbol{\gamma}. \tag{6}$$

It is to be emphasized that a possibility to use the linearized version, i.e. the models (5) and (6) leads to the equivalence between estimators of the parameter $\boldsymbol{\gamma}$ from the model (2) and from the model

$$\begin{pmatrix} \hat{\boldsymbol{\beta}}(t_1) \\ \vdots \\ \hat{\boldsymbol{\beta}}(t_m) \end{pmatrix} - \boldsymbol{\beta}_0 \sim_{mk} \left(\mathbf{G}\delta\boldsymbol{\gamma}, \begin{pmatrix} \mathrm{Var}[\hat{\boldsymbol{\beta}}(t_1)], & \mathbf{0}, & \dots, & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0}, & \mathbf{0}, & \dots, & \mathrm{Var}[\hat{\boldsymbol{\beta}}(t_m)] \end{pmatrix} \right)$$

(cf. Lemma 3.3). From the viewpoint of practice, it is very important and thus, except another reasons, a problem of linearization becomes important as well.

Let the rank of the matrix $\mathbf{F}(t_i)$ be $r[\mathbf{F}(t_i)] = k < n$, i.e. $\mathrm{r}(\mathbf{F}) = mk$, $\mathrm{r}(\mathbf{G}_{mk,s}) = s < mk$, $\mathrm{r}(\mathbf{FG}) = s < mn$ and let the matrix Σ be positive definite (p.d.) and known. In the case $\Sigma = \sigma^2\mathbf{V}$, the known matrix $\mathbf{V}$ is p.d. and $\sigma^2 \in (0,\infty)$ is an unknown parameter. It is assumed that the matrix Σ is either known, or it is of the form $\Sigma = \sigma^2\mathbf{V}$ and the matrix $\mathbf{V}$ is known.

If the number r of the epochs is smaller than s/k, then the vector parameter $\boldsymbol{\gamma}$ cannot be estimated on the basis $\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_r$, however, the vectors $\delta\boldsymbol{\beta}(t_1), \dots, \delta\boldsymbol{\beta}(t_r)$ can be estimated.

Thus it seems to be natural to determine the estimators $\delta\hat{\boldsymbol{\beta}}(t_i)$ after each epoch $i = 1, \dots, r$, until the number of epochs enables us to determine the estimator $\delta\hat{\boldsymbol{\gamma}}$. These estimators can be calculated either on the basis of the estimators $\delta\hat{\boldsymbol{\beta}}(t_1), \dots, \delta\hat{\boldsymbol{\beta}}(t_m)$, or on the basis of the observation vectors $\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_m$. It will be shown that these estimators are the same. (Lemma 3.3).

The main problem is whether a calculation can be proceeded in the framework of the linearized version (2) of the model (1), i.e. whether the terms $\boldsymbol{\kappa}$ and $\boldsymbol{\tau}$ can be neglected. The solution will be given in the form of some sufficient conditions which are inspired by measures of nonlinearity introduced by Bates and Watts [1].

The following lemmas will be necessary for the consideration. In these lemmas, for the sake of simplicity the model

$$\boldsymbol{Y} \sim N_n\left(\boldsymbol{f}_0 + \mathbf{F}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\omega}(\delta\boldsymbol{\beta}), \boldsymbol{\Sigma}\right)$$

is considered. Here $\boldsymbol{f}_0$ is a given vector, $\mathbf{F}$ is an $n\times k$ matrix with the rank $\mathrm{r}(\mathbf{F}) = k < n$, $\boldsymbol{\omega}(\delta\boldsymbol{\beta}) = \left(\omega_1(\delta\boldsymbol{\beta}),\dots,\omega_n(\delta\boldsymbol{\beta})\right)'$, $\omega_i(\delta\boldsymbol{\beta}) = \delta\boldsymbol{\beta}'\mathbf{H}_i\delta\boldsymbol{\beta}$, $i = 1,\dots,n$, $\mathbf{H}_i$, $i = 1,\dots,n$, are given $k\times k$ symmetric matrices and $\boldsymbol{\Sigma}$ is a given p.d. matrix. (It means that a quadratization of a nonlinear model $\boldsymbol{Y} \sim_n \left(\boldsymbol{f}(\boldsymbol{\beta}), \boldsymbol{\Sigma}\right)$, $\boldsymbol{\beta} \in \mathbb{R}^k$ is under consideration.)

LEMMA 2.1. *Let the symbol $\chi^2_{n-k}(\delta)$ mean the random variable with chi-square distribution with $n-k$ degrees of freedom and with the parameter of noncentrality equal to δ. If*

$$\delta\boldsymbol{\beta}'\mathbf{C}\delta\boldsymbol{\beta} \le \frac{2\sqrt{\delta_{\max}}}{K^{(\mathrm{int})}},$$

where $\mathbf{C} = \mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}$, $\delta_{\max}$ is a solution of the equation

$$P\left\{\chi^2_{n-k}(\delta_{\max}) \ge \chi^2_{n-k}(1-\alpha)\right\} = \alpha + \varepsilon,$$

($\chi^2_{n-k}(1-\alpha)$ is the $(1-\alpha)$-quantile of the central chi-square distribution with $n-k$ degrees of freedom), then

$$P\left\{(\boldsymbol{Y} - \boldsymbol{f}_0)'\boldsymbol{\Sigma}^{-1}\mathbf{M}_{\mathbf{F}}^{\boldsymbol{\Sigma}^{-1}}(\boldsymbol{Y} - \boldsymbol{f}_0) \le \chi^2_{n-k}(1-\alpha)\right\} \ge 1 - (\alpha + \varepsilon).$$

Here $K^{(\mathrm{int})}$ is the Bates and Watts intrinsic measure of nonlinearity (cf. [1]),

$$K^{(\mathrm{int})} = \sup\left\{\frac{\sqrt{\boldsymbol{\omega}'(\delta\boldsymbol{\beta})\boldsymbol{\Sigma}^{-1}\mathbf{M}_{\mathbf{F}}^{\boldsymbol{\Sigma}^{-1}}\boldsymbol{\omega}(\delta\boldsymbol{\beta})}}{\delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\delta\boldsymbol{\beta}} : \delta\boldsymbol{\beta} \in \mathbb{R}^k\right\}.$$

Proof. See [2] and [4]. □

LEMMA 2.2. *If*

$$\delta\boldsymbol{\beta}'\mathbf{C}\delta\boldsymbol{\beta} \le \frac{2c}{K^{(\mathrm{par})}},$$

then

$$(\forall \boldsymbol{h} \in \mathbb{R}^k)\left(\left|E\left[\boldsymbol{h}'\mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{Y} - \boldsymbol{f})\right] - \boldsymbol{h}'\delta\boldsymbol{\beta}\right| \le \sqrt{\boldsymbol{h}'\mathbf{C}^{-1}\boldsymbol{h}}\right).$$

Here

$$K^{(\mathrm{par})} = \sup\left\{\frac{\sqrt{\omega'(\delta\boldsymbol{\beta})\boldsymbol{\Sigma}^{-1}\mathbf{P}_{\mathbf{F}}^{\boldsymbol{\Sigma}^{-1}}\omega(\delta\boldsymbol{\beta})}}{\delta\boldsymbol{\beta}'\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\delta\boldsymbol{\beta}} : \ \delta\boldsymbol{\beta}\in\mathbb{R}^k\right\}$$

is the Bates and Watts parametric measure of nonlinearity.

Proof. See [2] and [4]. □

Remark 2.1. If $\hat{\sigma}^2 = \boldsymbol{Y}'(\mathbf{M}_{\mathbf{F}}\mathbf{V}\mathbf{M}_{\mathbf{F}})^{+}\boldsymbol{Y}/(n-k)$, then $E(\hat{\sigma}^2) = \sigma^2 + \sigma^2\delta/(n-k)$ and thus approximately

$$\sqrt{\sigma^2 + \sigma^2\frac{\delta}{n-k}} = \sigma\Big(1 + \frac{\delta}{n-k}\Big)^{1/2} \approx \sigma + \frac{\sigma\delta}{2(n-k)}\,.$$

Lemma 2.3. *Let $\boldsymbol{\Sigma} = \sigma^2\mathbf{V}$, $\mathbf{V}$ be a given $n\times n$ matrix and σ^2 is an unknown parameter $\sigma^2\in(0,\infty)$. Let $\mathbf{C}_0 = \mathbf{F}'\mathbf{V}^{-1}\mathbf{F}$.*

If

$$\delta\boldsymbol{\beta}'\mathbf{C}_0\delta\boldsymbol{\beta} \le \frac{2\sqrt{2(n-k)\varepsilon}}{K_0^{(\mathrm{int})}}\,,$$

then

$$\sigma\delta/\big[2(n-k)\big] \le \varepsilon\,,$$

where

$$\delta = \frac{1}{4}\omega'(\delta\boldsymbol{\beta})(\mathbf{M}_{\mathbf{F}}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{F}})^{+}\omega(\delta\boldsymbol{\beta})\,,\qquad K^{(\mathrm{int})} = \sigma K_0^{(\mathrm{int})}\,.$$

Here the value ε is chosen by a user in order to bound the bias in the estimator of σ.

Proof. See in [2] and [4]. □

Remark 2.2. The linearization regions

$$\mathcal{O}_a = \left\{\delta\boldsymbol{\beta} : \ \delta\boldsymbol{\beta}'\mathbf{C}\delta\boldsymbol{\beta} \le \frac{2\sqrt{\delta_{\max}}}{K^{(\mathrm{int})}}\right\} \qquad \text{(Lemma 2.1)},$$

$$\mathcal{O}_b = \left\{\delta\boldsymbol{\beta} : \ \delta\boldsymbol{\beta}'\mathbf{C}\delta\boldsymbol{\beta} \le \frac{2c}{K^{(\mathrm{par})}}\right\} \qquad \text{(Lemma 2.2)},$$

$$\mathcal{O}_d = \left\{\delta\boldsymbol{\beta} : \ \delta\boldsymbol{\beta}'\mathbf{C}_0\delta\boldsymbol{\beta} \le \frac{2\sqrt{2(n-k)\varepsilon}}{K_0^{(\mathrm{int})}}\right\} \qquad \text{(Lemma 2.3)}$$

are of practical use in such a case only, if the confidence region for $\delta\boldsymbol{\beta}$, i.e.

$$\mathcal{E}_{1-\alpha} = \big\{\delta\boldsymbol{\beta} : \ (\delta\boldsymbol{\beta} - \delta\hat{\boldsymbol{\beta}})'\mathbf{C}(\delta\boldsymbol{\beta} - \delta\hat{\boldsymbol{\beta}}) \le \chi_k^2(1-\alpha)\big\}$$

for a sufficiently small α, is included in them. For example in the case $\mathcal{O}_a$ it means

$$\chi_k^2(1-\alpha) \ll \frac{2\sqrt{\delta_{\max}}}{K^{(\mathrm{int})}}\,.$$

If this strong inequality is valid for $\mathcal{O}_a$, then the model has a weak nonlinearity in the intrinsic curvature. If $\chi_k^2(1-\alpha) \ll \frac{2c}{K^{(\mathrm{par})}}$, then the model has a weak nonlinearity for the bias of the estimator of the vector parameter $\boldsymbol{\beta}$. If $\sigma^2\chi_{n-k}^2(1-\alpha) \ll \frac{2\sqrt{2(n-k)\varepsilon}}{K_0^{(\mathrm{int})}}$, then the model has a weak nonlinearity for the bias of the estimator of the parameter σ.

3. Solution in linear model

The notation BLUE (best linear unbiased estimator) in the regular model $\boldsymbol{Y} \sim_n (\mathbf{F}\boldsymbol{\beta}, \boldsymbol{\Sigma})$, $\boldsymbol{\beta} \in \mathbb{R}^k$ (i.e. $\mathrm{r}(\mathbf{F}) = k < n$ and $\boldsymbol{\Sigma}$ is p.d.) means the estimator $\hat{\boldsymbol{\beta}} = (\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F})^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}$. If $\boldsymbol{\Sigma} = \sigma^2\mathbf{V}$, then $\hat{\boldsymbol{\beta}} = (\mathbf{F}'\mathbf{V}^{-1}\mathbf{F})^{-1}\mathbf{F}'\mathbf{V}^{-1}\boldsymbol{Y}$.

The following two lemmas are well known and therefore they are given without proofs.

LEMMA 3.1. *The BLUEs of the vectors $\delta\boldsymbol{\beta}(t_1),\dots,\delta\boldsymbol{\beta}(t_m)$ in the model* (5) *are*

$$\begin{aligned}\delta\hat{\boldsymbol{\beta}}(t_i) &= \left[\mathbf{F}'(t_i)\boldsymbol{\Sigma}^{-1}\mathbf{F}(t_i)\right]^{-1}\mathbf{F}'(t_i)\boldsymbol{\Sigma}^{-1}\{\boldsymbol{Y}_i - \boldsymbol{f}[\boldsymbol{\beta}(t_i,\boldsymbol{\gamma}^{(0)})]\}\\ &= \left[\mathbf{F}'(t_i)\mathbf{V}^{-1}\mathbf{F}(t_i)\right]^{-1}\mathbf{F}'(t_i)\mathbf{V}^{-1}\{\boldsymbol{Y}_i - \boldsymbol{f}[\boldsymbol{\beta}(t_i,\boldsymbol{\gamma}^{(0)})]\}\,, \qquad i=1,\dots,m\,.\end{aligned}$$

LEMMA 3.2. *The BLUE of $\delta\boldsymbol{\gamma}$ in* (2) *is*

$$\begin{aligned}\delta\hat{\boldsymbol{\gamma}} &= \left[\mathbf{G}'\mathbf{F}'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{F}\mathbf{G}\right]^{-1}\mathbf{G}'\mathbf{F}'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\,\mathrm{vec}\Big((\boldsymbol{Y}_1,\dots,\boldsymbol{Y}_m)\\ &\qquad - \{\boldsymbol{f}[\boldsymbol{\beta}(t_1,\boldsymbol{\gamma}^{(0)})],\dots,\boldsymbol{f}[\boldsymbol{\beta}(t_m,\boldsymbol{\gamma}^{(0)})]\}\Big)\\ &= \left[\sum_{i=1}^m \mathbf{G}'(t_i)\mathbf{F}'(t_i)\boldsymbol{\Sigma}^{-1}\mathbf{F}(t_i)\mathbf{G}(t_i)\right]^{-1}\sum_{i=1}^m \mathbf{G}'(t_i)\mathbf{F}'(t_i)\boldsymbol{\Sigma}^{-1}\times\\ &\qquad\times\{\boldsymbol{Y}_i - \boldsymbol{f}[\boldsymbol{\beta}(t_i,\boldsymbol{\gamma}^{(0)})]\}\,.\end{aligned}$$

LEMMA 3.3. *The BLUE $\delta\hat{\hat{\boldsymbol{\gamma}}}$ of $\delta\boldsymbol{\gamma}$ in the model* (6) *based on the estimators $\delta\hat{\boldsymbol{\beta}}(t_1),\dots,\delta\hat{\boldsymbol{\beta}}(t_m)$ is $\delta\hat{\boldsymbol{\gamma}}$ from Lemma* 3.2.

Proof. Let $\mathbf{C}(t_i) = \mathbf{F}'(t_i)\Sigma^{-1}\mathbf{F}(t_i)$, $i = 1,\dots,m$, $\mathbf{C} = \mathbf{F}'(\mathbf{I}\otimes\Sigma^{-1})\mathbf{F}$. In the linear models (5) and (6) $\delta\gamma = \mathbf{F}\begin{pmatrix}\delta\beta(t_1)\\ \vdots\\ \delta\beta(t_m)\end{pmatrix}$. Thus

$$\begin{pmatrix}\delta\hat\beta(t_1)\\ \vdots\\ \delta\hat\beta(t_m)\end{pmatrix} \sim N_{mk}\left(\begin{pmatrix}\delta\beta(t_1)\\ \vdots\\ \delta\beta(t_m)\end{pmatrix}, \begin{pmatrix}\mathbf{C}^{-1}(t_1), & 0, & \dots, & 0\\ \dots & \dots & \dots & \dots\\ 0, & 0, & \dots, & \mathbf{C}^{-1}(t_m)\end{pmatrix}\right)$$
$$\sim N_{mk}(\mathbf{G}\delta\gamma, \mathbf{C}^{-1})$$

and

$$\begin{aligned}\delta\hat{\hat\gamma} &= (\mathbf{G}'\mathbf{C}\mathbf{G})^{-1}\mathbf{G}'\mathbf{C}\begin{pmatrix}\delta\hat\beta(t_1)\\ \vdots\\ \delta\hat\beta(t_m)\end{pmatrix}\\ &= \left[\sum_{i=1}^m \mathbf{G}'(t_i)\mathbf{F}'(t_i)\Sigma^{-1}\mathbf{F}(t_i)\mathbf{G}(t_i)\right]^{-1}\sum_{i=1}^m \mathbf{G}'(t_i)\mathbf{C}(t_i)\delta\hat\beta(t_i)\\ &= \left[\sum_{i=1}^m \mathbf{G}'(t_i)\mathbf{F}'(t_i)\Sigma^{-1}\mathbf{F}(t_i)\mathbf{G}(t_i)\right]^{-1}\sum_{i=1}^m \mathbf{G}'(t_i)\mathbf{C}\mathbf{C}^{-1}(t_i)\mathbf{F}'(t_i)\Sigma^{-1}\times\\ &\qquad\times\left\{\boldsymbol{Y}_i - \boldsymbol{f}[\beta(t_i,\gamma^{(0)})]\right\} = \delta\hat\gamma .\end{aligned}$$

□

In the following lemma several estimators of the parameter σ^2 are given. These estimators are best in the considered models in the following sense. They are unbiased and if the observation vector $\boldsymbol{Y}$ is normally distributed, they have the smallest variance among all unbiased estimators in the considered model. (Cf. also [4; Theorem 4.1.1].)

Lemma 3.4. *The best unbiased estimator $\hat\sigma^2$ of σ^2 is*

(i) *in the model* (5),

$$\hat\sigma^2(5) = [\mathrm{vec}(\underline{\boldsymbol{Y}} - \underline{\boldsymbol{f}})]'[\mathbf{M}_{\mathbf{F}}(\mathbf{I}\otimes\mathbf{V})\mathbf{M}_{\mathbf{F}}]^{+}\,\mathrm{vec}(\underline{\boldsymbol{Y}} - \underline{\boldsymbol{f}})/[m(n-k)],$$

where $\underline{\boldsymbol{Y}} = (\boldsymbol{Y}_1,\dots,\boldsymbol{Y}_m)$, $\underline{\boldsymbol{f}} = (\boldsymbol{f}[\beta(t_1,\gamma^{(0)})],\dots,\boldsymbol{f}[\beta(t_m,\gamma^{(0)})])$,

(ii) *in the model* (2),

$$\hat\sigma^2(2) = [\mathrm{vec}(\underline{\boldsymbol{Y}} - \underline{\boldsymbol{f}})]'[\mathbf{M}_{\mathbf{FG}}(\mathbf{I}\otimes\mathbf{V})\mathbf{M}_{\mathbf{FG}}]^{+}\,\mathrm{vec}(\underline{\boldsymbol{Y}} - \underline{\boldsymbol{f}})/(mn-s),$$

(iii) *in the model* $\delta\hat\beta \sim N_{mk}(\mathbf{G}\delta\gamma, \sigma^2\mathbf{C}_0^{-1})$, $\delta\hat\beta = \begin{pmatrix}\delta\hat\beta(t_1)\\ \vdots\\ \delta\hat\beta(t_m)\end{pmatrix}$,

$$\hat\sigma^2(\mathrm{iii}) = \delta\hat\beta'\{\mathbf{M}_{\mathbf{G}}\mathbf{C}_0^{-1}\mathbf{M}_{\mathbf{G}}\}^{+}\delta\hat\beta/(mk-s);$$

here $\mathbf{C}_0 = \mathbf{F}'(\mathbf{I}\otimes\mathbf{V}^{-1})\mathbf{F}$.

The estimators $\hat{\sigma}^2(\mathrm{iii})$ *and* $\hat{\sigma}^2(5)$ *are stochastically independent and*

$$\hat{\sigma}^2(2) = \big[(nk-s)\hat{\sigma}^2(\mathrm{iii}) + m(n-k)\hat{\sigma}^2(5)\big]/(mn-s)\,.$$

Proof. In the model $\boldsymbol{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V})$, $\boldsymbol{\beta}\in\mathbb{R}^k$, $\mathrm{r}(\mathbf{X}) = k < n$, $\mathbf{V}$ is p.d., the well-known formula

$$\hat{\sigma}^2 = \boldsymbol{Y}'(\mathbf{M_X}\mathbf{V}\mathbf{M_X})^+\boldsymbol{Y}/(n-k)$$

has been used for the estimation of σ^2 (here $(\mathbf{M_X}\mathbf{V}\mathbf{M_X})^+$ is the Moore-Penrose inverse of the matrix $\mathbf{M_X}\mathbf{V}\mathbf{M_X}$, cf. [6]). Further $\boldsymbol{Y}'\mathbf{A}_{n,n}\boldsymbol{Y}$ and $\boldsymbol{Y}'\mathbf{B}_{n,n}\boldsymbol{Y}$ are independent if and only if $\mathbf{AVB} = \mathbf{0}$ (cf. [5]).

Since

$$\hat{\sigma}^2(\mathrm{iii}) = \big[\mathrm{vec}(\underline{\boldsymbol{Y}} - \underline{\boldsymbol{f}})\big]'(\mathbf{I}\otimes\mathbf{V}^{-1})\mathbf{F}\big\{\mathbf{M_G}\big[\mathbf{F}'(\mathbf{I}\otimes\mathbf{V}^{-1})\mathbf{F}\big]^{-1}\mathbf{M_G}\big\}^+ \times \\ \times\, \mathbf{F}'(\mathbf{I}\otimes\mathbf{V}^{-1})\,\mathrm{vec}(\underline{\boldsymbol{Y}} - \underline{\boldsymbol{f}})/(mk-s)\,,$$

it is sufficient to prove

$$(\mathbf{I}\otimes\mathbf{V}^{-1})\mathbf{F}\mathbf{C}_0^{-1}\big\{\mathbf{M_G}\big[\mathbf{F}'(\mathbf{I}\otimes\mathbf{V}^{-1})\mathbf{F}\big]^{-1}\mathbf{M_G}\big\}^+\mathbf{F}'(\mathbf{I}\otimes\mathbf{V}^{-1})\,\times \\ \times\,(\mathbf{I}\otimes\mathbf{V})\big[\mathbf{M_F}(\mathbf{I}\otimes\mathbf{V})\mathbf{M_F}\big]^+ = \mathbf{0}\,.$$

The last equality is obviously valid, since

$$\mathbf{F}'(\mathbf{I}\otimes\mathbf{V}^{-1})(\mathbf{I}\otimes\mathbf{V})\big[\mathbf{M_F}(\mathbf{I}\otimes\mathbf{V})\mathbf{M_F}\big]^+ = \mathbf{0}\,.$$

Thus the estimators $\hat{\sigma}^2(\mathrm{iii})$ and $\hat{\sigma}^2(5)$ are independent. Further

$$\begin{aligned}
&\big[\mathbf{M_F}(\mathbf{I}\otimes\mathbf{V})\mathbf{M_F}\big]^+ + (\mathbf{I}\otimes\mathbf{V}^{-1})\mathbf{F}\mathbf{C}_0^{-1}(\mathbf{M_G}\mathbf{C}_0^{-1}\mathbf{M_G})^+\mathbf{C}_0^{-1}\mathbf{F}(\mathbf{I}\otimes\mathbf{V}^{-1})\\
&= (\mathbf{I}\otimes\mathbf{V}^{-1}) - (\mathbf{I}\otimes\mathbf{V}^{-1})\mathbf{F}\mathbf{C}_0^{-1}\mathbf{F}'(\mathbf{I}\otimes\mathbf{V}^{-1})\\
&\qquad + \mathbf{I}\otimes\mathbf{V}^{-1}\mathbf{F}\mathbf{C}_0^{-1}\big[\mathbf{C}_0 - \mathbf{C}_0\mathbf{G}(\mathbf{G}'\mathbf{C}_0\mathbf{G})^{-1}\mathbf{G}'\mathbf{C}_0\big]\mathbf{C}_0^{-1}\mathbf{F}'(\mathbf{I}\otimes\mathbf{V}^{-1})\\
&= (\mathbf{I}\otimes\mathbf{V}^{-1}) - (\mathbf{I}\otimes\mathbf{V}^{-1})\mathbf{F}\mathbf{G}\big[\mathbf{G}'\mathbf{F}'(\mathbf{I}\otimes\mathbf{V}^{-1})\mathbf{F}\mathbf{G}\big]^{-1}\mathbf{G}'\mathbf{F}'(\mathbf{I}\otimes\mathbf{V}^{-1})
\end{aligned}$$

and

$$\big[\mathbf{M_{FG}}(\mathbf{I}\otimes\mathbf{V})\mathbf{M_{FG}}\big]^+ = \mathbf{I}\otimes\mathbf{V}^{-1} - (\mathbf{I}\otimes\mathbf{V}^{-1})\mathbf{F}\mathbf{G}\big[\mathbf{G}'\mathbf{F}'(\mathbf{I}\otimes\mathbf{V}^{-1})\mathbf{F}\mathbf{G}\big]^{-1}\mathbf{G}'\mathbf{F}'(\mathbf{I}\otimes\mathbf{V}^{-1})$$

thus

$$(mn-s)\hat{\sigma}^2(2) = (mk-s)\hat{\sigma}^2(\mathrm{iii}) + (mn-mk)\hat{\sigma}^2(5)\,.$$

□

Lemma 3.5. *If the number of epochs is r, then $\hat\sigma^2(5)$ from Lemma 3.4 is*

$$\hat\sigma^2(5) = \frac{1}{r}\sum_{i=1}^{r}\hat\sigma^2(\boldsymbol{Y}_i)\,,$$

$$\hat\sigma^2(\boldsymbol{Y}_i) = \big[\boldsymbol{Y}_i - \boldsymbol{f}(t_i)\big]'\big(\mathbf{M}_{\mathbf{F}(t_i)}\mathbf{V}\mathbf{M}_{\mathbf{F}(t_i)}\big)^{+}\big[\boldsymbol{Y}_i - \boldsymbol{f}(t_i)\big]\,, \qquad i = 1,\dots,r\,.$$

P r o o f.

$$\operatorname{vec}(\underline{\boldsymbol{Y}} - \underline{\boldsymbol{f}})'[\mathbf{M}_{\mathbf{F}}(\mathbf{I}\otimes\mathbf{V})\mathbf{M}]^{+}\operatorname{vec}(\underline{\boldsymbol{Y}} - \underline{\boldsymbol{f}})/[r(n-k)]$$

$$= \operatorname{vec}(\underline{\boldsymbol{Y}} - \underline{\boldsymbol{f}})'\left[\mathbf{I}\otimes\mathbf{V}^{-1} - (\mathbf{I}\otimes\mathbf{V}^{-1})\begin{pmatrix}\mathbf{F}(t_1), & \mathbf{0}, & \dots, & \mathbf{0}\\ \dots & \dots & \dots & \dots\\ \mathbf{0}, & \mathbf{0}, & \dots, & \mathbf{F}(t_r)\end{pmatrix}\right. \times$$

$$\left.\times\begin{pmatrix}\mathbf{C}(t_1), & \mathbf{0}, & \dots, & \mathbf{0}\\ \dots & \dots & \dots & \dots\\ \mathbf{0}, & \mathbf{0}, & \dots, & \mathbf{C}(t_r)\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{F}'(t_1), & \mathbf{0}, & \dots, & \mathbf{0}\\ \dots & \dots & \dots & \dots\\ \mathbf{0}, & \mathbf{0}, & \dots, & \mathbf{F}'(t_r)\end{pmatrix}(\mathbf{I}\otimes\mathbf{V}^{-1})\right]\times$$

$$\times\operatorname{vec}(\underline{\boldsymbol{Y}} - \underline{\boldsymbol{f}})/[r(n-k)]$$

$$= \frac{1}{r}\sum_{i=1}^{r}[\boldsymbol{Y}(t_i) - \boldsymbol{f}_0(t_i)]'[\mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{F}(t_i)\mathbf{C}^{-1}(t_i)\mathbf{F}'(t_i)\mathbf{V}^{-1}][\boldsymbol{Y}(t_i) - \boldsymbol{f}_0(t_i)]/(n-k)$$

$$= \frac{1}{r}\sum_{i=1}^{r}\hat\sigma^2(\boldsymbol{Y}_i)\,.$$

□

Remark 3.1. Lemma 3.3 enables us to determine the estimator $\delta\hat{\boldsymbol{\gamma}}$ either on the basis of the estimators $\delta\hat{\boldsymbol{\beta}}(t_1),\dots,\delta\hat{\boldsymbol{\beta}}(t_m)$, or on the basis of the observation vectors $\boldsymbol{Y}_1,\dots,\boldsymbol{Y}_m$, in the case that the linearization is possible. It is suitable from the viewpoint of a numerical calculation and a check of a numerical reliability. Lemma 3.4 enables us to determine several estimators of σ^2. A sequence of estimators $\hat\sigma^2(\boldsymbol{Y}_i)$ after each epoch is a good check of a stableness of a precision of measurement. A statistical comparison of $\hat\sigma^2(\text{iii})$ and $\hat\sigma^2(5)$ is a good check of the proper choice of the functions $\beta_i(t) = \beta_i(t,\boldsymbol{\gamma})$, $i = 1,\dots,k$, which must express the time course of deformations adequately.

If the model (1) is linear (the terms $\boldsymbol{\kappa}$ and $\boldsymbol{\tau}$ can be neglected), then there is not necessary to find $\boldsymbol{\gamma}^{(0)}$, i.e. also $\beta_i^{(0)}(t) = \beta_i(t,\boldsymbol{\gamma}^{(0)})$, and Lemma 3.3 and Lemma 3.4 can be used.

In what follows, conditions will be found under which the terms $\boldsymbol{\kappa}$ and $\boldsymbol{\tau}$ can be neglected without any serious deterioration of considered estimators.

4. Criteria of a linearization

In this section some generalization of ideas given by Lemmas 2.1, 2.2 and 2.3 is presented.

THEOREM 4.1. *If the intrinsic measures of nonlinearity in* (1), (3) *and* (4) *are* $K^{(\mathrm{int})}_{\boldsymbol{f}(\boldsymbol{\beta})}$, $K^{(\mathrm{int})}_{\boldsymbol{f}}$ *and* $K^{(\mathrm{int})}_{\boldsymbol{\beta}}$, *respectively, and analogously* $K^{(\mathrm{par})}_{\boldsymbol{f}(\boldsymbol{\beta})}$, $K^{(\mathrm{par})}_{\boldsymbol{f}}$ *and* $K^{(\mathrm{par})}_{\boldsymbol{\beta}}$ *are parametric measures of nonlinearity, then*

(i) $K^{(\mathrm{int})}_{\boldsymbol{f}(\boldsymbol{\beta})} \le K^{(\mathrm{int})}_{\boldsymbol{f}} + K^{(\mathrm{int})}_{\boldsymbol{\beta}} + K^{(\mathrm{par})}_{\boldsymbol{f}}$,

(ii) $K^{(\mathrm{par})}_{\boldsymbol{f}(\boldsymbol{\beta})} \le K^{(\mathrm{par})}_{\boldsymbol{f}} + K^{(\mathrm{par})}_{\boldsymbol{\beta}}$.

Proof.

(i) We have

$$K^{(\mathrm{int})}_{\boldsymbol{f}(\boldsymbol{\beta})} = $$
$$= \sup\left\{ \frac{\sqrt{[\boldsymbol{\kappa}(\mathbf{G}\delta\boldsymbol{\gamma}) + \mathbf{F}\boldsymbol{\tau}(\delta\boldsymbol{\gamma})]'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{M}_{\mathbf{FG}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})}[\boldsymbol{\kappa}(\mathbf{G}\delta\boldsymbol{\gamma}) + \mathbf{F}\boldsymbol{\tau}(\delta\boldsymbol{\gamma})]}}{\delta\boldsymbol{\gamma}'\mathbf{G}'\mathbf{F}'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{FG}\delta\boldsymbol{\gamma}} : \delta\boldsymbol{\gamma}\in\mathbb{R}^s \right\}.$$

Since

$$\begin{aligned}(\boldsymbol{\kappa} + \mathbf{F}\boldsymbol{\tau})'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{M}_{\mathbf{FG}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})}(\boldsymbol{\kappa} + \mathbf{F}\boldsymbol{\tau}) \\ \le \left(\sqrt{\boldsymbol{\tau}'\mathbf{F}'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{M}_{\mathbf{FG}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})}\mathbf{F}\boldsymbol{\tau}} + \sqrt{\boldsymbol{\kappa}'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{M}_{\mathbf{FG}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})}\boldsymbol{\kappa}}\right)^2,\end{aligned}$$

$$\begin{aligned}K^{(\mathrm{int})}_{\boldsymbol{f}(\boldsymbol{\beta})} &\le \sup\left\{ \frac{\sqrt{\boldsymbol{\tau}'\mathbf{F}'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{M}_{\mathbf{FG}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})}\mathbf{F}\boldsymbol{\tau}}}{\delta\boldsymbol{\gamma}'\mathbf{G}'\mathbf{C}\mathbf{G}\delta\boldsymbol{\gamma}} : \delta\boldsymbol{\gamma}\in\mathbb{R}^s \right\} \\ &+ \sup\left\{ \frac{\sqrt{\boldsymbol{\kappa}'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{M}_{\mathbf{FG}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})}\boldsymbol{\kappa}}}{\delta\boldsymbol{\gamma}'\mathbf{G}'\mathbf{C}\mathbf{G}\delta\boldsymbol{\gamma}} : \delta\boldsymbol{\gamma}\in\mathbb{R}^s \right\}.\end{aligned}$$

Let $\mathbf{H}$ be a $km \times (km - s)$ matrix with the property $\mathrm{r}(\mathbf{H}) = km - s$ and $\mathbf{H}'\mathbf{G} = \mathbf{0}$. Then

$$\begin{aligned}\mathbf{P}_{\mathbf{FG}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})} &= \mathbf{P}_{\mathbf{FM_H}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})} = \mathbf{FM_H}(\mathbf{M_H}\mathbf{C}\mathbf{M_H})^{+}\mathbf{M_H}\mathbf{F}'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1}) \\ &= \mathbf{F}\big[\mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{H}(\mathbf{H}'\mathbf{C}^{-1}\mathbf{H})^{-1}\mathbf{H}'\mathbf{C}^{-1}\big]\mathbf{F}'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1}) \\ &= \mathbf{P}_{\mathbf{F}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})} - \mathbf{P}_{\mathbf{FC}^{-1}\mathbf{H}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})}\end{aligned}$$

and thus

$$\mathbf{M}_{\mathbf{FG}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})} = \mathbf{M}_{\mathbf{F}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})} + \mathbf{P}_{\mathbf{FC}^{-1}\mathbf{H}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})}.$$

Since $\mathbf{M}_{\mathbf{F}}^{(\mathbf{I}\otimes\Sigma^{-1})}\mathbf{F}=\mathbf{0}$,

$$K_{\boldsymbol{f}(\boldsymbol{\beta})}^{(\mathrm{int})}\leq\sup\left\{\frac{\sqrt{\boldsymbol{\tau}'\mathbf{F}'(\mathbf{I}\otimes\Sigma^{-1})\mathbf{P}_{\mathbf{FC}^{-1}\mathbf{H}}^{(\mathbf{I}\otimes\Sigma^{-1})}\mathbf{F}\boldsymbol{\tau}}}{\delta\gamma'\mathbf{G}'\mathbf{C}\mathbf{G}\delta\gamma}:\ \delta\gamma\in\mathbb{R}^{s}\right\}$$
$$+\sup\left\{\frac{\sqrt{\boldsymbol{\kappa}'(\mathbf{I}\otimes\Sigma^{-1})\mathbf{M}_{\mathbf{F}}^{(\mathbf{I}\otimes\Sigma^{-1})}\boldsymbol{\kappa}}}{\delta\gamma'\mathbf{G}'\mathbf{C}\mathbf{G}\delta\gamma}:\ \delta\gamma\in\mathbb{R}^{s}\right\}$$
$$+\sup\left\{\frac{\sqrt{\boldsymbol{\kappa}'(\mathbf{I}\otimes\Sigma^{-1})\mathbf{P}_{\mathbf{FC}^{-1}\mathbf{H}}^{(\mathbf{I}\otimes\Sigma^{-1})}\boldsymbol{\kappa}}}{\delta\gamma'\mathbf{G}'\mathbf{C}\mathbf{G}\delta\gamma}:\ \delta\gamma\in\mathbb{R}^{s}\right\}.$$

Further

$$\begin{aligned}
&\boldsymbol{\tau}'\mathbf{F}'(\mathbf{I}\otimes\Sigma^{-1})\mathbf{P}_{\mathbf{FC}^{-1}\mathbf{H}}^{(\mathbf{I}\otimes\Sigma^{-1})}\mathbf{F}\boldsymbol{\tau}\\
&=\boldsymbol{\tau}'\mathbf{F}'(\mathbf{I}\otimes\Sigma^{-1})\mathbf{FC}^{-1}\mathbf{H}\left[\mathbf{H}'\mathbf{C}^{-1}\mathbf{F}'(\mathbf{I}\otimes\Sigma^{-1})\mathbf{FC}^{-1}\mathbf{H}\right]^{-1}\mathbf{H}'\mathbf{C}^{-1}\mathbf{F}'(\mathbf{I}\otimes\Sigma^{-1})\mathbf{F}\boldsymbol{\tau}\\
&=\boldsymbol{\tau}'\mathbf{H}(\mathbf{H}'\mathbf{C}^{-1}\mathbf{H})^{-1}\mathbf{H}'\boldsymbol{\tau}\\
&=\boldsymbol{\tau}'\mathbf{C}^{1/2}\mathbf{C}^{-1/2}\mathbf{H}(\mathbf{H}'\mathbf{C}^{-1}\mathbf{H})^{-1}\mathbf{H}'\mathbf{C}^{-1/2}\mathbf{C}^{1/2}\boldsymbol{\tau}\\
&=\boldsymbol{\tau}'\mathbf{C}^{1/2}\mathbf{P}_{\mathbf{C}^{-1/2}\mathbf{H}}\mathbf{C}^{1/2}\boldsymbol{\tau}=\boldsymbol{\tau}'\mathbf{C}^{1/2}\mathbf{M}_{\mathbf{C}^{1/2}\mathbf{G}}\mathbf{C}^{1/2}\boldsymbol{\tau}=\boldsymbol{\tau}'\mathbf{C}\mathbf{M}_{\mathbf{G}}^{\mathbf{C}}\boldsymbol{\tau}
\end{aligned}$$

and

$$K_{\boldsymbol{\beta}}^{(\mathrm{int})}=\sup\left\{\frac{\sqrt{\boldsymbol{\tau}'\mathbf{C}\mathbf{M}_{\mathbf{G}}^{\mathbf{C}}\boldsymbol{\tau}}}{\delta\gamma'\mathbf{G}'\mathbf{C}\mathbf{G}\delta\gamma}:\ \delta\gamma\in\mathbb{R}^{s}\right\}.$$

Thus

$$K_{\boldsymbol{f}(\boldsymbol{\beta})}^{(\mathrm{int})}\leq K_{\boldsymbol{f}}^{(\mathrm{int})}+K_{\boldsymbol{\beta}}^{(\mathrm{int})}+K_{\boldsymbol{f}}^{(\mathrm{par})},$$

since

$$\sup\left\{\frac{\sqrt{\boldsymbol{\kappa}'(\mathbf{G}\delta\gamma)(\mathbf{I}\otimes\Sigma^{-1})\mathbf{M}_{\mathbf{F}}^{(\mathbf{I}\otimes\Sigma^{-1})}\boldsymbol{\kappa}(\mathbf{G}\delta\gamma)}}{\delta\gamma'\mathbf{G}'\mathbf{C}\mathbf{G}\delta\gamma}:\ \delta\gamma\in\mathbb{R}^{s}\right\}$$
$$\leq\sup\left\{\frac{\sqrt{\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{I}\otimes\Sigma^{-1})\mathbf{M}_{\mathbf{F}}^{(\mathbf{I}\otimes\Sigma^{-1})}\boldsymbol{\kappa}(\delta\boldsymbol{\beta})}}{\delta\boldsymbol{\beta}'\mathbf{C}\delta\boldsymbol{\beta}}:\ \delta\boldsymbol{\beta}\in\mathbb{R}^{mk}\right\}=K_{\boldsymbol{f}}^{(\mathrm{int})}$$

and

$$\sup\left\{\frac{\sqrt{\boldsymbol{\kappa}'(\mathbf{G}\delta\gamma)(\mathbf{I}\otimes\Sigma^{-1})\mathbf{P}_{\mathbf{FC}^{-1}\mathbf{H}}^{(\mathbf{I}\otimes\Sigma^{-1})}\boldsymbol{\kappa}(\mathbf{G}\delta\gamma)}}{\delta\gamma'\mathbf{G}'\mathbf{C}\mathbf{G}\delta\gamma}:\ \delta\gamma\in\mathbb{R}^{s}\right\}$$
$$\leq\sup\left\{\frac{\sqrt{\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{I}\otimes\Sigma^{-1})\mathbf{P}_{\mathbf{F}}^{(\mathbf{I}\otimes\Sigma^{-1})}\boldsymbol{\kappa}(\delta\boldsymbol{\beta})}}{\delta\boldsymbol{\beta}'\mathbf{C}\delta\boldsymbol{\beta}}:\ \delta\boldsymbol{\beta}\in\mathbb{R}^{mk}\right\}=K_{\boldsymbol{f}}^{(\mathrm{par})}$$

(ii) Analogously

$$K^{(\mathrm{par})}_{\boldsymbol{f}(\boldsymbol{\beta})} =$$

$$= \sup\left\{ \frac{\sqrt{(\boldsymbol{\kappa}+\mathbf{F}\boldsymbol{\tau})'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{P}_{\mathbf{FG}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})}(\boldsymbol{\kappa}+\mathbf{F}\boldsymbol{\tau})}}{\delta\boldsymbol{\gamma}'\mathbf{G}'\mathbf{C}\mathbf{G}\delta\boldsymbol{\gamma}} : \delta\boldsymbol{\gamma}\in\mathbb{R}^s \right\}$$

$$\leq \sup\left\{ \frac{\sqrt{\left(\sqrt{\boldsymbol{\kappa}'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{P}_{\mathbf{FG}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})}\boldsymbol{\kappa}} + \sqrt{\boldsymbol{\tau}'\mathbf{F}'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{P}_{\mathbf{FG}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})}\mathbf{F}\boldsymbol{\tau}}\right)^2}}{\delta\boldsymbol{\gamma}'\mathbf{G}'\mathbf{C}\mathbf{G}\delta\boldsymbol{\gamma}} : \delta\boldsymbol{\gamma}\in\mathbb{R}^s \right\}$$

$$\leq \sup\left\{ \frac{\sqrt{\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{P}_{\mathbf{F}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})}\boldsymbol{\kappa}(\delta\boldsymbol{\beta})}}{\delta\boldsymbol{\beta}'\mathbf{C}\delta\boldsymbol{\beta}} : \delta\boldsymbol{\beta}\in\mathbb{R}^{km} \right\}$$

$$+ \sup\left\{ \frac{\sqrt{\boldsymbol{\tau}'(\mathbf{G}\delta\boldsymbol{\gamma})\mathbf{C}\mathbf{P}_{\mathbf{G}}^{\mathbf{C}}\boldsymbol{\tau}(\mathbf{G}\delta\boldsymbol{\gamma})}}{\delta\boldsymbol{\gamma}'\mathbf{G}'\mathbf{C}\mathbf{G}\delta\boldsymbol{\gamma}} : \delta\boldsymbol{\gamma}\in\mathbb{R}^s \right\}$$

$$= K^{(\mathrm{par})}_{\boldsymbol{f}} + K^{(\mathrm{par})}_{\boldsymbol{\beta}},$$

since

$$\boldsymbol{\tau}'\mathbf{F}'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{F}\mathbf{G}(\mathbf{G}'\mathbf{C}\mathbf{G})^{-1}\mathbf{G}'\mathbf{F}'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{F}\boldsymbol{\tau} = \boldsymbol{\tau}'\mathbf{C}\mathbf{G}(\mathbf{G}'\mathbf{C}\mathbf{G})^{-1}\mathbf{G}'\mathbf{C}\boldsymbol{\tau} = \boldsymbol{\tau}'\mathbf{C}\mathbf{P}_G^C\boldsymbol{\tau}.$$

□

Let $\mathbf{F}_1 = \mathbf{F}(t_1) = \dots = \mathbf{F}(t_m)$, $\mathbf{C}_1 = \mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}_1$, $\mathbf{G}_1 = \mathbf{G}(t_1) = \dots = \mathbf{G}(t_m)$, $\boldsymbol{\kappa}_1 = \boldsymbol{\kappa}(t_1) = \dots = \boldsymbol{\kappa}(t_m)$, $\boldsymbol{\tau}_1 = \boldsymbol{\tau}(t_1) = \dots = \boldsymbol{\tau}(t_m)$. Then

$$\boldsymbol{\tau}'(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})\mathbf{M}_{\mathbf{FG}}^{(\mathbf{I}\otimes\boldsymbol{\Sigma}^{-1})}\boldsymbol{\tau} = (\mathbf{1}'\otimes\boldsymbol{\tau}_1')\big[\mathbf{I}\otimes(\boldsymbol{\Sigma}^{-1}\mathbf{M}_{\mathbf{F}_1\mathbf{G}_1}^{\boldsymbol{\Sigma}^{-1}})\big](\mathbf{1}+\boldsymbol{\tau}_1) = m\boldsymbol{\tau}_1'\boldsymbol{\Sigma}^{-1}\mathbf{M}_{\mathbf{F}_1\mathbf{G}_1}^{\boldsymbol{\Sigma}^{-1}}\boldsymbol{\tau}_1$$

and $\boldsymbol{\tau}'\mathbf{G}'\mathbf{C}\mathbf{G}\boldsymbol{\tau} = m\boldsymbol{\tau}_1'\mathbf{G}_1'\mathbf{C}_1\mathbf{G}_1\boldsymbol{\tau}_1$. (Here $\mathbf{1} = (1,\dots,1)'\in\mathbb{R}^m$.)

The intrinsic measure of nonlinearity for m epochs in this case is $K^{(\mathrm{int})}_{m,\boldsymbol{f}(\boldsymbol{\beta})} = \frac{1}{\sqrt{m}}K^{(\mathrm{int})}_{1,\boldsymbol{f}(\boldsymbol{\beta})}$ and analogously $K^{(\mathrm{par})}_{m,\boldsymbol{f}(\boldsymbol{\beta})} = \frac{1}{\sqrt{m}}K^{(\mathrm{par})}_{1,\boldsymbol{f}(\boldsymbol{\beta})}$, $K^{(\mathrm{int})}_{m,\boldsymbol{f}} = \frac{1}{\sqrt{m}}K^{(\mathrm{int})}_{1,\boldsymbol{f}}$, etc..

The confidence ellipsoid for $\delta\boldsymbol{\beta} = \big(\delta\boldsymbol{\beta}'(t_1),\dots,\delta\boldsymbol{\beta}'(t_m)\big)'$ in this case is

$$\mathcal{E}_{1-\alpha} = \{\delta\boldsymbol{\beta} : (\delta\boldsymbol{\beta}-\delta\hat{\boldsymbol{\beta}})'(\mathbf{I}\otimes\mathbf{C}_1)(\delta\boldsymbol{\beta}-\delta\hat{\boldsymbol{\beta}}) \leq \chi^2_{mk}(1-\alpha)\}$$
$$= \Big\{\delta\boldsymbol{\beta} : \sum_{i=1}^m \big[\delta\boldsymbol{\beta}(t_i)-\delta\hat{\boldsymbol{\beta}}(t_i)\big]'(\mathbf{I}\otimes\mathbf{C}_1)\big[\delta\boldsymbol{\beta}(t_i)-\delta\hat{\boldsymbol{\beta}}(t_i)\big] \leq \chi^2_{mk}(1-\alpha)\Big\}.$$

How the number m of epochs influences the inclusion of the confidence region into linearization regions can be characterized by the ratio $\chi^2_{km}(1-\alpha)/\sqrt{m\delta_{\max}}$ (cf. Table 4.1), where $P\{\chi^2_{m(n-k)}(\delta_{\max}) \geq \chi^2_{m(n-k)}(1-\alpha)\} = \alpha+\varepsilon$, and by the ratio $\chi^2_{km}(1-\alpha)/\sqrt{m}$ (cf. Table 4.2).

Table 4.1. $k = 6$, $n-k = 5$, $\alpha = 0.05$, $\varepsilon = 0.05$.

m	1	2	3	4
$\chi^2_{km}(1-\alpha)/\sqrt{m\delta_{\max}}$	12.35	12.29	12.27	12.41

Table 4.2. $k = 6$, $n-k = 5$, $\alpha = 0.05$.

m	1	2	3	4
$\chi^2_{km}(1-\alpha)/\sqrt{m}$	11.10	12.94	14.40	15.70

The quantity $K^{(\mathrm{par})}$ seems to be essentially more dangerous than $K^{(\mathrm{int})}$. Nevertheless it seems to be useful to check the inequalities $\chi^2_k(1-\alpha) \ll 2c/K^{(\mathrm{par})}_{\boldsymbol{f}(t_i)}$ and $\sigma^2\chi^2_k(1-\alpha) \ll 2\sqrt{2(n-k)\varepsilon}/K^{(\mathrm{int})}_{0,\boldsymbol{f}(t_i)}$ before each epoch.

When the number m of epochs is sufficiently large in order to estimate the vector γ, it can be checked whether

$$\chi^2_s(1-\alpha) \ll \frac{2\sqrt{\delta_{\max}}}{K^{(\mathrm{int})}_{\boldsymbol{f}} + K^{(\mathrm{int})}_{\boldsymbol{\beta}} + K^{(\mathrm{par})}_{\boldsymbol{f}}},$$

$$\chi^2_s(1-\alpha) \ll \frac{2c}{K^{(\mathrm{par})}_{\boldsymbol{f}} + K^{(\mathrm{par})}_{\boldsymbol{\beta}}},$$

$$\chi^2_s(1-\alpha) \ll \frac{2\sqrt{2(mn-s)\varepsilon}}{K^{(\mathrm{int})}_{0,\boldsymbol{f}} + K^{(\mathrm{int})}_{0,\boldsymbol{\beta}} + K^{(\mathrm{par})}_{0,\boldsymbol{f}}}.$$

The value $\delta_{\max}$ is a solution of the equation $P\{\chi^2_{mn-s}(\delta_{\max}) \geq \chi^2_{mn-s}(1-\alpha)\} = \alpha+\varepsilon$.

The inequality $\chi^2_s(1-\alpha) \ll \left[2\sqrt{\delta_{\max}}/(K^{(\mathrm{int})}_{1,\boldsymbol{f}} + K^{(\mathrm{int})}_{1,\boldsymbol{\beta}} + K^{(\mathrm{par})}_{1,\boldsymbol{f}})\right]\sqrt{m}$, etc. can be checked for the first orientation.

If the given inequalities are satisfied, then Lemmas 3.1, 3.2, 3.3, 3.4, 3.5 and Remark 3.1 can be used without any essential deterioration of estimators.

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