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Dedicated to Professor Tibor Katriňák

ALGEBRAS ON SUBINTERVALS OF BL-ALGEBRAS, PSEUDO BL-ALGEBRAS AND BOUNDED RESIDUATED ℓ-MONOIDS

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ABSTRACT. Let M be a BL-algebra or a pseudo BL-algebra or a bounded residuated ℓ-monoid and let $a \le b$, $a, b \in M$. We endow the subinterval $[a, b]$ with algebraic structure to form an algebra of the same kind as the original one. Obtained results generalize ones presented in [CHAJDA, I.—KÜHR, J.: *A note on interval MV-algebras*, Math. Slovaca **56** (2006), 47–52] and [CHAJDA, I.—KÜHR, J.: *GMV-algebras and meet-semilattices with sectionally antitone permutations*, Math. Slovaca (To appear)], [JAKUBÍK, J.: *On interval subalgebras of generalized MV-algebras*, Math. Slovaca **56** (2006) (To appear)] for MV-algebras and pseudo MV- (GMV-) algebras, respectively. We also study restrictions of Bosbach states on such subinterval algebras. We show that for a commutative case it is necessary to introduce one additional condition and for a non-commutative case it is necessary to introduce two conditions, left and right one, in order that the restriction of a state can define a state on the subinterval. We prove that BL-algebras always satisfy the additional condition.

1. Introduction

Many valued logics are modelled by many kinds of algebras. BL-algebras were introduced by P. Hájek [14] as an algebraic model of the so called *basic logic*, i.e. a fuzzy logic of continuous t-norms and their residua.

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Various kinds of generalizations of BL-algebras were studied. One line of generalization follows from omitting the commutativity condition. Non-commutative BL-algebras, called pseudo BL-algebras, were introduced by A. Di Nola, G. Georgescu, A. Iorgulescu in [7], [8]. Particular examples of pseudo BL-algebras are pseudo MV-algebras. These were introduced by G. Georgescu, A. Iorgulescu [13], and independently, under the name GMV-algebras, by J. Rachůnek [16]. In [7; Proposition 3.27] it was shown that pseudo MV-algebras are pseudo BL-algebras that satisfy the identity $(x^{\sim})^{-} = x = (x^{-})^{\sim}$, where $x^{-} := x \to 0$ and $x^{\sim} := x \rightsquigarrow 0$.

Another line of generalization of BL-algebras stems from omitting the prelinearity condition. Such a generalization was studied by A. Dvurečenskij, J. Rachůnek [10], [11], [12] and the resulting algebras are called (commutative) bounded residuated ℓ-monoids ((commutative) bounded $R\ell$-monoids, for short). These monoids are also known as *bounded integral generalized BL-algebras*, see [2]. For the sake of completeness, we note that commutative bounded $R\ell$-monoids satisfying the prelinearity condition are BL-algebras and bounded $R\ell$-monoids with the prelinearity are pseudo BL-algebras.

Recently, I. Chajda, J. Kühr [3], [4] and J. Jakubík [15] solved the problem of introducing an MV-algebra structure ([3]) and a pseudo MV-algebra (GMV-algebra) structure ([4], [15]) on subintervals of the algebras. The natural question arises:

> *Is it possible to do it also for BL-algebras, pseudo BL-algebras and (commutative) bounded $R\ell$-monoids?*

A positive answer is in Section 3.

Sections 4 and 5 contain an analysis of the properties of special conditions, namely, P(a) or PL(a) and PR(a). With the use of the representational theorem of BL-algebras due to P. Hájek [14] it is shown that P(a) is satisfied in each BL-algebra M and any $a \in M$.

A motivation for the study of such identities is the fact that they are sufficient for restricting existing Bosbach states on the whole algebra also on subinterval algebras as it is shown in Section 6, which is dedicated to the study of some properties of Bosbach states.

2. Basic definitions

In this part, we define some notions used in this paper. We start with the definitions of the concerned structures.

BL-algebras, as an algebraization of a basic logic, are due to P. Hájek [14].

Definition 2.1. An algebra $(M; \odot, \wedge, \vee, \rightarrow, 0, 1)$ of type $(2,2,2,2,0,0)$ is said to be a *BL-algebra* if the following conditions are satisfied for all elements $x, y, z \in M$:

(i) $(M; \wedge, \vee, 0, 1)$ is a bounded lattice with the smallest element 0 and the greatest element 1,

(ii) $(M; \odot, 1)$ is a commutative monoid with a unit 1,

(iii) $x \leq y \rightarrow z$ iff $x \odot y \leq z$,

(iv) $x \wedge y = x \odot (x \rightarrow y)$,

(v) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

A non-commutative generalization of BL-algebras, introduced by A. Di Nola, G. Georgescu, A. Iorgulescu in [7], [8], is defined as follows.

Definition 2.2. An algebra $(M; \odot, \wedge, \vee, \rightarrow, \rightsquigarrow, 0, 1)$ of type $(2,2,2,2,2,0,0)$ is a *pseudo BL-algebra* if the following conditions are satisfied for all $x, y, z \in M$:

(i) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice with the smallest element 0 and the largest element 1,

(ii) $(M; \odot, 1)$ is a monoid with unit 1,

(iii) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$,

(iv) $x \wedge y = (x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y)$,

(v) $(x \rightarrow y) \vee (y \rightarrow x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$.

In Definitions 2.1 and 2.2 the property (iii) is called an *adjointness*, (iv) a *divisibility* and (v) a *prelinearity*.

We use the following abbreviations: $x^- := x \rightarrow 0$ and $x^\sim := x \rightsquigarrow 0$ and for BL-algebras there are the following important subvarieties.

A BL-algebra satisfying the identity $x^{--} = x$ is said to be a *Łukasiewicz BL-algebra* and it is an MV-algebra. A BL-algebra satisfying the following two identities:

(i) $x \wedge x^- = 0$,

(ii) $z^{--} \odot (x \odot z \rightarrow y \odot z) \leq x \rightarrow y$

is said to be a *product* BL-algebra. Finally, the identity $x \odot x = x$ defines a *Gödel BL-algebra.* Gödel linear BL-algebras are obtained as ordinal sums (pastings) of trivial $\{0,1\}$ MV-algebras.

For a pseudo BL-algebra we can (due to [8]) assume the following subvarieties. A pseudo BL-algebra is said to be *good*, if it satisfies the identity $x^{-\sim} = x^{\sim-}$. As it was mentioned earlier, pseudo MV-algebras are good BL-algebras satisfying identity $x = x^{-\sim} = x^{\sim-}$. A *product* pseudo BL-algebra is a pseudo BL-algebra with the following identities:

(i) $x \wedge x^- = 0 = x \wedge x^\sim$,
(ii) $z^{--} \odot (x \odot z \to y \odot z) \le x \to y$,
(iii) $z^{\sim\sim} \odot (z \odot x \rightsquigarrow z \odot y) \le x \rightsquigarrow y$.

Further generalization of the concept of a (pseudo) BL-algebra which omits a prelinearity condition, introduced by A. Dvurečenskij and J. Rachůnek in [10], [11], for commutative case, and in [12], for non-commutative case, are the notions of a *commutative bounded residuated ℓ-monoid* (generalization of a BL-algebra) and a *bounded residuated ℓ-monoid* (generalization of a pseudo BL-algebra). In the same way as in [10], [11], [12] we will refer to these structures as (*commutative*) *bounded $R\ell$-monoids* for short.

We say that a pseudo BL-algebra (bounded $R\ell$-monoid) is *commutative* if $\to = \rightsquigarrow$. (This implies also commutativity of the operation $\odot$ and vice-versa.)

Previously defined algebraic structures form a hierarchy: a bounded $R\ell$-monoid which is commutative is a commutative bounded $R\ell$-monoid, a bounded $R\ell$-monoid with the prelinearity condition is a pseudo BL-algebra and a commutative pseudo BL-algebra is a BL-algebra.

If it is not said otherwise, we assume that M is one of the previously defined algebras.

The following relations of product and residua are obvious:

$$a \odot b,\ b \odot a \le a \wedge b \le a, b \quad \text{and} \quad b \le a \to b,\ a \rightsquigarrow b \qquad \text{for any} \quad a, b \in M\,.$$

The residua $\to$ and $\rightsquigarrow$ satisfy the following monotonicity rules:

If $a \le b$, then $x \to a \le x \to b$, $x \rightsquigarrow a \le x \rightsquigarrow b$, $a \to x \ge b \to x$ and $a \rightsquigarrow x \ge b \rightsquigarrow x$ for $a, b, x \in M$.

The operation $\odot$ is distributive over lattice operations $\wedge$, $\vee$, i.e.

$$a \odot (x \vee y) \odot b = (a \odot x \odot b) \vee (a \odot y \odot b)\,,$$
$$a \odot (x \wedge y) \odot b = (a \odot x \odot b) \wedge (a \odot y \odot b)\,.$$

An element a of M is called an *idempotent* if $a \odot a = a$. Using divisibility and adjointness, we can prove that if a is an idempotent, then

$$x \odot a = a \odot x = a \wedge x \qquad \text{for any} \quad x \in M\,, \tag{1}$$

i.e. idempotent elements commute (in product $\odot$) with any other elements. Indeed, $a \odot x = a \wedge (a \odot x) = (a \odot a) \wedge (a \odot x) = a \odot (a \wedge x) = a \odot \big(a \odot (a \rightsquigarrow x)\big) = a \odot (a \rightsquigarrow x) = a \wedge x$, in the same manner we can prove, that $x \odot a = x \wedge a$.

We recall the definition of an ordinal sum (or a *pasting*) of a finite number of bounded $R\ell$-monoids ([12]). For an infinite number, later in Section 5, we introduce two different concepts, one due to P. Hájek [14] and the other due to P. Agliano, F. Montagna [1]. Although the original definitions in [14] and [1] were introduced for BL-algebras and for hoops, respectively, we use the adaption for bounded $R\ell$-monoids.

Let $(M_i; \odot_i, \wedge_i, \vee_i, \rightarrow_i, \rightsquigarrow_i, 0_i, 1_i)$, $i = 1, 2, \ldots, n$, be bounded $R\ell$-monoids such that $M_i \cap M_{i+1} = \{1_i, 0_{i+1}\}$ is a singleton for $1 \le i \le n-1$ and $M_i \cap M_j = \emptyset$, for $|i-j| \ge 2$. We identify the element 0_{i+1} with 1_i and set $0 = 0_1$, $1 = 1_n$, $M = M_1 \cup M_2 \cup \cdots \cup M_n$. On M we define operations $\odot$, $\vee$, $\wedge$, $\rightarrow$ and $\rightsquigarrow$ as follows: if the arguments are in the same M_i, then operations coincide with operations in M_i. For $x \in M_i \setminus M_j$ and $y \in M_j \setminus M_i$, $i < j$, we define $x \odot y = x = y \odot x$, $x \rightarrow y = x \rightsquigarrow y = 1$, $y \rightarrow x = y \rightsquigarrow x = x$. Then $(M; \odot, \wedge, \vee, \rightarrow, \rightsquigarrow, 0, 1)$, denoted also as $\bigsqcup_{i=1}^{n} M_i$, is an *ordinal sum* of bounded $R\ell$-monoids M_i. It is again a bounded $R\ell$-monoid. This definition can be naturally adapted to all other three types of algebras, however, pasting of (pseudo) BL-algebras need not be again a (pseudo) BL-algebra.

3. Algebras on subintervals

In this part, we introduce a bounded $R\ell$-monoid structure defined on the subintervals $[a, b]$ of the interval $[0, 1]$. First we introduce the structure on the subintervals of the type $[a, 1]$, then for the $[0, a]$ type and finally, by the combination of the previous two types, we obtain desired results for arbitrary intervals $[a, b]$. We note, that a similar approach was taken in [3], [4], [15].

THEOREM 3.1. *Let $(M; \odot, \wedge, \vee, \rightarrow, \rightsquigarrow, 0, 1)$ be a bounded $R\ell$-monoid. Then algebra $M_a^1 := \big([a, 1]; \odot_a^1, \wedge, \vee, \rightarrow, \rightsquigarrow, a, 1\big)$ is a bounded $R\ell$-monoid, where $x \odot_a^1 y = (x \odot y) \vee a$. Moreover, if M satisfies the prelinearity condition, then also M_a^1 satisfies it.*

P r o o f. It is obvious that $\big([a, 1]; \wedge, \vee, a, 1\big)$ is a bounded lattice with the smallest element a and the greatest element 1. The facts that 1 is a unit in M and a trivial observation $x = x \vee a$ for any $x \in [a, 1]$ imply that 1 is a unit in $\big([a, 1]; \odot_a^1, 1\big)$. Associativity of $\odot_a^1$ is proved with the use of distributivity of the product operation over lattice operations and the fact that $a \odot x, x \odot a \le a$. Divisibility is implied by divisibility in M. If $x \odot_a^1 y = x \odot y \vee a \le z$, then $x \odot y \le z$ and $x \le y \rightarrow z$, $y \le x \rightsquigarrow z$. The converse, $x \le y \rightarrow z$ implies $x \odot y \le z$ and for $z \ge a$, $x \odot y \vee a \le z \vee a = z$. The implication $y \le x \rightsquigarrow z$ is proved in analogous manner. Prelinearity on a subset $[a, 1]$ follows from the prelinearity of M. $\square$

Trivially, $M = M_0^1$, $\{1\} = M_1^1$.

THEOREM 3.2. *Let $(M; \odot, \wedge, \vee, \rightarrow, \rightsquigarrow, 0, 1)$ be a bounded $R\ell$-monoid and let $a \in M$. Then $M_0^a := \big([0, a]; \odot_0^a, \wedge, \vee, \rightarrow_0^a, \rightsquigarrow_0^a, 0, a\big)$, where*

$$x \odot_0^a y = x \odot (a \rightsquigarrow y), \quad x \rightarrow_0^a y = (x \rightarrow y) \odot a \quad \textit{and} \quad x \rightsquigarrow_0^a y = a \odot (x \rightsquigarrow y),$$

is a bounded $R\ell$-monoid. Moreover, if M satisfies the prelinearity condition, the same is true for M_0^a.

Proof. Obviously, $([0,a];\wedge,\vee,0,a)$ is a bounded lattice with the smallest element 0 and the greatest element a. First we prove that a is a unit with respect to the $\odot_0^a$. Indeed, $x \odot_0^a a = x \odot (a \rightsquigarrow a) = x \odot 1 = x$ and $a \odot_0^a x = a \odot (a \rightsquigarrow x) = a \wedge x = x$ for any $x \in [0,a]$. For any $x, y \in [0,a]$ the product $\odot_0^a$ can be expressed in the following ways: $x \odot_0^a y = x \odot (a \rightsquigarrow y) = (a \wedge x) \odot (a \rightsquigarrow y) = a \odot (a \rightsquigarrow x) \odot (a \rightsquigarrow y) = (a \to x) \odot a \odot (a \rightsquigarrow y) = (a \to x) \odot (a \to y) \odot a = (a \to x) \odot (y \wedge a) = (a \to x) \odot y$. Using the previous facts and associativity of $\odot$, we can prove associativity of a product $\odot_0^a$. $x \odot_0^a (y \odot_0^a z) = x \odot_0^a \big(y \odot (a \rightsquigarrow z)\big) = (a \to x) \odot \big(y \odot (a \rightsquigarrow z)\big) = (a \to x) \odot (y \wedge a) \odot (a \rightsquigarrow z) = (a \to x) \odot (a \to y) \odot a \odot (a \rightsquigarrow z) = (x \odot_0^a y) \odot (a \rightsquigarrow z) = (x \odot_0^a y) \odot_0^a z$. Therefore, $\big([0,a]; \odot_0^a, a\big)$ is a monoid with a unit a.

The divisibility condition, for any $x, y \in [0,a]$, can be proved as follows: $x \odot_0^a (x \rightsquigarrow_0^a y) = (a \to x) \odot \big(a \odot (x \rightsquigarrow y)\big) = (a \wedge x) \odot (x \rightsquigarrow y) = x \odot (x \rightsquigarrow y) = x \wedge y$ and $(x \to_0^a y) \odot_0^a x = \big((x \to y) \odot a\big) \odot (a \rightsquigarrow x) = (x \to y) \odot (a \wedge x) = (x \to y) \odot x = x \wedge y$, where we used the divisibility of M.

Let us assume that $x \odot_0^a y \le z$, i.e. $x \odot (a \rightsquigarrow y), (a \to x) \odot y \le z$. From adjointness of M we have $a \rightsquigarrow y \le x \rightsquigarrow z$ and $a \to x \le y \to z$. Multiplying the first inequality by a from the left and the second by a from the right implies: $y = a \wedge y \le a \odot (x \rightsquigarrow y) = x \rightsquigarrow_0^a y$ and $x = a \wedge x \le (x \to y) \odot a = x \to_0^a y$. For the converse, let us assume $x \le y \to_0^a z = (y \to z) \odot a$ $\big(y \le x \rightsquigarrow_0^a z = a \odot (x \rightsquigarrow z)\big)$. Then $x \odot_0^a y = x \odot (a \rightsquigarrow y) \le (y \to z) \odot a \odot (a \rightsquigarrow y) = (y \to z) \odot (a \wedge y) = (y \to z) \odot y \le y \wedge z \le z$ for $y \le a$ $\big(x \odot_0^a y = (a \to x) \odot y \le (a \to x) \odot a \odot (x \rightsquigarrow z) = (a \wedge x) \odot (x \rightsquigarrow z) = x \odot (x \rightsquigarrow z) = x \wedge z \le z$ for $x \le a\big)$.

Let M satisfy the prelinearity condition. Then $(x \rightsquigarrow_0^a y) \vee (y \rightsquigarrow_0^a x) = \big(a \odot (x \rightsquigarrow y)\big) \vee \big(a \odot (y \rightsquigarrow x)\big) = a \odot \big((x \rightsquigarrow y) \vee (y \rightsquigarrow x)\big) = a \odot 1 = a$. The equality for $\to$ is proved in the similar manner. □

Trivially, $M = M_0^1$, $\{0\} = M_0^0$.

In the particular cases, e.g. a is an idempotent, is the operation $x \to_0^a y$ simply defined as $(x \to y) \wedge a$. However, it is not the case for arbitrary elements a.

EXAMPLE 3.3. The operation $\to_0^a$ cannot be defined as $x \to_0^a y = a \wedge (x \to y)$, e.g. if a is not an idempotent. Let us consider a BL-algebra whose product operation corresponds to the product t-norm and the subinterval $\big[0, \frac{1}{2}\big]$ of the unit real interval. Then

$$x \to_0^{\frac{1}{2}} y = \frac{1}{2} \wedge (x \to y) = \begin{cases} \frac{1}{2} & \text{for } x < 2y\,, \\ \frac{y}{x} & \text{for } x \ge 2y\,. \end{cases}$$

Due to the divisibility condition, $x \wedge y = x \odot_0^{\frac{1}{2}} (x \to_0^{\frac{1}{2}} y) = x \odot_0^{\frac{1}{2}} \frac{1}{2} = x$ for $x < 2y$, but considering $y < x < 2y$, we obtain $x \wedge y = y \ne x$, i.e. the contradiction.

We can obtain an algebraic structure on the subinterval $[a,b]$, $a \le b$, by applying Theorems 3.1 and 3.2. The order of the application is not important, i.e. both results lead to the same operations, as the following computations show:
$[0,1] \to [a,1] \to [a,b]$:

$[a,1]$: $x \odot_a^1 y = (x \odot y) \vee a$, $x \to_a^1 y = x \to y$, $x \rightsquigarrow_a^1 y = x \rightsquigarrow y$.

For any $x, y \in [a,b]$ we have the following inequalities.

$$\begin{aligned}(x \to y) \odot b &\ge (x \to y) \odot x = x \wedge y \ge a\,,\\ b \odot (x \rightsquigarrow y) &\ge x \odot (x \rightsquigarrow y) = x \wedge y \ge a\,.\end{aligned}$$

$[a,b]$:

$$\begin{aligned}x \odot_a^b y &= x \odot_a^1 (b \rightsquigarrow_a^1 y) = x \odot_a^1 (b \rightsquigarrow y) = \big(x \odot (b \rightsquigarrow y)\big) \vee a\,,\\ x \to_a^b y &= (x \to_a^1 y) \odot_a^1 b = (x \to y) \odot_a^1 b = \big((x \to y) \odot b\big) \vee a = (x \to y) \odot b\,,\\ x \rightsquigarrow_a^b y &= b \odot_a^1 (x \rightsquigarrow_a^1 y) = b \odot_a^1 (x \rightsquigarrow y) = \big(b \odot (x \rightsquigarrow y)\big) \vee a = b \odot (x \rightsquigarrow y)\,.\end{aligned}$$

$[0,1] \to [0,b] \to [a,b]$:

$[0,b]$: $x \odot_0^b y = x \odot (b \rightsquigarrow y)$, $x \to_0^b y = (x \to y) \odot b$, $x \rightsquigarrow_0^b y = b \odot (x \rightsquigarrow y)$.

$[a,b]$:

$$\begin{aligned}x \odot_a^b y &= (x \odot_0^b y) \vee a = \big(x \odot (b \rightsquigarrow y)\big) \vee a\,,\\ x \to_a^b y &= x \to_0^b y = (x \to y) \odot b\,,\\ x \rightsquigarrow_a^b y &= x \rightsquigarrow_0^b y = b \odot (x \rightsquigarrow y)\,.\end{aligned}$$

COROLLARY 3.4. *Let $(M; \odot, \wedge, \vee, \to, \rightsquigarrow, 0, 1)$ be a bounded $R\ell$-monoid and let $a, b \in M$ such that $a \le b$. Then*

$$M_a^b := \big([a,b]; \odot_a^b, \wedge, \vee, \to_a^b, \rightsquigarrow_a^b, a, b\big)$$

is a bounded $R\ell$-monoid, where

$$x \odot_a^b y = \big(x \odot (b \rightsquigarrow y)\big) \vee a\,, \quad x \to_a^b y = (x \to y) \odot b \quad \textit{and} \\ x \rightsquigarrow_a^b y = b \odot (x \rightsquigarrow y)\,.$$

Moreover, if M satisfies a prelinearity condition, the same is true for M_a^b.

Obtained operations correspond to ones presented in [3] and [4], [15] for MV-algebras and pseudo MV- (GMV-) algebras, respectively. We only check the non-commutative case from [4] because J a k u b í k's result is equivalent to it (see [15]). We will use a group representation of pseudo MV-algebras due to A. D v u r e č e n s k i j [9], i.e. $M = \Gamma(G,u) = \{x \in G :\ 0 \le x \le u\}$, where G is a lattice ordered group with a strong unit u and MV-operations are defined as follows: $x \oplus y = (x+y) \wedge u$, $x^- = u - x$ and $x^\sim = -x + u$.

We recall this result ([4]):

For $a, b \in M$, $a < b$, the operations on GMV-algebra over $[a,b]$ are defined as follows: $x \oplus_{ab} y = \big(x \oplus (y \odot a^{\sim})\big) \wedge b$, $x^{-_{ab}} = (x^{-} \odot b) \oplus a$, $x^{\sim_{ab}} = a \oplus (b \odot x^{\sim})$.

Then

$$\begin{aligned} x \oplus_{ab} y &= \big(x \oplus (y \odot a^{\sim})\big) \wedge b = \big(x \oplus \big(y \odot (-a+u)\big)\big) \wedge b \\ &= \big(x \oplus \big(((-a+u) - u + y) \vee 0\big)\big) \wedge b = \big(x \oplus (-a+y) \vee 0\big) \wedge b \\ &= \big((x + (-a+y) \vee 0\big) \wedge u\big) \wedge b = \big((x - a + y) \vee x\big) \wedge b \\ &= (x - a + y) \wedge b \qquad \text{for} \quad x, y \in [a,b]\,. \end{aligned}$$

$$\begin{aligned} x^{-_{ab}} &= (x^{-} \odot b) \oplus a = \big((u - x) \odot b\big) \oplus a = \big((b - u + (u - x)) \vee 0\big) \oplus a \\ &= \big((b - x) \vee 0\big) \oplus a = \big((b - x) \vee 0 + a\big) \wedge u = \big((b - x + a) \vee a\big) \wedge u \\ &= (b - x + a) \wedge u = b - x + a \qquad \text{for} \quad x, y \in [a,b]\,. \end{aligned}$$

$$\begin{aligned} x^{\sim_{ab}} &= a \oplus (b \odot x^{\sim}) = a \oplus \big(b \odot (-x+u)\big) = a \oplus \big(((-x+u) - u + b\big) \vee 0\big) \\ &= a \oplus \big((-x+b) \vee 0\big) = \big(a + \big((-x+b) \vee 0\big)\big) \wedge u \\ &= \big((a - x + b) \vee a\big) \wedge u = (a - x + b) \wedge u \\ &= a - x + b \qquad \text{for} \quad x, y \in [a,b]\,. \end{aligned}$$

Our results, when expressed in the group representation language, imply the following:

$$\begin{aligned} x^{-_{ab}} &= x \to_a^b a = (x \to a) \odot b = (x^{-} \oplus a) \odot b = \big((u - x + a) \wedge u\big) \odot b \\ &= \big(b - u + (u - x + a) \wedge u\big) \vee 0 = \big((b - x + a) \wedge b\big) \vee 0 = (b - x + a) \vee 0 \\ &= b - x + a \qquad \text{for} \quad x \in [a,b]\,. \end{aligned}$$

$$\begin{aligned} x^{\sim_{ab}} &= x \rightsquigarrow_a^b a = b \odot (x \rightsquigarrow a) = b \odot (a \oplus x^{\sim}) = b \odot \big((a - x + u) \wedge u\big) \\ &= \big((a - x + u) \wedge u - u + b\big) \vee 0 = \big((a - x + b) \wedge b\big) \vee 0 = (a - x + b) \vee 0 \\ &= a - x + b \qquad \text{for} \quad x \in [a,b]\,. \end{aligned}$$

$$\begin{aligned} x \odot_a^b y &= x \odot (b \rightsquigarrow y) \vee a = \big(x \odot (y \oplus b^{\sim})\big) \vee a = \big(x \odot \big((y - b + u) \wedge u\big)\big) \vee a \\ &= \big(\big((y - b + u) \wedge u - u + x\big) \vee 0\big) \vee a = \big(\big((y - b + x) \wedge x\big) \vee 0\big) \vee a \\ &= (y - b + x) \wedge x \vee a = (y - b + x) \vee a \qquad \text{for} \quad x, y \in [a,b]\,. \end{aligned}$$

$$\begin{aligned} x \oplus_a^b y &:= \left(x^{-ab} \odot_a^b y^{-ab}\right)^{\sim ab} = \left((b-x+a) \odot_a^b (b-y+a)\right)^{\sim ab} \\ &= \left(\left((b-y+a) - b + (b-x+a)\right) \vee a\right)^{\sim ab} \\ &= \left((b-y+a-x+a) \vee a\right)^{\sim ab} \\ &= a - \left((b-y+a-x+a) \vee a\right) + b \\ &= \left(\left(a + (-a+x-a+y-b)\right) \wedge 0\right) + b \\ &= (x-a+y) \wedge b \qquad \text{for} \quad x, y \in [a,b]\,. \end{aligned}$$

When b in Corollary 3.4 is an idempotent, then operations become as follows:

$$\begin{aligned} x \odot_a^b y &= x \odot (b \rightsquigarrow y) \vee a = (b \wedge x) \odot (b \rightsquigarrow y) \vee a = (b \to x) \odot b \odot (b \rightsquigarrow y) \vee a \\ &= (b \to x) \odot b \odot b \odot (b \rightsquigarrow y) \vee a = (b \wedge x) \odot (b \wedge y) \vee a \\ &= (x \odot y) \vee a \qquad \text{for} \quad x, y \le b\,, \end{aligned}$$

$$x \to_a^b y = (x \to y) \odot b = (x \to y) \wedge b \qquad \text{and}$$

$$x \rightsquigarrow_a^b y = b \odot (x \rightsquigarrow y) = (x \rightsquigarrow y) \wedge b\,.$$

Moreover, if a is also an idempotent, then $x \odot_a^b y = x \odot y$.

4. Identities $\mathrm{PL}(a,x,y)$, $\mathrm{PR}(a,x,y)$ and $\mathrm{P}(a,x,y)$

Let us consider the following identities:

$$a \to \left((x \to y) \odot a\right) = x \to y\,, \qquad \mathrm{PL}(a,x,y)$$

$$a \rightsquigarrow \left(a \odot (x \rightsquigarrow y)\right) = x \rightsquigarrow y\,. \qquad \mathrm{PR}(a,x,y)$$

In the commutative case $\mathrm{PL}(a,x,y) \equiv \mathrm{PR}(a,x,y) := \mathrm{P}(a,x,y)$:

$$a \to \left(a \odot (x \to y)\right) = x \to y\,. \qquad \mathrm{P}(a,x,y)$$

Let M be a bounded $R\ell$-monoid and let $a \in M$ be a fixed element. We say that M *satisfies the condition* $\mathrm{PL}(a)$ ($\mathrm{PR}(a)$, $\mathrm{P}(a)$) if the identity $\mathrm{PL}(a,x,y)$ ($\mathrm{PR}(a,x,y)$, $\mathrm{P}(a,x,y)$) hold for any $x, y \le a$.

The monotonicity of residua ensures for the expressions in identities $\mathrm{PL}(a)$ and $\mathrm{PR}(a)$ the following lower and upper bounds:

$$x \to y \le a \to \left((x \to y) \odot a\right) \le a \to (x \to y) = (a \odot x) \to y\,, \tag{2}$$

$$x \rightsquigarrow y \le a \rightsquigarrow \left(a \odot (x \rightsquigarrow y)\right) \le a \rightsquigarrow (x \rightsquigarrow y) = (x \odot a) \rightsquigarrow y\,. \tag{3}$$

Thus by (1), if a is an idempotent, then $\mathrm{PL}(a)$ and $\mathrm{PR}(a)$ are satisfied.

For lower bounds of residua we have for any $x, y \in [0,a]$:

$$x \to y \ge x \to 0 \ge a \to 0\,, \tag{4}$$

$$x \rightsquigarrow y \ge x \rightsquigarrow 0 \ge a \rightsquigarrow 0\,. \tag{5}$$

However, the replacing of the condition $\mathrm{P}(a)$ with either

$$a \to (a \odot x) = x \qquad \text{for any} \quad a \geq x \geq a \to 0\,,$$

or

$$a \to (a \odot x) = (a \to 0) \vee x \qquad \text{for any} \quad x \in [0,a]$$

weakens the original condition as the following example shows.

EXAMPLE 4.1. Let us consider $M = \{0, a, 1\}$, the 3-element linear Gödel BL-algebra, i.e. M is an ordinal sum of two copies of $\{0,1\}$ MV-algebras. Then a is an idempotent, so $\mathrm{P}(a)$ holds and $a \to 0 = 0$, but $a \to (a \odot a) = a \to a = 1 \neq a = a \vee (a \to 0) = a \vee 0$.

If L is a pseudo MV-algebra, then the conditions $\mathrm{PL}(a)$ and $\mathrm{PR}(a)$ are satisfied. Let us recall,

$$x \rightsquigarrow y = y \oplus x^{\sim}\,, \qquad x \to y = x^{-} \oplus y\,, \qquad a^{\sim} = a \rightsquigarrow 0\,, \qquad a^{-} = a \to 0\,.$$

Using (4) and (5), for $z \equiv x \to y$ or $z \equiv x \rightsquigarrow y$, we obtain:

$$\begin{aligned} a \to (z \odot a) &= a^{-} \oplus (z \odot a) = a^{-} \oplus (z \odot a^{-\sim}) = a^{-} \vee z = z\,,\\ a \rightsquigarrow (a \odot z) &= (a \odot z) \oplus a^{\sim} = (a^{\sim -} \odot z) \oplus a^{\sim} = a^{\sim} \vee z = z\,. \end{aligned}$$

Please note, that we have used the GMV definition ([16]) of a product, i.e. $x \odot y = (x^{-} \oplus y^{-})^{\sim}$ instead of a converse definition of the product ([13]), i.e. $y \odot x = (x^{-} \oplus y^{-})^{\sim}$. (In the latter case, a divisibility becomes $x \odot (x \to y) = (x \rightsquigarrow y) \odot y$, etc..)

Moreover, for BL-algebras (by [5; Lemma 4.1]) if M is either a linearly ordered Łukasiewicz BL-algebra (an MV-chain) or a linearly ordered product BL-algebra (a product chain), then the equation $x \to (x \odot y) = (x \to 0) \vee y$ is satisfied for any $x, y \in M$. Applying the result for $x \equiv a$, $y \equiv x \to y$ and using the lower bounds for $x \to y$, $x, y \in [0,a]$, ((4), (5)), we obtain that $\mathrm{P}(a)$ holds for MV-chains and product BL-chains.

Let us recall, that Gödel BL-algebras satisfy the identity $x \odot x = x$, therefore each element of a Gödel BL-algebra is an idempotent and the condition $\mathrm{P}(a)$ trivially holds for any a. Moreover, in Gödel BL-algebras we have $a \to (a \odot x) = a \to x$ for any $a, x \in M$. Due to [8; Corollary 2.30], all Gödel pseudo BL-algebras, i.e. pseudo BL-algebras satisfying identity $x \odot x = x$, are commutative, i.e. BL-algebras.

Using the similar technique as in [5; Lemma 4.1] we can prove that linear product pseudo BL-algebras satisfy the conditions $\mathrm{PL}(a)$ and $\mathrm{PR}(a)$ for any a. First we prove that the following identities are satisfied for any $a, x \in M$:

$$a \to (x \odot a) = (a \to 0) \vee x\,, \tag{6}$$

$$a \rightsquigarrow (a \odot x) = (a \rightsquigarrow 0) \vee x\,. \tag{7}$$

Indeed, if $a = 0$, it holds trivially. Let us consider $a \neq 0$, then from $a \wedge a^- = 0 = a \wedge a^\sim$ and linearity of order we obtain that $a^- = a^\sim = 0$ and therefore $a^{--} = a^{\sim\sim} = 1$. Using the second and third identity, where $z \equiv a$, $x \equiv 1$ and $y \equiv x$, we have $z^{--} \odot \big((x \odot z) \to (y \odot z)\big) \leq x \to y \equiv 1 \odot \big((1 \odot a) \to (x \odot a)\big) = a \to (x \odot a) \leq 1 \to x = x \leq x \vee (a \to 0)$ and $z^{\sim\sim} \odot \big((z \odot x) \rightsquigarrow (z \odot y)\big) \leq x \rightsquigarrow y \equiv 1 \odot \big((a \odot 1) \rightsquigarrow (a \odot x)\big) = a \rightsquigarrow (a \odot x) \leq 1 \rightsquigarrow x = x \leq x \vee (a \rightsquigarrow 0)$. The converse inequalities stems from the monotonicity of residua and the adjointness, i.e. $a \to (x \odot a) \geq x, a \to 0$ and $a \rightsquigarrow (a \odot x) \geq x, a \rightsquigarrow 0$. So again assuming $x \equiv x \to y$ in (6) and $x \equiv x \rightsquigarrow y$ in (7) and the use of (4), (5) conclude the validity of the conditions $\mathrm{PL}(a)$ and $\mathrm{PR}(a)$ for any $a \in M$.

The previous reasoning is summarized in the following proposition.

Proposition 4.2. *Let M be either*

(i) *a linear product BL-algebra,*

or

(ii) *a Gödel BL-algebra,*

or

(iii) *a linear Łukasiewicz BL-algebra,*

or

(iv) *an MV-algebra,*

or

(v) *a linear product pseudo BL-algebra,*

or

(vi) *a pseudo MV-algebra*

and let $a \in M$. Then in the cases (i)–(iv), *M satisfies the condition* $\mathrm{P}(a)$ *and in the cases* (v) *and* (vi) *M satisfies* $\mathrm{PL}(a)$ *and* $\mathrm{PR}(a)$.

5. Algebraic constructions and the conditions $\mathrm{P}(a)$, $\mathrm{PL}(a)$ and $\mathrm{PR}(a)$

In this part, we perform a further analysis of the conditions $\mathrm{P}(a)$, $\mathrm{PL}(a)$ and $\mathrm{PR}(a)$ with respect to algebraic constructions of new algebras such as forming a subalgebra, an isomorphic image, a direct and subdirect product and an ordinal sum (pasting). Unfortunately, homomorphic images in general do not preserve validity of the conditions $\mathrm{P}(a)$, $\mathrm{PL}(a)$ and $\mathrm{PR}(a)$. As a consequence of P. Hájek's representational theorem ([14]) of BL-algebras we conclude that $\mathrm{P}(a)$ is satisfied for any BL-algebra M and all $a \in M$.

We start with direct and subdirect products. The componentwise definition of the operations in direct or subdirect products of bounded $R\ell$-monoids implies the following propositions.

PROPOSITION 5.1. *Let $(M_i : i \in I)$ be a non-empty system of commutative bounded $R\ell$-monoids such that for each $i \in I$ and $a \in M_i$ the condition* $\mathrm{P}(a)$ *is satisfied and let M be either* (i) *a direct, or* (ii) *a subdirect product of M_i, $i \in I$. Then for any $a \in M$ the condition* $\mathrm{P}(a)$ *holds.*

PROPOSITION 5.2. *Let $(M_i : i \in I)$ be a non-empty system of bounded $R\ell$-monoids satisfying the conditions* $\mathrm{PL}(a)$ *and* $\mathrm{PR}(a)$ *for any $i \in I$ and $a \in M_i$ and let M be either* (i) *a direct, or* (ii) *a subdirect product of M_i, $i \in I$. Then M also satisfies the conditions* $\mathrm{PL}(a)$ *and* $\mathrm{PR}(a)$ *for any $a \in M$.*

The conditions $\mathrm{P}(a,x,y)$, $\mathrm{PL}(a,x,y)$ and $\mathrm{PR}(a,x,y)$ are in the form of the identities, therefore they remain valid also for subalgebras.

PROPOSITION 5.3. *Let M be a commutative bounded $R\ell$-monoid satisfying the condition* $\mathrm{P}(a)$ *for all $a \in M$ and let $S \subset M$ be its subalgebra. Then S satisfies the condition* $\mathrm{P}(a,x,y)$ *for all $a \in S$ and all $x, y \in S \cap [0,a]$.*

PROPOSITION 5.4. *Let M be a bounded $R\ell$-monoid satisfying the conditions* $\mathrm{PL}(a)$ *and* $\mathrm{PR}(a)$ *for all $a \in M$ and let $S \subset M$ be its subalgebra. Then S satisfies the conditions* $\mathrm{PL}(a,x,y)$, $\mathrm{PR}(a,x,y)$ *for all $a \in S$ and all $x, y \in S \cap [0,a]$.*

Let A, B be bounded $R\ell$-monoids and let $h\colon A \to B$ be a surjective homomorphism, i.e. a surjective mapping preserving operations $\wedge$, $\vee$, $\odot$, $\to$, $\rightsquigarrow$ and constants 0, 1. Then for any $a, x, y \in B$, there exists u_a, u_x, u_y from A such that $h(u_a) = a$, $h(u_x) = x$ and $h(u_y) = y$. Although $a \to \big((x \to y) \odot a\big) = h\big(u_a \to \big((u_x \to u_y) \odot u_a\big)\big)$ and $a \rightsquigarrow \big(a \odot (x \rightsquigarrow y)\big) = h\big(u_a \rightsquigarrow \big(u_a \odot (u_x \rightsquigarrow u_y)\big)\big)$, it need not be true that $u_a \to \big((u_x \to u_y) \odot u_a\big) = u_x \to u_y$ (or $u_a \rightsquigarrow \big(u_a \odot (u_x \rightsquigarrow u_y)\big) = u_x \rightsquigarrow u_y$), because $a \geq x, y$ does not in general imply $u_a \geq u_x, u_y$. We can only prove $h(u_a \wedge u_x) = h(u_x)$, but not in general (e.g. without assuming injectivity of h) $u_a \wedge u_x = u_x$. Thus, if h is an isomorphism, then the conditions $\mathrm{PL}(a)$ and $\mathrm{PR}(a)$ are preserved.

For the finite ordinal sums we have the following.

PROPOSITION 5.5. *Let M be an ordinal sum of commutative bounded $R\ell$-monoids $M_1, \dots, M_n$. Let a be an element of M_j for some j, $1 \leq j \leq n$. Then, if M_j satisfies the condition* $\mathrm{P}(a)$, *then the condition* $\mathrm{P}(a,x,y)$ *holds for all $x, y \in [0,a] \subseteq M$, i.e.* $\mathrm{P}(a)$ *holds in M.*

Proof. For $a \in M$ there are three cases:

(A) $a \in M_j \setminus \{0_j, 1_j\}$,
(B) $a = 0_j$,
(C) $a = 1_j$.

For $j = 1$ all three cases hold due to the assumption concerning the algebra M_j. Let us assume $j > 1$. We can exclude the case (B) from our reasoning, because it is equivalent to the case (C) for the index $j-1$. For the cases (A) and (C), each element u of the interval $[0,a]$ is one of the following

(i) $u \in M_j \setminus M_{j-1} = M_j \setminus \{0_j\}$,
(ii) $u \in M_j \cap M_{j-1} = 0_j = 1_{j-1}$,
(iii) $u \in M_k \setminus M_j$, $k \le j-1$.

The analysis of all the nine possibilities is summarized in the table:

$x \longrightarrow y$	$M_j \setminus M_{j-1}$	$M_j \cap M_{j-1}$	$M_l \setminus M_j$, $l \le j-1$
$M_j \setminus M_{j-1}$	1 or $\in M_j \setminus \{1_j\}$	$\in M_j \setminus \{1_j\}$	$y \in M_l \setminus M_j$
$M_j \cap M_{j-1}$	$1(\le)$	$1(=)$	$y \in M_l \setminus M_j$
$M_k \setminus M_j$, $k \le j-1$	$1(\le)$	$1(\le)$	1 or $\in M_t \setminus M_j$, $t = \min(k,l)$

There are the following possibilities:

- Case 1: $x \to y = 1$.

$$a \to \big(a \odot (x \to y)\big) = a \to (a \odot 1) = a \to a = 1 = (x \to y).$$

- Case 2: $x \to y \in M_i \setminus M_j$ for $i < j$.

$$a \to \big(a \odot (x \to y)\big) = a \to (x \to y) = x \to y.$$

- Case 3: $x \to y \in M_j$.
  The condition implies that $x, y \in M_j \cap [0,a]$ and we can use the assumption of the proposition for M_j.

□

PROPOSITION 5.6. *Let M be an ordinal sum of bounded $R\ell$-monoids M_i, $i = 1,\dots,n$. Let a be an element of M_j for some j, $1 \le j \le n$. Then, if M_j satisfies the conditions* $\mathrm{PL}(a)$ *and* $\mathrm{PR}(a)$, *then the conditions* $\mathrm{PL}(a,x,y)$, $\mathrm{PR}(a,x,y)$ *hold for all $x, y \in [0,a] \subseteq M$, i.e.* $\mathrm{PL}(a)$, $\mathrm{PR}(a)$ *hold in M.*

Proof. A similar analysis as in the commutative case gives us the following three cases:

- Case 1: $x \to y = x \rightsquigarrow y = 1$.

$$a \to \big((x \to y) \odot a\big) = a \to (1 \odot a) = a \to a = 1 = x \to y,$$
$$a \rightsquigarrow \big(a \odot (x \rightsquigarrow y)\big) = a \rightsquigarrow (a \odot 1) = a \rightsquigarrow a = 1 = x \rightsquigarrow y.$$

- Case 2: $x \to y, x \rightsquigarrow y \in M_i \setminus M_j$ for $j > i$.

$$a \to \big((x \to y) \odot a\big) = a \to (x \to y) = x \to y\,,$$
$$a \rightsquigarrow \big(a \odot (x \rightsquigarrow y)\big) = a \rightsquigarrow (x \rightsquigarrow y) = x \rightsquigarrow y\,.$$

- Case 3: $x \to y, x \rightsquigarrow y \in M_j$.
  The conditions imply $x, y \in M_j \cap [0, a]$ and we can use the assumption of the proposition for M_j.

$\square$

For infinite ordinal sums we can consider two kinds of definitions of an ordinal sum. One is due to P. Hájek [14] and the second is due to P. Agliano, F. Montagna [1].

The essence of P. Hájek's definition is in the extension to a linearly ordered set with the smallest and greatest element for which the concept of the successor is used. In contrast, P. Agliano, F. Montagna [1] are using only linearly ordered set with the smallest element and they identify all elements 1_i, $i \in I$, to the single element 1. The concept of successor is not needed for their approach. We introduce the extension of the definitions for bounded $R\ell$-monoids.

Definition 5.7 (Hájek). Let $(I, \leq)$ be a chain with a least element 0 and a largest element 1. For each $\alpha \in I$, we define α^+, the *upper neighbour* of α, if it exists, i.e. $\alpha^+ = \beta$ iff $\beta > \alpha$ and there is no γ such that $\alpha < \gamma < \beta$. Otherwise, we define $\alpha^+ = \alpha$. Let $(M_\alpha : \alpha \in I)$ be a system of bounded $R\ell$-monoids such that for each α, α is the least, α^+ the greatest element of M_α and $\big(M_\alpha - \{\alpha, \alpha^+\}\big) \cap \big(M_\beta - \{\beta, \beta^+\}\big) = \emptyset$ for any $\alpha, \beta \in I$, $\alpha \neq \beta$. Then the *ordinal sum* of the system $(M_\alpha : \alpha \in I)$, denoted M, is the algebra $(M; \odot, \wedge, \vee, \to, \rightsquigarrow, 0, 1)$, where the sets and operations are defined as follows:

$$M = \bigcup M_\alpha\,,$$

$$x \leq y \quad \text{for } x \in M_\alpha\,,\ y \in M_\beta \qquad \text{iff} \qquad \alpha < \beta \text{ or } \alpha = \beta \text{ and } x \leq_\alpha y\,,$$

$$x \odot y = \begin{cases} x \odot_\alpha y\,, & x, y \in M_\alpha\,, \\ \min(x, y)\,, & x \in M_\alpha\,,\ y \in M_\beta\,,\ \alpha \neq \beta\,, \end{cases}$$

$$x \to y = \begin{cases} 1\,, & x \leq y \\ x \to_\alpha y\,, & x, y \in M_\alpha\,, \\ y\,, & x \in M_\alpha \setminus M_\beta\,,\ y \in M_\beta \setminus M_\alpha\,,\ \alpha > \beta\,, \end{cases}$$

$$x \rightsquigarrow y = \begin{cases} 1\,, & x \leq y\,, \\ x \rightsquigarrow_\alpha y\,, & x, y \in M_\alpha\,, \\ y\,, & x \in M_\alpha \setminus M_\beta\,,\ y \in M_\beta \setminus M_\alpha\,,\ \alpha > \beta\,. \end{cases}$$

DEFINITION 5.8 (AGLIANO, MONTAGNA). Let $(I,\leq)$ be a linearly ordered set with the smallest element 0 and let M_α be a system of bounded $R\ell$-monoids such that $M_\alpha \cap M_\beta = \{1\}$, for $\alpha,\beta \in I$, $\alpha \neq \beta$. Then the *ordinal sum* of the system $(M_\alpha : \alpha \in I)$ is the algebra $(\bigcup M_\alpha; \wedge, \vee, \odot, \to, \rightsquigarrow, 0_0, 1)$, where the operations are defined as follows:

$$\begin{array}{ll} x \leq y & \text{if} \quad x \leq_\alpha y\,, \quad \alpha \in I\,, \quad x,y \in M_\alpha\,, \\ x \leq y & \text{if} \quad x \in M_\alpha\,, \quad y \in M_\beta\,, \quad \alpha < \beta\,, \end{array}$$

$$x \odot y = \begin{cases} x \odot_\alpha y\,, & x,y \in M_\alpha\,, \\ y\,, & x \in A_\alpha\,, \ y \in A_\beta \setminus \{1\}\,, \ \alpha > \beta\,, \\ x\,, & x \in A_\alpha \setminus \{1\}\,, \ y \in A_\beta\,, \ \alpha < \beta\,, \end{cases}$$

$$x \to y = \begin{cases} x \to_\alpha y\,, & x,y \in M_\alpha\,, \\ y\,, & x \in A_\alpha\,, \ y \in A_\beta \setminus \{1\}\,, \ \alpha > \beta\,, \\ 1\,, & x \in A_\alpha \setminus \{1\}\,, \ y \in A_\beta\,, \ \alpha < \beta\,, \end{cases}$$

$$x \rightsquigarrow y = \begin{cases} x \rightsquigarrow_\alpha y\,, & x,y \in M_\alpha\,, \\ y\,, & x \in A_\alpha\,, \ y \in A_\beta \setminus \{1\}\,, \ \alpha > \beta\,, \\ 1\,, & x \in A_\alpha \setminus \{1\}\,, \ y \in A_\beta\,, \ \alpha < \beta\,. \end{cases}$$

Same as for a finite case, also infinite ordinal sums preserve the conditions $\mathrm{P}(a)$, $\mathrm{PL}(a)$, $\mathrm{PR}(a)$.

PROPOSITION 5.9. *Let $(I,\leq)$ be a linearly ordered set with the smallest element 0 and the greatest element 1 and let $(M_\alpha : \alpha \in I)$ be a system of bounded $R\ell$-monoids satisfying the conditions $\mathrm{PL}(a)$, $\mathrm{PR}(a)$ for any $\alpha \in I$ and $a \in M_\alpha$. Let M be the ordinal sum of the system M_α, $\alpha \in I$, in the sense of Hájek. Then for any $a \in M$ the conditions $\mathrm{PL}(a)$, $\mathrm{PR}(a)$ hold.*

Proof. Let a be an element of M. Then there exists $\alpha \in I$ such that $a \in M_\alpha$. Let x, y be elements of $[0,a] \subseteq M$. Using the same reasoning as in Proposition 5.5, we conclude that there are the following three cases:

(i) $x \to y$, $x \rightsquigarrow y \in M_\alpha$,

(ii) $x \to y = 1 = x \rightsquigarrow y$,

(iii) $x \to y$, $x \rightsquigarrow y \in M_\beta \setminus M_\alpha$, $\beta < \alpha$.

Again, the case (ii) is trivial, the case (i) follows from the assumption concerning M_α and the case (iii) from the definition of the operations. $\square$

PROPOSITION 5.10. *Let $(I, \leq)$ be a linearly ordered set with the smallest element 0, let $(M_\alpha : \alpha \in I)$ be a system of bounded $R\ell$-monoids such that $M_\alpha \cap M_\beta = \{1\}$, $\alpha, \beta \in I$, $\alpha \neq \beta$, and that* $\mathrm{PL}(a)$, $\mathrm{PR}(a)$ *are satisfied for all $\alpha \in I$ and any $a \in M_\alpha$. Then the ordinal sum in the sense of Agliano, Montagna of the system $(M_\alpha : \alpha \in I)$, denoted M, satisfies the conditions* $\mathrm{PL}(a)$ *and* $\mathrm{PR}(a)$ *for any $a \in M$.*

Proof. Any $a \in M$ is either 1, or $a \in M_\alpha \setminus \{1\}$ for some $\alpha \in I$. The case $a = 1$ is trivial. Let us assume $a \neq 1$ and let $x, y \in [0, a]$. For the element $u \in [0, a]$ there are the following two possibilities: $u \in M_\alpha \setminus \{1\}$, or $u \in M_\beta \setminus \{1\}$, $\beta < \alpha$. The analysis of four possible combinations leads to the following cases:

(i) $x \to y = 1 = x \rightsquigarrow y$,
(ii) $x \to y$, $x \rightsquigarrow y \in M_\alpha \setminus \{1\}$
(iii) $x \to y$, $x \rightsquigarrow y \in M_\beta \setminus \{1\}$, $\beta < \alpha$.

Using the definition of the operations we conclude the proof. □

The application of the previous result on the Hájek representation theory of BL-algebras leads to the fact that the condition $\mathrm{P}(a)$ is satisfied in any BL-algebra M and any $a \in M$.

COROLLARY 5.11. *The ordinal sum M of a system of BL-algebras $(M_i : i \in I)$ satisfying the condition* $\mathrm{P}(a)$ *for any $a \in M_i$, $i \in I$, satisfies again the condition* $\mathrm{P}(a)$ *for any a.*

COROLLARY 5.12. *A linearly ordered BL-algebra M satisfies the condition* $\mathrm{P}(a)$ *for any $a \in M$.*

Proof. Due to [14], each linearly ordered BL-algebra can be isomorphically embedded into a linear saturated BL-algebra, and each saturated linearly ordered BL-algebra is the ordinal sum of product of Łukasiewicz BL-chains (or Gödel BL-chains [5]). Due to Proposition 4.2, each BL-chain of the previous two types satisfies the condition $\mathrm{P}(a, x, y)$. Corollary 5.11 and Proposition 5.3 conclude the proof. □

THEOREM 5.13. *Let M be an arbitrary BL-algebra. Then M satisfies* $\mathrm{P}(a)$ *for any $a \in M$.*

Proof. Due to [5], each BL-algebra is a subdirect product of BL-chains. Corollary 5.12 and Proposition 5.1 conclude proof. □

6. Bosbach states

Let us recall the definition of Bosbach states.

DEFINITION 6.1. Let M be a bounded $R\ell$-monoid. Then a mapping $s\colon M \to [0,1]$ is said to be a *Bosbach state* if the following conditions are satisfied:

(i) $s(0) = 0,\ s(1) = 1$,
(ii) $s(x) + s(x \to y) = s(y) + s(y \to x)$,
(iii) $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$.

PROPOSITION 6.2. *Let M be a bounded $R\ell$-monoid, let $a, b \in M$ fixed elements and let s be a Bosbach state on M. Then*

(i) *if $s(a) \neq 1$, then $s^1_a(x) = \frac{s(x)-s(a)}{1-s(a)}$ is a Bosbach state on M^1_a;*

(ii) *if $s(a) \neq 0$ and M satisfies the conditions* $\mathrm{PL}(a)$ *and* $\mathrm{PR}(a)$, *then $s^a_0(x) = \frac{s(x)}{s(a)}$ is a Bosbach state on M^a_0;*

(iii) *if $a \leq b$, $s(a) \neq s(b)$ and the conditions* $\mathrm{PL}(b,x,y)$, $\mathrm{PR}(b,x,y)$ *hold for any $x, y \in [a,b]$, then $s^b_a(x) = \frac{s(x)-s(a)}{s(b)-s(a)}$ is a Bosbach state on M^b_a.*

Proof.
Case (i):
The condition (i) of the definition of a Bosbach's state holds trivially. Let us compute:

$$\begin{aligned} s^1_a(x) + s^1_a(x \to^1_a y) = s^1_a(x) + s^1_a(x \to y) &= \frac{s(x)-s(a)}{1-s(a)} + \frac{s(x \to y) - s(a)}{1-s(a)} \\ &= \frac{s(x) + s(x \to y) - 2s(a)}{1-s(a)} = \frac{s(y) + s(y \to x) - 2s(a)}{1-s(a)} \\ &= \frac{s(y)-s(a)}{1-s(a)} + \frac{s(y \to x) - s(a)}{1-s(a)} = s^1_a(y) + s^1_a(y \to x) \\ &= s^1_a(y) + s^1_a(y \to^1_a x). \end{aligned}$$

The case for $\rightsquigarrow^1_a$ is proved analogously.
Case (ii):
When assuming $\mathrm{PL}(a,x,y)$ or $\mathrm{PR}(a,x,y)$ we obtain for s the following equalities:

$$\begin{aligned} s\big((x \to y) \odot a\big) &= s(a) + s(x \to y) - 1, \\ s\big(a \odot (x \rightsquigarrow y)\big) &= s(a) + s(x \rightsquigarrow y) - 1. \end{aligned}$$

Indeed,

$$\begin{aligned} &s(a) + s\big(a \to \big((x \to y) \odot a\big)\big) \\ &\qquad = s(a) + s(x \to y) = s\big((x \to y) \odot a\big) + s\big(\big((x \to y) \odot a\big) \to a\big) \\ &\qquad = s\big((x \to y) \odot a\big) + 1\,, \end{aligned}$$

$$\begin{aligned} &s(a) + s\big(a \rightsquigarrow \big(a \odot (x \rightsquigarrow y)\big)\big) \\ &\qquad = s(a) + s(x \rightsquigarrow y) = s\big(a \odot (x \rightsquigarrow y)\big) + s\big(\big(a \odot (x \rightsquigarrow y)\big) \rightsquigarrow a\big) \\ &\qquad = s\big(a \odot (x \rightsquigarrow y)\big) + 1\,. \end{aligned}$$

Therefore,

$$\begin{aligned} s_0^a(x) + s_0^a(x \to_0^a y) &= s_0^a(x) + s_0^a\big((x \to y) \odot a\big) = \frac{s(x)}{s(a)} + \frac{s((x \to y) \odot a)}{s(a)} \\ &= \frac{s(x) + s(a) + s(x \to y) - 1}{s(a)} = \frac{s(y) + s(a) + s(y \to x) - 1}{s(a)} \\ &= \frac{s(y)}{s(a)} + \frac{s((y \to x) \odot a)}{s(a)} = s_0^a(y) + s_0^a(y \to_0^a x)\,. \end{aligned}$$

For the residuum $\rightsquigarrow_0^a$ we can proceed analogously. Trivially, $s_0^a(a) = 1$ and $s_0^a(0) = 0$.

Case (iii):

Residua in $[a,b]$ are the same as for the algebras $[0,b]$, therefore we can use the same approach as in the case (ii). □

If M is a BL-algebra, then $\mathrm{P}(a)$ holds for any $a \in M$ and we have the following corollary:

COROLLARY 6.3. *Let M be a BL-algebra, let $a, b \in M$ be fixed elements and let s be a Bosbach state on M. Then*

(i) *if $s(a) \neq 1$, then $s_a^1(x) = \frac{s(x)-s(a)}{1-s(a)}$ is a Bosbach state on M_a^1;*

(ii) *if $s(a) \neq 0$, then $s_0^a(x) = \frac{s(x)}{s(a)}$ is a Bosbach state on M_0^a;*

(iii) *if $a \leq b$, $s(a) \neq s(b)$, then $s_a^b(x) = \frac{s(x)-s(a)}{s(b)-s(a)}$ is a Bosbach state on M_a^b.*

We say that a non-empty subset F of M is a *filter*, if

(i) $x, y \in F \implies x \odot y \in F$,

(ii) $x \in F$, $y \in M$, $y \geq x \implies y \in F$.

A *filter generated by an element* $a \in M$, denoted $F(a)$, is the smallest filter in M containing the element a. Trivially, $F(a) = \{x \in M : a^m \leq x \text{ for some } m \in \mathbb{N}\}$, where $a^m := a \odot \cdots \odot a$ (m-times) and $a^0 := 1$.

PROPOSITION 6.4. *Let M be a bounded $R\ell$-monoid and let s be a Bosbach state on M. Let $a \in M$ be an element such that $F(a) = M$. Then $s(a) \neq 1$.*

P r o o f. For any $a, x \in M$ we have the inequality $x \leq a \to (x \odot a)$, therefore $s(x) \leq s\big(a \to (x \odot a)\big)$.

We claim, $s(a^n) \geq ns(a) - (n-1)$, where $a^n := a \odot \cdots \odot a$. We can proceed by mathematical induction. For $n = 1$ it is obvious. Let us assume that the inequality is true for any $k \leq n$. Then for $n+1$ we obtain $s(a^n) + s(a^n \to a^{n+1}) = 1 + s(a^{n+1})$, so $s(a^n) - s(a^{n+1}) = 1 - s(a^n \to a^{n+1}) \leq 1 - s(a)$ and $s(a^n) + s(a) - 1 \leq s(a^{n+1})$. Due to the assumption for n, $s(a^n) \geq ns(a) - (n-1)$, we can conclude $(n+1)s(a) - n \leq s(a^n) + s(a) - 1 \leq s(a^{n+1})$.

Using the inequality, if $s(a) = 1$, then $s(a^k) \geq ks(a) - (k-1) = k - (k-1) = 1$, i.e. $s(a^k) = 1$ for any k. Let $F(a) = M$, so there exists an integer $m \in \mathbb{N}$ such that $a^m = 0$ and $0 = s(0) = s(a^m) = 1$ that contradicts our assumption. $\square$

PROPOSITION 6.5. *Let M be a good bounded $R\ell$-monoid, i.e. $x^{-\sim} = x^{\sim-}$ for any $x \in M$. Then $F(a^-) = M$ iff $F(a^\sim) = M$.*

P r o o f. Let $F(a^-) = M$, then there exists a natural number k such that $(a^-)^k = 0$. For the implication $F(a^-) = M \implies F(a^\sim) = M$, it is sufficient to prove, that this condition implies $(a^\sim)^k = 0$. We prove that if $(a^-)^k = 0$, then $(a^-)^{k-m} \odot (a^\sim)^m = 0$ for any $0 \leq m \leq k$, where $x^0 =: 1$. We proceed with induction over m. For $m = 0$, it is trivial. Let us assume that $k \geq m \geq 1$ and it holds for $m-1$, i.e. $(a^-)^{k-m+1} \odot (a^\sim)^{m-1} = 0$. Then $a^- \odot (a^-)^{k-m} \odot (a^\sim)^{m-1} = 0$ iff $(a^-)^{k-m} \odot (a^\sim)^{m-1} \leq a^{-\sim} = a^{\sim-}$ iff $(a^-)^{k-m} \odot (a^\sim)^m = 0$. The other direction, i.e. $F(a^\sim) = M \implies F(a^-) = M$, is proved with a dual reasoning. $\square$

An element $a \in M$ is said to be *strong* if $F(a^-) = M$ or $F(a^\sim) = M$.

Note 6.6. The motivation for the notion of a strong element stems from the notion of a strong unit in ℓ-groups. One of the characterizations of the strong units is the following. An element u of ℓ-group G is a strong unit iff the ℓ-ideal generated by u is the whole group G.

Applying Propositions 6.4 and 6.2, if a is a strong element of a bounded $R\ell$-monoid M and if s is a Bosbach state on M and the conditions PL(a), PR(a) are satisfied, then either $s(a^-) \neq 1$, or $s(a^\sim) \neq 1$, i.e. $s(a) \neq 0$ and $s_0^a(x) = \frac{s(x)}{s(a)}$ is a Bosbach state on M_0^a.

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