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ON ISOMORPHISMS OF INNER PRODUCT SPACES

DAVID BUHAGIAR* — EMANUEL CHETCUTI**

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ABSTRACT. In this paper, we show that if $S_1$ and $S_2$ are two separable, real inner product spaces such that $P(S_1)$ is algebraically isomorphic to $P(S_2)$, where $P(S)$ denotes the modular lattice of finite and cofinite dimensional subspaces of an inner product space $S$, then $S_1$ and $S_2$ are isomorphic as inner product spaces. The proof makes use of Gleason’s theorem. We also remark that, as a consequence of this, if for two separable, real inner product spaces $S_1$ and $S_2$, the respective complete lattices of strongly closed subspaces are isomorphic, then $S_1$ and $S_2$ are unitarily equivalent. In particular, if we just restrict ourselves to complete inner product spaces, we obtain the classical Wigner’s theorem ([WIGNER, E. P.: Group Theory and its Applications to Quantum Mechanics of Atomic Spectra, Acad. Press. Inc., New York, 1959]).

1. Introduction

For an inner product space $S$, let $P(S)$ (see [3]) denote the family of finite and cofinite dimensional subspaces of $S$. The idea is to show that if $S_1$ and $S_2$ are two separable real inner product spaces such that $P(S_1)$ is isomorphic to $P(S_2)$, then $S_1$ and $S_2$ are isomorphic as inner product spaces.

We say that $P(S_1)$ is isomorphic to $P(S_2)$ when there exists a bijective mapping $\psi: P(S_1) \rightarrow P(S_2)$ such that:

1. $\psi(S_1) = S_2$;
2. $\psi(A^{s_1}) = (\psi(A))^{s_2}$ for all $A \in P(S_1)$;

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1 A subspace $A$ of $S$ is cofinite dimensional if there exists a finite dimensional subspace $M$ of $S$ such that $A = M^\perp$. 
(3) $\psi(A \lor B) = \psi(A) \lor \psi(B)$, whenever $A, B \in P(S_1)$ and $A \subseteq B^\perp_{S_1}$;
(4) $\psi^{-1}$ satisfies (1), (2) and (3).

2. Preliminary results

We first prove that $P(S)$ is an orthomodular lattice.

**Proposition 2.1.** $P(S)$ is an orthomodular lattice with the largest and smallest elements being $S$ and $\{0\}$ respectively.

**Proof.** First we show that $P(S)$ is a lattice. If $A$ and $B$ are either both finite or cofinite dimensional, then obviously we have $A \lor B = A + B$. If $A$ is finite and $B$ is cofinite dimensional, then, by noting that

$$(A + B)^\perp = A^\perp \cap B^\perp \subseteq B^\perp,$$

it follows that $A + B$ is cofinite dimensional. (The other case is the same).

We now show that $P(S)$ is orthomodular. Let $A \subseteq B$ be elements of $P(S)$. We certainly have that $A \oplus (B \cap A^\perp) \subseteq B$. Moreover, since $A \subseteq B$, we have $B = B \cap (A \oplus A^\perp) \subseteq (B \cap A) \oplus (B \cap A^\perp) = A \oplus (B \cap A^\perp)$. $\square$

In [6], the family of complete-cocomplete subspaces of an inner product space, denoted by $C(S)$, was defined and investigated. It was shown that the structure of $C(S)$ can be very different for different separable inner product spaces. It is evident that $P(S)$ is a suborthomodular lattice of $C(S)$, and using an argument similar to that used in [1], one can easily show that $P(S)$ admits no $\sigma$-additive states.

**Lemma 2.2.** Let $A \in P(S_1)$, $\dim A = n < \infty$, then $\dim \psi(A) = n$.

**Proof.** Let $\{e_i : i \leq n\}$ be an ONB for $A$. Then

$$\psi(A) = \psi\left( \bigvee_{i \leq n} [e_i] \right) = \bigvee_{i \leq n} \psi([e_i]).$$

Since for $i \neq j$, we have $\psi([e_i]) \perp \psi([e_j])$, it follows that $\dim A \leq \dim \psi(A)$. On the other hand, let $\{f_i : i \in I\}$ be a MONS in $\psi(A)$. Then $\psi(A) = \bigvee_{i \in I} [f_i]$. Let $I_0 \subseteq I$ such that $|I_0| = n$. Then

$$A = \psi^{-1}(\psi(A)) = \psi^{-1}\left( \bigvee_{i \in I_0} [f_i] \lor \bigvee_{i \in I \setminus I_0} [f_i] \right) = \bigvee_{i \in I_0} \psi^{-1}([f_i]) \lor \psi^{-1}\left( \bigvee_{i \in I \setminus I_0} [f_i] \right),$$

and therefore $\dim A \geq \dim \psi(A)$. $\square$
As a consequence of Lemma 2.2, we have that atoms in $P(S_1)$ are mapped onto the atoms of $P(S_2)$. Since $S_1$ is separable, we can always find an orthonormal basis $\{e_i\}$ of $S_1$ in $S_2$, see [3], [5]. For every $i \in \mathbb{N}$, let $f_i$ be a unit vector in $S_2$ such that $\psi([e_i]) = [f_i]$.

For every atom in $P(S_1)$, choose a representative vector — i.e., a unit vector in $S_1$ which spans the atom. For the atom $[e_i]$, the representative is chosen to be $e_i$, and to make the proof of Lemma 2.9 free of unnecessary awkward notation, we also take the representative of the following atoms to be as follows:

$$[e_i + e_j] \rightarrow y_{ij} = \frac{1}{\sqrt{2}}(e_i + e_j) \quad i, j \in \mathbb{N};$$

$$[e_k + e_{k+1} + \cdots + e_l] \rightarrow z_{kl} = \frac{1}{\sqrt{l - k + 1}} \sum_{i=k}^{l} e_i \quad l > k \in \mathbb{N}.$$
This fundamental and highly non-trivial result is of crucial importance for the probabilistic theory of $L(H)$ and has many generalization and applications (see, for example [3]).

We shall need the following proposition ([2]).

**Lemma 2.4.** Let $S$ be any inner product space, and suppose that $s_1$, $s_2$ are two (finitely-additive) states on $P(S)$ such that:

(i) $s_1(M) = s_2(M) = 1$ for some $M \subset S$, $M$ finite dimensional;
(ii) $s_1(K) = s_2(K)$ for all $K \subset M$.

Then $s_1(L) = s_2(L)$ holds for all $L \in P(S)$.

**Proof.** It suffices to show that $s_1([x]) = s_2([x])$ holds for all $x \in S$. Let $x \in S$, $||x|| = 1$, be arbitrary. If $x \in M$, result follows by hypothesis. Suppose that $x \notin M$. Let $N$ be a finite dimensional subspace of $S$, of dimension at least equal to 3, including $M$ and $x$. We certainly have that $s_1|_N$ and $s_2|_N$ are states on $L(N)$, and therefore, by Gleason’s theorem, there exist finite orthonormal sequences \( \{e_i : i \leq n\} \) and \( \{f_i : i \leq n\} \) \((n = \dim N)\) in $N$ such that

\[
\begin{align*}
    s_1|_N(K) &= s_1(K) = \sum_{i \leq n} s([e_i]) \langle P_K e_i, e_i \rangle, \\
    s_2|_N(K) &= s_2(K) = \sum_{i \leq n} s([f_i]) \langle P_K f_i, f_i \rangle 
\end{align*}
\]

for all $K \subset N$.

Let $z \in M^\perp_N$. Then

\[
\begin{align*}
    0 &= s_1([z]) = \sum_{i \leq n} s([e_i]) \langle P_{[z]} e_i, e_i \rangle, \\
    0 &= s_2([z]) = \sum_{i \leq n} s([f_i]) \langle P_{[z]} f_i, f_i \rangle.
\end{align*}
\]

This implies that

\[
\begin{align*}
    z \in \text{span}\{e_i : i \leq n\}^\perp_N, \\
    z \in \text{span}\{f_i : i \leq n\}^\perp_N.
\end{align*}
\]

Hence, \( \{e_i : i \leq n\} \subset M \) and \( \{f_i : i \leq n\} \subset M \).

But

\[
    x = P_M x + P_{M^\perp} x = x_M + x_{M^\perp},
\]

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and therefore,
\[
\begin{align*}
s_1([x]) &= \sum_{i<n} s([e_i]) \langle P_{[x]} e_i, e_i \rangle \\
&= \sum_{i<n} s([e_i]) |\langle x, e_i \rangle|^2 \\
&= \sum_{i<n} s([e_i]) |\langle x_M, e_i \rangle|^2 \\
&= \sum_{i<n} s([e_i]) \|x_M\|^2 \left| \left\langle \frac{x_M}{\|x_M\|}, e_i \right\rangle \right|^2 \\
&= \|x_M\|^2 \sum_{i<n} s([e_i]) \langle P_{[x]} e_i, e_i \rangle \\
&= \|x_M\|^2 s_1([x_M]) .
\end{align*}
\]

Similarly, \( s_2([x]) = \|x_M\|^2 s_2([x_M]) \). Then,
\[
\begin{align*}
s_1([x]) &= \|x_M\|^2 s_1([x_M]) \\
&= \|x_M\|^2 s_2([x_M]) \quad \text{(by hypothesis)} \\
&= s_2([x]) .
\end{align*}
\]

This completes the proof. \( \square \)

**Corollary 2.5.** If \( s \) is a state on \( P(S) \) that lives on an atom (i.e. there exists a unit vector \( u \in S \) such that \( s([u]) = 1 \)), then \( s \) is determined by
\[
s(N) = \langle P_N u, u \rangle .
\]

**Corollary 2.6.** The state \( \hat{s}_{e_i} \) defined in equation (2.1) satisfies:
\[
\hat{s}_{e_i}(N) = \langle P_N f_i, f_i \rangle \tag{2.2}
\]
for all \( N \in P(S_2) \).

**Lemma 2.7.** Let \( 0 \neq x = \sum_{i \in \mathbb{N}} \alpha_i e_i \in \mathcal{S}^+_1 \). Then for every \( i \in \mathbb{N} \), we have:
\[
\langle \hat{x}, f_i \rangle = \pm \alpha_i .
\]

**Proof.** This follows from the following equalities:
\[
|\alpha_i|^2 = s_{e_i}([x]) = \hat{s}_{e_i}([\hat{x}]) = \langle \hat{x}, f_i \rangle^2 .
\]
\( \square \)
**DEFINITION 2.1.** For any unit vector $x \in \mathcal{S}_1^+$ and $i \in \mathbb{N}$ satisfying $\langle x, e_i \rangle \neq 0$, define

$$\beta(x, i) = \frac{\langle \hat{x}, f_i \rangle}{\langle x, e_i \rangle} (= \pm 1).$$

When $\langle x, e_i \rangle = 0$, we set $\beta(x, i) = 1$.

**Lemma 2.8.** For any unit vector $x \in \mathcal{S}_1^+$, the vector $\hat{x}$ can be expressed in terms of the $f_i$'s as follows:

$$\hat{x} = \sum_{i \in \mathbb{N}} \beta(x, i) \alpha_i f_i.$$

**Proof.** First we observe that

$$
\hat{x} = \hat{x}_{\psi(A_n)} + \hat{x}_{(\psi(A_n))^\perp}
= \sum_{i \leq n} \langle \hat{x}, f_i \rangle f_i + \hat{x}_{(\psi(A_n))^\perp}
= \sum_{i \leq n} \beta(x, i) \alpha_i f_i + \hat{x}_{(\psi(A_n))^\perp}.
$$

Therefore

$$
\|\hat{x}_{(\psi(A_n))^\perp}\|^2 = \|\hat{x}\|^2 - \sum_{i \leq n} |\alpha_i|^2
= 1 - \sum_{i \leq n} |\alpha_i|^2 \to 0 \quad \text{as} \quad n \to \infty.
$$

**Lemma 2.9.** Let $x \in \mathcal{S}_1^+$. If $\langle x, e_i \rangle \neq 0$ and $\langle x, e_j \rangle \neq 0$, then

$$
\frac{\beta(x, i)}{\beta(x, j)} = \frac{\beta(y_{ij}, i)}{\beta(y_{ij}, j)}.
$$

**Proof.** Recall that

$$y_{ij} = \frac{1}{\sqrt{2}} e_i + \frac{1}{\sqrt{2}} e_j \in \mathcal{S}_1^+.$$

It is not difficult to see that

$$\hat{y}_{ij} = \frac{1}{\sqrt{2}} \beta(y_{ij}, i) f_i + \frac{1}{\sqrt{2}} \beta(y_{ij}, j) f_j.$$
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We have
\[(\alpha_i + \alpha_j)^2 = |(y_{ij}, x)|^2 = s_{y_{ij}}([x]) = \hat{s}_{y_{ij}}(\hat{x}) = |(\hat{y}_{ij}, \hat{x})|^2 = \left(\frac{\beta(x, i)\beta(y_{ij}, i)\alpha_i}{\sqrt{2}} + \frac{\beta(x, j)\beta(y_{ij}, j)\alpha_j}{\sqrt{2}}\right)^2.\]

Since the field is real, it follows that
\[\beta(x, i)\beta(y_{ij}, i) = \beta(x, j)\beta(y_{ij}, j),\]
and therefore,
\[\frac{\beta(x, i)}{\beta(x, j)} = \frac{\beta(y_{ij}, i)}{\beta(y_{ij}, j)}.\]

\[\square\]

3. Main result

Let 0 ≠ x ∈ \mathbb{S}_1^+ be arbitrary and let k be the smallest natural number satisfying \langle x, e_k \rangle ≠ 0. For any j ∈ \mathbb{N} satisfying \langle x, e_j \rangle ≠ 0, by Lemma 2.9, we have
\[\frac{\beta(x, j)}{\beta(x, k)} = \frac{\beta(y_{kj}, j)}{\beta(y_{kj}, k)}.\]

This implies that
\[\beta(x, j) = \beta(x, k)\frac{\beta(z_{1j}, j)}{\beta(z_{1j}, k)}.\]

But since, from Lemma 2.9,
\[\frac{\beta(z_{1j}, k)}{\beta(z_{1j}, 1)} = \frac{\beta(z_{1k}, k)}{\beta(z_{1k}, 1)},\]
we have that
\[\beta(x, j) = \beta(x, k)\frac{\beta(z_{1k}, 1)}{\beta(z_{1k}, k)}\frac{\beta(z_{1j}, j)}{\beta(z_{1j}, 1)}.\]

For any j ∈ \mathbb{N}, define:
\[\gamma_j = \frac{\beta(z_{1j}, j)}{\beta(z_{1j}, 1)}.\]

Thus, we have that
\[\hat{x} = \beta(x, k)\frac{\beta(z_{1k}, 1)}{\beta(z_{1k}, k)}\sum_{i ∈ \mathbb{N}} \gamma_i \alpha_i f_i.\]
So if we define $U: \mathcal{S}^+ \rightarrow S_2$ by
\[
U(x) = \begin{cases} 
U \left( \sum_{i \in \mathbb{N}} \alpha_i e_i \right) = \sum_{i \in \mathbb{N}} \gamma_i \alpha_i f_i & \text{if } x \neq 0, \\
0 & \text{if } x = 0, 
\end{cases} 
\tag{3.1}
\]
we certainly have that $U$ is well defined on $\mathcal{S}^+$ and moreover, it is an injection into $S_2$. We now prove the claim put in the abstract.

**Theorem 3.1.** Let $S_1$ and $S_2$ be two separable real inner product spaces. Then, $P(S_1)$ is isomorphic to $P(S_2)$ if and only if $S_1$ and $S_2$ are isomorphic as inner product spaces.

**Proof.** If $S_1$ is isomorphic to $S_2$, then we obviously have that $P(S_1)$ is isomorphic to $P(S_2)$. Suppose that $P(S_1)$ is isomorphic to $P(S_2)$ as understood in the beginning of this note. We show that there exists a bijective operator $T$ from $S_1$ onto $S_2$ such that $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in S_1$.

Define $T: S_1 \rightarrow S_2$ by:
\[
T(v) = T(\lambda x) \quad \text{for some unique } x \in \mathcal{S}^+ \\
= \lambda U(x)
\]
where $U$ is as defined in equation (3.1). Clearly $T$ is a bijection between $S_1$ and $S_2$. We show that $T$ is linear. From the same definition, it is clear that for every $\rho \in \mathbb{R}$, $T(\rho v) = \rho T(v)$. Let $v, w \in S_1$. Put $\delta_i = \langle v, e_i \rangle$ and $\theta_i = \langle w, e_i \rangle$. Then
\[
T(v + w) = T \left( \sum_{i \in \mathbb{N}} (\delta_i + \theta_i) e_i \right) \\
= T \left( \kappa \|v + w\| \left[ \frac{\kappa}{\|v + w\|} \sum_{i \in \mathbb{N}} (\delta_i + \theta_i) e_i \right] \right),
\]
where $\kappa = \pm 1$ so that $\left[ \frac{\kappa}{\|v + w\|} \sum_{i \in \mathbb{N}} (\delta_i + \theta_i) e_i \right] \in \mathcal{S}^+$. Then we have
\[
T(v + w) = \kappa \|v + w\| U \left( \sum_{i \in \mathbb{N}} \left( \frac{\kappa \delta_i}{\|v + w\|} + \frac{\kappa \theta_i}{\|v + w\|} \right) e_i \right) \\
= \kappa \|v + w\| \sum_{i \in \mathbb{N}} \left( \frac{\kappa \gamma_i \delta_i}{\|v + w\|} + \frac{\kappa \gamma_i \theta_i}{\|v + w\|} \right) f_i \\
= \sum_{i \in \mathbb{N}} \gamma_i \delta_i f_i + \sum_{i \in \mathbb{N}} \gamma_i \theta_i f_i \\
= T(v) + T(w).
\]
This completes the proof. □

Let $F(S)$ denote the complete lattice of strongly closed subspaces of $S$ and $E(S)$ the orthomodular poset of splitting subspaces of $S$. We recall that

$$P(S) \subseteq C(S) \subseteq E(S) \subseteq F(S).$$

**COROLLARY 3.2.** The following statements are equivalent:

1. $S_1$ is isomorphic to $S_2$ (as inner product spaces);
2. $P(S_1)$ is isomorphic to $P(S_2)$ (as orthomodular lattices);
3. $C(S_1)$ is isomorphic to $C(S_2)$ (as orthomodular posets);
4. $E(S_1)$ is isomorphic to $E(S_2)$ (as orthomodular posets);
5. $F(S_1)$ is isomorphic to $F(S_2)$ (as complete lattices).

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*Department of Mathematics
University of Malta
Msida MSD.06
MALTA
E-mail: david.buhagiar@um.edu.mt

**Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK-814 73 Bratislava
SLOVAKIA
E-mail: chetcuti@mat.savba.sk