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THE $3x+1$ PROBLEM, GENERALIZED PASCAL TRIANGLES AND CELLULAR AUTOMATA

IVAN KOREC[1]

ABSTRACT. Iterations of the 3x+1 problem are encoded using a 7-state one way cellular automaton, or, equivalently, a generalized Pascal triangle associated to a 7-element algebra.

1. Introduction

Let $\mathbb{N}$ denote the set of nonnegative integers and define for $y \in \mathbb{N}$

$$T(y) = \frac{3y+1}{2} \quad \text{if } y \text{ is odd}, \quad T(y) = \frac{y}{2} \quad \text{if } y \text{ is even}.$$

Further, denote $T^0(y) = y$ and $T^{n+1}(y) = T(T^n(y))$ for every $n, y \in \mathbb{N}$. The sequence $(T^0(y), T^1(y), T^2(y), \dots)$ will be called the T-trajectory of y. There are several unsolved hypotheses connected with the iterations of T; for example:

3X+1 CONJECTURE. *For every positive integer y there is n such that $T^n(y) = 1$.*

DIVERGENT TRAJECTORY CONJECTURE (ON $\mathbb{N}$). *There is no $y \in \mathbb{N}$ such that $\lim_{n\to\infty} T^n(y) = \infty$.*

FINITE CYCLES CONJECTURE (ON $\mathbb{N}$). *There are only finitely many $y \in \mathbb{N}$ such that there is $n > 0$ such that $T^n(y) = y$.*

Of course, the first hypothesis (called also Syracuse problem, Collatz-Kakutani problem, etc.) implies the second and the third one. All three hypotheses seem to be very hard. For references and history see [8] (where, however, the second

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and the third hypothesis are formulated for the set $\mathbb{Z}$ of all integers instead of $\mathbb{N}$).

Generalized Pascal triangles (GPT; they will be defined below in a slightly modified form with respect to that of [5] and [6]) are structures formed similarly to the classical Pascal triangle, but instead of 0 and the addition of integers operations of a finite algebra $\mathcal{A}$ are used; $\mathcal{A}$ will have one binary operation and one constant. (The term "algebra" is used in the sense of univeral algebra; the binary operation need not be commutative or associative etc.)

GPT can be interpreted as computations of one-dimensional cellular automata (from finite initial configurations); hence e. g. many algorithmical problems concerning them can be shown undecidable.

Here two algebras (consisting of 7 or 8 elements) will be constructed and some structural questions about their GPT will be shown equivalent with the above hypotheses. So we can conclude that these questions about GPT are also hard. On the other hand, GPT can help us to visualize some results concerning the $3x+1$ problem. In terms of cellular automata, two nearest-neighbour one-dimensional cellular automata with 7 and 8 states are constructed whose behaviour is very close to the $3x+1$ problem; in particular, the 8-element automaton is nilpotent if and only if the $3x+1$ conjecture holds. A similar result is contained in [1], where, however, only a quasi-cellular automaton is constructed.

2. Generalized Pascal triangles and cellular automata

For every $n \in \mathbb{N}$ we denote

$$\mathbf{D}_n = \{(x,y) \in \mathbb{N} \times \mathbb{N};\ x + y \geq n - 1\}.$$

If $\mathbf{A}$ is an alphabet (i. e., a finite nonempty set), then $\mathbf{A}^+$ will denote the set of all nonempty words in the alphabet $\mathbf{A}$. The length of a word w will be denoted $|w|$; it must be distinguished from the absolute value of a real by the context. The i-th symbol of w will be denoted by $w(i)$; the starting symbol is $w(0)$ and hence the last symbol is $w(|w|-1)$. Also some further usual notations from the theory of formal languages will be used.

By an algebra we shall always understand an algebra $\mathcal{A} = (\mathbf{A}; *, \mathrm{o})$ of signature $(2,0)$ and satisfying the identity $\mathrm{o} * \mathrm{o} = \mathrm{o}$. We shall usually consider finite algebras; the exceptions will be explicitly mentioned.

DEFINITION 2.1. *To every algebra $\mathcal{A} = (\mathbf{A}; *, \mathrm{o})$ and every $w \in \mathbf{A}^+$, the function $G = \mathrm{GPT}(\mathcal{A}, w)$ will be the mapping $G\colon \mathbf{D}_{|w|} \to \mathbf{A}$ defined by the* *formulae*

$$G(x,y)=\begin{cases} w(x) & \text{if } \ x+y=|w|-1\,,\\ \mathrm{o} * G(0,y-1) & \text{if } \ x=0\,,\ y\geq |w|\,,\\ G(x-1,0)*\mathrm{o} & \text{if } \ y=0\,,\ x\geq |w|\,,\\ G(x-1,y)*G(x,y-1) & \text{if } \ x+y\geq |w|\,,\ x>0\,,\ y>0\,. \end{cases}$$

The functions of the form $\mathrm{GPT}(\mathcal{A},w)$ *for a finite algebra* $\mathcal{A}$ *and a word* $w\in\mathbf{A}^+$ *will be called generalized Pascal triangles* (*abbreviation: GPT*).

As an illustrative example, let us imagine the classical Pascal triangle written in the usual way in a plane. Then it is natural to consider it as the function P with the domain $\mathbf{D}_1=\mathbb{N}\times\mathbb{N}$ and satisfying $P(x,y)=\binom{x+y}{x}$. The usual orientation of Pascal's triangle is in the plane obtained by rotating the axes $\frac{\pi}{4}$ clockwise, and by the reflection which reverses the direction of the axis y. So the positive x-axis is directed down and to the right, and the positive y-axis is directed down and to the left. (In the rest of the paper we always represent lattice points in the plane with its axes transformed in this fashion, so the cellular automata configuration on the line $x+y=t$ appears at a horizontal line.) The function P can be expressed as $\mathrm{GPT}(\mathcal{N},1)$, where $\mathcal{N}=(\mathbb{N};+,0)$ and $+$ is the usual addition on $\mathbb{N}$. This analogy explains the term "generalized Pascal triangle". However, P is not a GPT because the algebra $\mathcal{N}$ is not finite. Nice (and very often studied) examples of GPT are Pascal triangles modulo n, particularly if n is a prime or a prime power. They are obtained if the values of P are reduced modulo a positive integer n. We can express them in the form $\mathrm{GPT}(\mathcal{N}_n,1)$, where $\mathcal{N}_n=(\{0,1,\dots,n-1\};+,0)$ and $+$ denotes addition modulo n.

DEFINITION 2.3. *Let* $\mathcal{A}=(\mathbf{A};*,\mathrm{o})$ *be an algebra,* $w\in\mathbf{A}^+$ *and* $t\in\mathbb{N}$.

a) *The* t*-th* row *of* $G=\mathrm{GPT}(\mathcal{A},w)$, *notation:* $\mathrm{R}(\mathcal{A},w;t)$, *will be the word consisting of*

$$G(0,t+|w|-1),G(1,t+|w|-2),\dots,G(t+|w|-2,1),G(t+|w|-1,0).$$

b) *The* substantial part $\mathrm{SP}(\mathcal{A},w,t)$ *of the* t*-th row of* $G=\mathrm{GPT}(\mathcal{A},w)$ *will be the empty word if* $G(x,t-x)=\mathrm{o}$ *for all* $x\in\mathbb{Z}$; *otherwise* $\mathrm{SP}(\mathcal{A},w;t)$ *is the largest connected subword of this row in which both endpoints are distinct from* o.

c) $\mathrm{lmarg}(\mathcal{A},w;t)$ *and* $\mathrm{rmarg}(\mathcal{A},w;t)$ *will denote the number of* (*occurrences of*) o *in* $\mathrm{R}(\mathcal{A},w;t)$ *before and after* $\mathrm{SP}(\mathcal{A},w;t)$, *respectively; if* $\mathrm{SP}(\mathcal{A},w;t)$ *is* *empty then* $\operatorname{lmarg}(\mathcal{A},w;t) = |w| + t$ *and* $\operatorname{rmarg}(\mathcal{A},w;t) = 0$. (*Here* lmarg *and* rmarg *stands instead of "left margin" and "right margin", respectively.*)

The $\mathrm{GPT}(\mathcal{A},w)$ can be viewed as the computation of a one-dimensional cellular automaton, where the t-row corresponds to the configuration of the automaton at time t. In these terms the element o in the algebra represents the symbol "blank" (or the quiescent state), and the substantial part $\mathrm{SP}(\mathcal{A},w,t)$ is the shortest connected subword of the t-th row containing all the nonblank symbols, i.e. containing the interesting part of the configuration. Further, always

$$\mathrm{R}(\mathcal{A},w;t) = \mathrm{o}^{\operatorname{lmarg}(\mathcal{A},w;t)}\mathrm{SP}(\mathcal{A},w;t)\mathrm{o}^{\operatorname{rmarg}(\mathcal{A},w;t)}.$$

The author defined GPT to study the structure of real-time regular systolic trellis automata, see [2]. However, GPT also describe the computations of one-dimensional cellular automata (CA) from finite initial configurations. The 0-th row corresponds to the initial configuration, and every subsequent row corresponds to one step of computation. Particularly, GPT immediately correspond to computations of so called one-way CA, which are CA where the neighbourhood of a cell consists of the cell itself and its right neighbour, see [3]; of course, the left neighbour can be considered as well. For other types of neighbourhood it is suitable to consider several consecutive elements of a CA as one element of the algebra $\mathcal{A}$. Particularly, for the most usual 3-element neighbourhood pairs of cells will be considered. There are two partitions of the cells of a 1-dimensional CA into pairs of consecutive elements. One such partition will be considered in even moments and the other in odd moments of the discrete time.

DEFINITION 2.4. *A GPT G will be called* nilpotent *if $G(x,y) = \mathrm{o}$ for all but finitely many pairs $(x,y) \in \mathbf{D}$. An algebra $\mathcal{A} = (\mathbf{A}; *, \mathrm{o})$ will be called* nilpotent *if for every $w \in \mathbf{A}^+$ $\mathrm{GPT}(\mathcal{A},w)$ is nilpotent.*

DEFINITION 2.5. *A language L will be called a* simple linear language of degree at most k *if there are words $u_0, v_1, u_1, v_2, \dots, u_{k-1}, v_k, u_k$ such that*

$$L = \{u_0 v_1^i u_1 v_2^i u_2 \dots u_{k-1} v_k^i u_k \mid i > 0\}. \tag{2.5.1}$$

The set

$$\left\{ \frac{|v_1|}{|v_1 v_2 \dots v_k|}, \frac{|v_1 v_2|}{|v_1 v_2 \dots v_k|}, \dots, \frac{|v_1 v_2 \dots v_k|}{|v_1 v_2 \dots v_k|} \right\} \tag{2.5.2}$$

will be called a type of L *(here $|v_1 v_2 \dots v_k| \neq 0$ is assumed).*

A language L will be called simple semilinear language *(abbreviated: SSL language) of degree at most k if L is a disjoint union of finitely many simple linear languages of degree at most k. The union of (some) types of these linear languages will be* a type of L.

L will be called a SSL language of degree k if L is a SSL language of degree at most k but it is not a SSL language of degree at most $k-1$. L will be called a SSL language if it is an SSL language of degree at most k for some $k \in \mathbb{N}$.

A GPT G will be called SSL GPT (of degree [at most] k) if the set of its rows is a SSL language (of degree [at most] k). Analogously for types of GPT.

An algebra $\mathcal{A} = (\mathbf{A}; *, \mathrm{o})$ will be called an SSL algebra (of degree at most k) if for every $w \in \mathbf{A}^+$ $\mathrm{GPT}(\mathcal{A}, w)$ is SSL (of degree at most k). It will be called SSL algebra of degree k if it is of degree at most k but it is not of degree at most $k-1$. A finite set X of positive rationals not greater than 1 will be called a type of $\mathcal{A}$ if for every $w \in \mathbf{A}^+$ the set X is a type of $\mathrm{GPT}(\mathcal{A}, w)$.

A SSL language can have several types. The degree of a simple linear language is less than or equal to that the cardinality of any of its types. Notice that a SSL algebra need not be SSL one of a (finite) degree (and need not have a type) because the degrees of its GPT can be arbitrarily large. A nilpotent algebra is a SSL algebra of degree 1 but the converse does not hold.

Some of the above defined properties can be characterized as follows. $G = \mathrm{GPT}(\mathcal{A}, w)$ is a SSL GPT of degree at most 2 if and only if the set of its rows is a context-free language. G is a SSL GPT if and only if for every $c \in \mathbf{A}$ the set $G^{-1}\{c\}$ is definable in Presburger arithmetic. Since Presburger arithmetic is decidable many algorithmic problems which are undecidable for general GPT become decidable for SSL GPT (and for the corresponding cellular automata as well).

3. Simple semilinear GPT and the $3x+1$ problem

The operators DIV , MOD will denote the quotient and the remainder by integer division (as in the programming language PASCAL). Floors $\lfloor\ \rfloor$ will denote the integer part of a real number.

Definition 3.1. *Let $\mathcal{A}_1 = (\mathbf{A}_1; *, \mathrm{o})$, where $\mathbf{A}_1 = \{\mathrm{o}, 0, 1, 2, 3, 4, 5\}$ and the operation $*$ is defined by*

$$x * y = \begin{cases} (3x) \text{ MOD } 6 + (3y) \text{ DIV } 6 & \textit{if } x \neq \mathrm{o},\ y \neq \mathrm{o}, \\ \mathrm{o} & \textit{if } x = \mathrm{o},\ y \in \{\mathrm{o}, 0, 1\}, \\ (3y) \text{ DIV } 6 & \textit{if } x = \mathrm{o},\ y \in \{2,3,4,5\}, \\ 4 & \textit{if } x \in \{1,3,5\},\ y = \mathrm{o}, \\ \mathrm{o} & \textit{if } x \in \{0,2,4\},\ y = \mathrm{o}. \end{cases} \tag{3.1.1}$$

The algebra $\mathcal{A}_1$ and one of its associated GPT are displayed in Figure 1.

Algebra A:

*	o	0	1	2	3	4	5
o	o	o	o	1	1	2	2
0	o	0	0	1	1	2	2
1	4	3	3	4	4	5	5
2	o	0	0	1	1	2	2
3	4	3	3	4	4	5	5
4	o	0	0	1	1	2	2
5	4	3	3	4	4	5	5

'o' $\longrightarrow$ '.' in GPT

Initial word w='o102ooo101ooo51oooo52o'

GPT(A,w), rows 0..60, columns -40..36 :

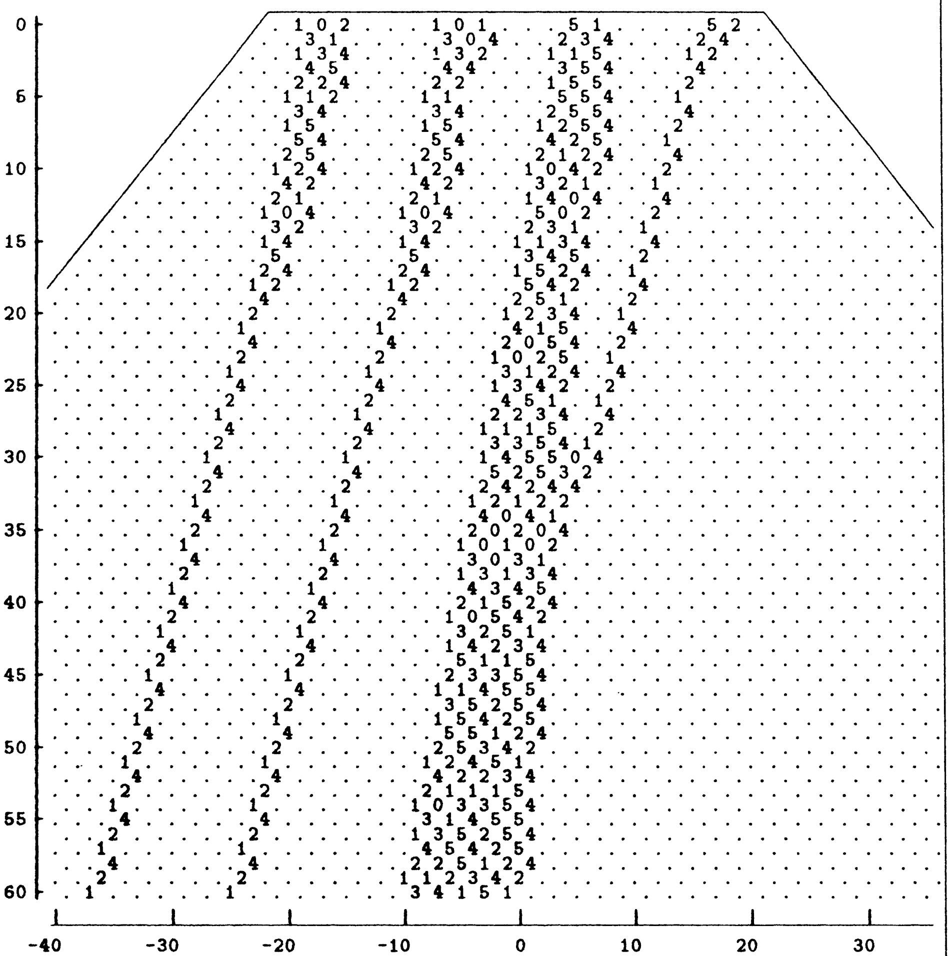

Figure 1.

Let us consider the leftmost maximal segment without o in each row. If we consider these segments as integers in number base 6 we obtain the sequence

$$38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 13, \dots .$$

Notice that every member x is followed by $3x+1$ if x is odd and by $\frac{x}{2}$ if x is even. The same holds for the sequence of k-th maximal segments, $k = 1, 2, 3, \dots$, provided that these segments exist and are not glued together with the other segments. We shall prove these facts below.

Assume that the substantial part of a row of $\mathrm{GPT}(\mathcal{A}_1, w)$ does not contain any o. (This assumption is not fulfilled in Figure 1; however, any maximal segment of a row not containing any o can be considered as well.)

If we understand o as a blank, then the way how a row of a $\mathrm{GPT}(\mathcal{A}_1, w)$ is formed from the previous row resembles multiplying by 3 in the number system with base 6. In such multiplication the digits 0 or 3 would arise without a carry, any carry is at most 2, and hence their sum is less than 6. Therefore the carry on one position cannot influence the carry on the next position. (The situation is very similar to that with multiplication by 5 in the usual decadic system, where 0 or 5 arises without a carry, and the carry can be at most 4.) There is a difference at the right end, where:

either 4 arises instead of 3; so $3u+1$ is obtained instead of $3u$;
or the rightmost zero is lost; so $3u$ is divided by 6, and $\frac{1}{2}u$ is obtained.

To formulate the result strictly, for every $w \in \left(\mathbf{A}_1 - \{\mathrm{o}\}\right)^+$ denote by $\mathrm{val}_6(w)$ the integer represented by w in the number system with base 6 (leading zeros are allowed). The above considerations give:

LEMMA 3.2. *Let* $w \in \mathbf{A}_1^+$ *and* $t \in \mathbb{N}$ *be such that* o *does not occur in* $\mathrm{SP}(\mathcal{A}_1, w; t)$ *and let* $u = \mathrm{val}_6\left(\mathrm{SP}(\mathcal{A}_1, w; t)\right)$. *Then* o *does not occur in* $\mathrm{SP}(\mathcal{A}_1, w; t+1)$ *and*

$$\mathrm{val}_6\left(\mathrm{SP}(\mathcal{A}_1, w; t+1)\right) = \begin{cases} 3u+1 & \textit{if } u \textit{ is odd,} \\ \frac{1}{2}u & \textit{if } u \textit{ is even.} \end{cases} \tag{3.2.1}$$

Proof. Let us consider all integers written in base 6. Let $u_n, u_{n-1}, \dots, u_1$, u_0 be the digits of $u = \mathrm{val}_6\left(\mathrm{SP}(\mathcal{A}_1, w; t)\right)$; denote also $u_{n+1} = 0$ and $u_{-1} = \mathrm{o}$. If u_0 is odd, let $v_n, v_{n-1}, \dots, v_1, v_0$ be the digits of $v = 3u+1$. The j-th digit (from the right end) of any $x \in \mathbb{N}$ can be expressed as $(x \text{ MOD } 6^{j+1}) \text{ DIV } 6^j$

or $(x \text{ DIV } 6^j) \text{ MOD } 6$ and therefore for $1 \le j \le n$ we have:

$$\begin{aligned}
v_j = (v \text{ MOD } 6^{j+1}) \text{ DIV } 6^j &= \Big(\Big(1 + \sum_{i=0}^{n} 3u_i.6^i\Big) \text{ MOD } 6^{j+1}\Big) \text{ DIV } 6^j \\
&= \Big(\Big(1 + \sum_{i=0}^{j} 3u_i.6^i\Big) \text{ MOD } 6^{j+1}\Big) \text{ DIV } 6^j \\
&= \Big(\Big(1 + \sum_{i=0}^{j} 3u_i.6^i\Big) \text{ DIV } 6^j\Big) \text{ MOD } 6 \\
&= \Big(\Big(\Big(1 + \sum_{i=0}^{j-2} 3u_i.6^i\Big) + \big(3u_{j-1}.6^{j-1} + 3u_j.6^j\big)\Big) \text{ DIV } 6^j\Big) \text{ MOD } 6 \\
&= ((3u_{j-1} + 18u_j) \text{ DIV } 6) \text{ MOD } 6 \\
&= ((3u_{j-1}) \text{ DIV } 6 + (3u_j)) \text{ MOD } 6 \\
&= (3u_j) \text{ MOD } 6 + (3u_{j-1}) \text{ DIV } 6 = u_j * u_{j-1}\,.
\end{aligned}$$

For v_0 we have

$$v_0 = \Big(\Big(1 + \sum_{i=0}^{n} 3u_i.6^i\Big) \text{ DIV } 6^0\Big) \text{ MOD } 6 = (3u_0 + 1) \text{ MOD } 6 = 4 = u_0 * \text{o}\,.$$

So we have obtained $v = 3u + 1 = \text{val}_6\big(\text{SP}(\mathcal{A}_1, w; t+1)\big)$. The proof for u_0 even is similar. However, since $u_0 * \text{o} = \text{o}$, it is suitable to denote the digits of $v = \frac{1}{2}u$ by $v_n, v_{n-1}, \dots, v_1$. Then $v_j = u_j * u_{j-1}$ can be proved analogously. □

There is a difference between (3.2.1) and the definition of T: the expression $3u + 1$ occurs here instead of $\dfrac{3u+1}{2}$. Therefore the sequence

$$\big(\text{val}_6(\text{SP}(\mathcal{A}_1, w; t))\,;\ \ t = 0, 1, 2, \dots\big) \tag{3.2.2}$$

is not exactly the T-trajectory of $\text{val}_6\big(\text{SP}(\mathcal{A}_1, w; 0)\big)$. However, this T-trajectory is contained in (3.2.2) as a subsequence. (To obtain it, all immediate successors of odd members must be removed from (3.2.2).) It is clear that this difference is not substantial for considerations below.

LEMMA 3.3. *Let* $w \in \big(\mathbf{A} - \{o\}\big)^+$, $w(0) \neq 0$ *and* $q = \liminf\limits_{n\to\infty} T^n\big(\text{val}_6(w)\big)$. *Then*

$$\|w\| = \lim_{t\to\infty} \frac{\text{lmarg}(\mathcal{A}, w; t)}{t} \tag{3.3.1}$$

exists and

(i) $\|w\| = \frac{1}{3}$ *if* $q = 1$,

(ii) $\log_6 2 - \frac{0.072}{q} < \|w\| < \log_6 2$ *if* $1 < q < \infty$,

(iii) $\|w\| = \log_6 2$ *if* $q = \infty$.

Proof. (Notice that the $3x+1$ conjecture implies $q = 1$ for all considered w.) Assume that w is fixed and for every $t \in \mathbb{N}$ denote

$$a_t = \mathrm{val}_6\big(\mathrm{SP}(\mathcal{A}, w; t)\big), \qquad b_t = a_t \cdot 6^{\mathrm{rmarg}(\mathcal{A}, w; t)};$$

notice that $b_t = \mathrm{val}_6\big(\mathrm{R}'(\mathcal{A}, w; t)\big)$, where $\mathrm{R}'(\mathcal{A}, w; t)$ is the t-th row of $\mathrm{GPT}(\mathcal{A}, w)$ with all o replaced by 0. Then for every $t \in \mathbb{N}$ we have

$$\mathrm{lmarg}(\mathcal{A}, w; t) + \lfloor \log_6 b_t \rfloor = |w| + t + 1.$$

Further denote

$$c_t = \begin{cases} 1 & \text{if } a_t \text{ is even,} \\ 1 + \dfrac{1}{3a_t} & \text{if } a_t \text{ is odd.} \end{cases}$$

Then $b_{t+1} = 3b_t c_t$ for all $t \in \mathbb{N}$. Now distinguish two cases.

If q is finite, then the sequence $(a_0, a_1, a_2, \ldots)$ is ultimately periodic and contains q infinitely many times. Let $a_r = a_s = q$ for some $r, s \in \mathbb{N}$, $r < s$, $s - r$ minimal. Then the fractional parts of $\log_6 b_r$, $\log_6 b_s$ are equal, $b_s = 3^{s-r} b_r c_r c_{r+1} \ldots c_{s-1}$ and hence

$$\begin{aligned} \|w\| &= \frac{\mathrm{lmarg}(\mathcal{A}, w; s) - \mathrm{lmarg}(\mathcal{A}, w; r)}{s - r} \\ &= \frac{\big(|w| + s + 1 - \lfloor \log_6 b_s \rfloor\big) - \big(|w| + r + 1 - \lfloor \log_6 b_r \rfloor\big)}{s - r} \\ &= \frac{s - r - \log_6 b_s + \log_6 b_r}{s - r} = \frac{(s-r)\cdot(1 - \log_6 3) - \log_6(c_r c_{r+1} \ldots c_{s-1})}{s - r} \\ &= \log_6 2 - \frac{1}{s - r} \sum_{t=r}^{s-1} \log_6 c_t . \end{aligned}$$

To estimate the last sum, denote by k the number of odd integers in the finite sequence $(a_r, a_{r+1}, \ldots, a_{s-1})$. Positive members of the estimated sum correspond to odd a_t and do not exceed $\log_6\Big(1 + \dfrac{1}{3q}\Big)$; the other members are equal to 0.

Since obviously $k > 0$ we immediately obtain $\|w\| < \log_6 2$. For the other inequality notice that

$$a_r = a_s = a_r \cdot 3^k c_r c_{r+1} \dots c_{s-1} \cdot \left(\frac{1}{2}\right)^{s-r-k} > a_r \cdot \frac{3^k}{2^{s-r-k}} = a_r \cdot \frac{6^k}{2^{s-r}} \,.$$

Therefore $6^k < 2^{s-r}$ and hence $\frac{k}{s-r} < \log_6 2$. So we obtain

$$\sum_{t=r}^{s-1} \log_6 c_t < \frac{k}{s-r} \log_6 \left(1 + \frac{1}{3q}\right) < \log_6 2 \cdot \frac{1}{\ln 6} \cdot \frac{1}{3q} < \frac{0.072}{q} \,,$$

and (ii) is proved.

The above consideration holds also for $q = 1$. In this case $s = r + 3$, $a_r = 1$, $a_{r+1} = 4$, $a_{r+2} = 2$, $b_s = 6^2 b_r$, hence $k = 1$ and $\|w\| = \frac{1}{3}$. Of course, this result also can be obtained in a much simpler way.

Now consider the case $q = \infty$. For every $t \in \mathbb{N}$ we have

$$\begin{aligned} \left|\operatorname{lmarg}(\mathcal{A}, w; t) - t \cdot \log_6 2\right| &= \left||w| + t + 1 - \lfloor \log_6 b_t \rfloor - t \cdot \log_6 2\right| \\ &\le \left||w| + t + 1 - \lfloor \log_6 (3^t b_0 c_0 c_1 \dots c_{t-1}) \rfloor - t \cdot \log_6 2\right| \\ &\le K(w) + \log_6 (c_0 c_1 \dots c_{t-1}) \,, \end{aligned}$$

where $K(w)$ does not depend on t. Therefore $\|w\| = \log_6 2$ will be proved (with extra to spare) if we show

$$c_0 c_1 \dots c_{t-1} \le \sqrt[9]{t} \qquad \text{for all } t \ge 2 \,. \tag{3.3.2}$$

The factors $c_r \ne 1$ here are pairwise distinct and are of the form $\frac{a+1}{a}$, where $a > 120$ and with at most one exception (corresponding to the first odd a_r) $a \equiv \pm 3 \pmod{18}$. Consider all $c_r \ne 1$ in descending order and estimate every c_r^9 by a product of nine consecutive numbers of the form $\frac{m+1}{m}$, $m = 90, 91, \dots$. So for $t \ge 2$ we obtain

$$(c_0 c_1 \dots c_{t-1})^9 < \prod_{m=91}^{99+9t} \frac{m+1}{m} = \frac{99 + 9t}{91} < t \,,$$

what immediately implies (3.3.2). □

LEMMA 3.4. *For every* $w \in (\mathbf{A}_1 - \{\mathrm{o}\})^+$ *and* $c, d \in \mathbb{N}$, $\mathrm{GPT}(\mathcal{A}_1, \mathrm{o}^c w \mathrm{o}^d)$ *is simple semilinear if and only if the* T*-trajectory of* $\mathrm{val}_6(w)$ *is ultimately periodic. Moreover, in this case* $\{\|w\|, 1\}$ *is its type and its degree is at most* 2.

Proof. The considered properties of GPT are not changed by deleting several of its initial rows. If $w \in \{\mathrm{o}, 0\}^+$, then after several steps rows consisting of only o are obtained. Therefore $\mathrm{GPT}(\mathcal{A}, w)$ is nilpotent, and hence its degree is 1 and its type is $\{1\}$. So we may assume that a nonzero digit occurs in w. The parts o^c, o^d of w cause only (additional) margins of widths c, d consisting of o in $\mathrm{GPT}(\mathcal{A}, w)$. Therefore we may assume $c = d = 0$ without loss of generality. Finally, the leading zeros vanish in the first several steps, and so we may assume $w(0) \neq 0$. Together, we may assume that w satisfies the assumptions of Lemma 3.3. Let us also use the notation from it and its proof.

If q is finite and $a_r = a_s$, $r < s$, then for every $t \geq r$

$$\mathrm{R}(\mathcal{A}, w; t+s-r) = \mathrm{o}^e \mathrm{R}(\mathcal{A}, w; t) \mathrm{o}^f, \qquad \text{where}$$

$$e = \mathrm{lmarg}(\mathcal{A}, w; s) - \mathrm{lmarg}(\mathcal{A}, w; r), \qquad f = \mathrm{rmarg}(\mathcal{A}, w; s) - \mathrm{rmarg}(\mathcal{A}, w; r);$$

for $t = r$ it can be verified directly, for $t > r$ by an easy induction. Therefore the set of rows of $\mathrm{GPT}(\mathcal{A}, w)$ can be written as a disjoint union of a finite language (consisting of the first s rows) and $s - r$ simple linear languages of degree 2, one for every $\mathrm{R}(\mathcal{A}, w; t)$, $s \leq t < 2s - r$. Every of them can be represented in the form (2.6.1), with

$$k = 2, \quad u_0, u_2 \in \{\mathrm{o}\}^*, \quad v_1 = \mathrm{o}^e, \quad u_1 = \mathrm{SP}(\mathcal{A}, w; t), \quad v_2 = \mathrm{o}^f,$$

and its type is $\left\{\frac{e}{e+f}, 1\right\} = \{\|w\|, 1\}$. (Notice that $r \leq t < s$ cannot be used instead of $s \leq t < 2s - r$ because in this case the number of o in $\mathrm{R}(\mathcal{A}, w; t)$ could be insufficient.)

If q is infinite, then for every $u \in (\mathbf{A} - \{o\})^+$, $u(0) \neq 0$ there is a row of $\mathrm{GPT}(\mathcal{A}, w)$ the substantial part $\mathrm{SP}(\mathcal{A}, w; t)$ of which begins with u. (Roughly speaking, because for every such u and every positive real β there are infinitely $n \in \mathbb{N}$ such that the initial digits of $\beta \cdot 3^n$ coincide with u. This fact follows from the irrationality of $\log_6 3$ and Example 2.1 of [7], where it is proved that for every irrational real α the sequence of fractional parts of $(n \cdot \alpha;\ n = 0, 1, 2, \dots)$ is uniformly distributed in $[0, 1)$.) Therefore $\mathrm{GPT}(\mathcal{A}, w)$ is not SSL because in every simple semilinear GPT the number of distinct row segments of length l grows only polynomially with l. □

THEOREM 3.5. (i) *The algebra $\mathcal{A}_1$ is a SSL algebra of degree 2 if and only if the $3x+1$ conjecture holds.*

(ii) *The algebra $\mathcal{A}_1$ is a SSL algebra if and only if the Divergent trajectory conjecture (on $\mathbb{N}$) holds.*

(iii) *The algebra $\mathcal{A}_1$ is a SSL algebra of a finite degree if and only if both the Divergent trajectory conjecture (on $\mathbb{N}$) and the Finite cycles conjecture (on $\mathbb{N}$) hold.*

P r o o f . We shall use notation from Lemma 3.3 and its proof, particularly $\|w\|$ defined in (3.3.1). Let us consider $\mathrm{GPT}(\mathcal{A}_1, w)$ for some fixed $w \in \mathbf{A}_1^+$. The word w can be written as

$$w = \mathrm{o}^{e(0)} w_1 \mathrm{o}^{e(1)} w_2 \mathrm{o}^{e(2)} \dots w_k \mathrm{o}^{e(k)}, \tag{3.5.1}$$

where $w_1, w_2, \dots, w_k \in (\mathbf{A}_1 - \{\mathrm{o}\})^+$ and $e(1), \dots, e(k-1)$ are positive. Analogously every row $\mathrm{R}(\mathcal{A}_1, w; t)$, $t \in \mathbb{N}$ can be divided into $K(t)$ segments from $\mathbf{A}_1^+$ and segments from $\{o\}^+$. The function $K(t)$ is non-increasing because $x * y \neq \mathrm{o}$ for all $x, y \in (\mathbf{A}_1 - \{\mathrm{o}\})^+$. Therefore we may assume that $K(t) = k$ for all $t \in \mathbb{N}$ (we leave out several first lines if necessary); analogously we may assume $w_i(0) \neq 0$ for all $i \in \{1, \dots, k\}$. If we imagine a blank instead any o, then $\mathrm{GPT}(\mathcal{A}_1, w)$ can be considered as a superposition of k distinct GPT which do not influence each other:

$$\mathrm{GPT}(\mathcal{A}_1, \mathrm{o}^{c(i)} w_i \mathrm{o}^{d(i)}), \qquad 1 \leq i \leq k, \tag{3.5.2}$$

(where $c(i)$, $d(i)$ can be expressed by $e(j)$, $|w_j|$). Particularly, $\mathrm{GPT}(\mathcal{A}_1, w)$ is SSL GPT if and only if all (3.5.2) are SSL GPT; in this case the union of their types is a type of $\mathrm{GPT}(\mathcal{A}_1, w)$. Finally, let us denote $q_i = \liminf\limits_{n \to \infty} T^n(w_i)$ for every i, $1 \leq i \leq k$. We shall use the relation between $\|w_i\|$ and q_i described in Lemma 3.3.

Let $\mathcal{A}_1$ be a SSL algebra of degree 2. To prove the $3x+1$ conjecture, take an arbitrary integer $y > 0$. Let w_2 be such that $val_6(w_2) = y$, $w_2(0) \neq 0$, let $w_1 = 1$ and let $w = w_1 \mathrm{o}^9 w_2$ (this is a special case of (3.5.1) for $k = 2$). Since $\mathrm{GPT}(\mathcal{A}_1, w)$ is a SSL GPT of degree at most 2 we have $\|w_2\| = \|w_1\| = \frac{1}{3}$, and hence $q_2 = 1$, which implies $T^n(y) = 1$ for some n. Conversely, let the $3x+1$ conjecture hold. Then for every word (3.5.1) we have $\|w_i\| = \frac{1}{3}$ for all i, $1 \leq i \leq k$. Then $\mathrm{GPT}(\mathcal{A}_1, w)$ is a SSL GPT of degree at most 2. (After the reductions above this degree is less than 2 only if $k = 0$, i. e. $w \in \{\mathrm{o}\}^+$.) Hence $\mathcal{A}_1$ is a SSL algebra of degree (at most) 2, and the first part of Theorem 3.5 is proved.

If the Divergent trajectory conjecture (on $\mathbb{N}$) holds then for every word $w \in \mathbf{A}_1^+$ all q_i are finite, and hence $\mathcal{A}_1$ is a SSL algebra. Conversely, let $\mathcal{A}_1$ is a SSL algebra. For every $y \in \mathbb{N}$ there is w such that $\mathrm{val}_6(w) = y$, and by Lemma 3.4 the T-trajectory of y is periodic. So the second part of Theorem 3.5 is proved.

For the third part we may assume that the Divergent trajectory conjecture (on $\mathbb{N}$) holds (and that $\mathcal{A}_1$ is a SSL algebra). If the Finite cycles conjecture (on $\mathbb{N}$) does not hold, then there is an increasing sequence of posiyive integers $(y_i;\ i = 1,2,3,\dots)$ such that $y_i = T^n(y_i)$ for some $n = n(i)$. Then there is a sequence of words $(w_i;\ i = 1,2,3,\dots)$ such that

$$\|w_1\| < \|w_2\| < \|w_3\| < \dots . \tag{3.5.3}$$

(By Lemma 3.3, the sequence (3.5.3) converges to $\log_6 2$.) If we take its first k members and choose $e(j)$ in (3.5.1) sufficiently large, then $\mathrm{GPT}(\mathcal{A}_1, w)$ will be a SSL GPT of degree k. Therefore the degree of $\mathcal{A}_1$ cannot be finite. Conversely, if $\mathcal{A}_1$ is a SSL algebra of no finite degree, then words $w_i \in \left(\mathbf{A} - \{\mathrm{o}\}\right)^+$ which satisfy (3.5.3) exist. Then $y_i = \mathrm{val}_6(w_i)$ form a counterexample for the Finite cycles conjecture (on $\mathbb{N}$). □

Remark 3.6. Since the $3x+1$ conjecture was verified up to 2^{40}, Lemmas 3.3 and 3.4 leave only an interval of length less than 10^{-13} for possible values of $\|w\|$ distinct from $\frac{1}{3}$ and 1. (Note: $\|w\| = 1$ is possible only for $w \in \{\mathrm{o}, 0\}^+$.)

4. Nilpotent GPT and the $3x+1$ problem

The nilpotency of a GPT is a simpler and more transparent property than simple semilinearity. Therefore it would be nice to replace the SSL algebra in Theorem 3.5 by a nilpotent algebra. We can do it for the first part of Theorem 3.5, but the cardinality of the algebra will increase to 8.

DEFINITION 4.1. *Let $\mathcal{A}_2 = (\mathbf{A}_2; \oplus, \mathrm{o})$, where $\mathbf{A}_2 = \{\mathrm{o}, 0, 1, 2, 3, 4, 5, \mathrm{I}\}$ and the operation $\oplus$ is defined by*

$$x \oplus y = \begin{cases} \mathrm{o} & \text{if } x = \mathrm{I},\ y = \mathrm{o}, \\ \mathrm{I} & \text{if } x = \mathrm{o},\ y = 2, \\ x' * y' & \text{otherwise}, \end{cases}$$

where $*$ is as in $\mathcal{A}_1$ and $x' = x$ if $x \neq \mathrm{I}$ and $\mathrm{I}' = 1$.

The algebra $\mathcal{A}_2$ and one of its GPT are displayed at Figure 2.

Algebra A:

*	o	0	1	2	3	4	5	I
o	o	o	o	I	1	2	2	o
0	o	0	0	1	1	2	2	0
1	4	3	3	4	4	5	5	3
2	o	0	0	1	1	2	2	0
3	4	3	3	4	4	5	5	3
4	o	0	0	1	1	2	2	0
5	4	3	3	4	4	5	5	3
I	o	3	3	4	4	5	5	3

'o' ⟶ '.' in GPT

Initial word **w='001122oo5354o1425oo133'**

GPT(A,w), rows 0..59, columns -40..36 :

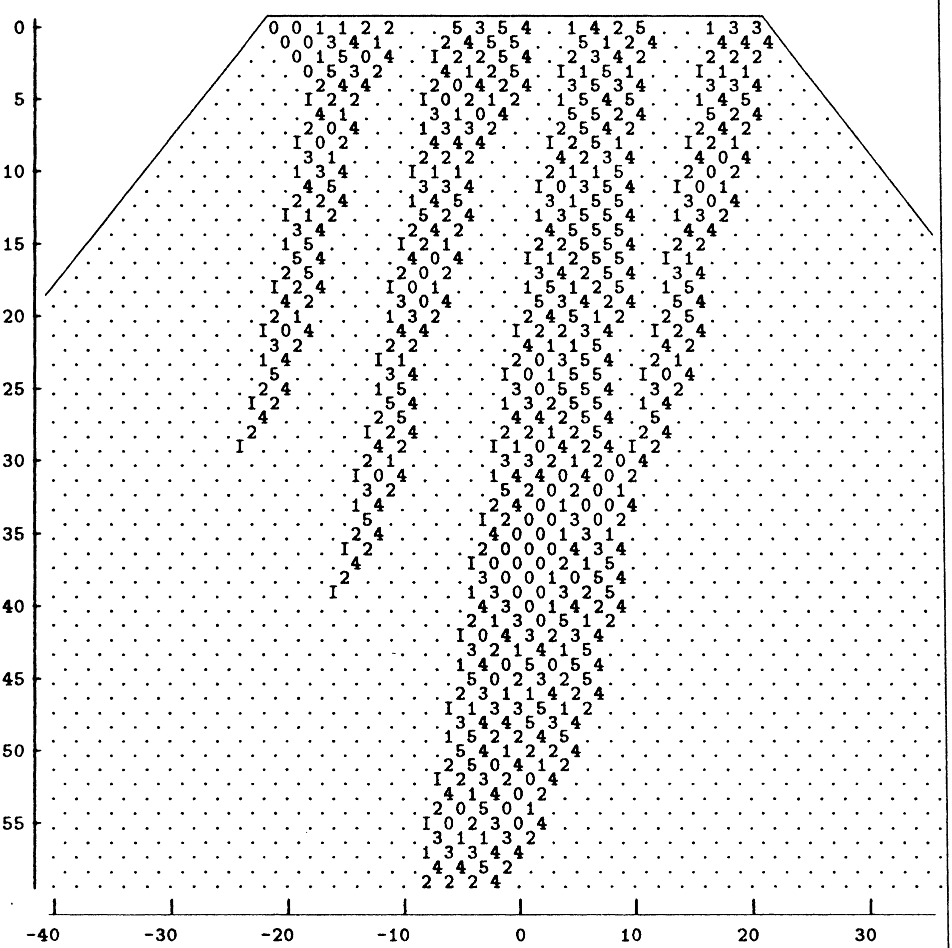

Figure 2.

Algebra A:

*	o	0	1	2	3	4	5	F
o	o	o	o	1	1	2	2	o
0	o	0	0	1	1	2	2	2
1	F	3	3	4	4	5	5	5
2	o	0	0	1	1	2	2	2
3	4	3	3	4	4	5	5	5
4	o	0	0	1	1	2	2	2
5	4	3	3	4	4	5	5	3
F	o	3	3	4	4	5	5	2

'o' → '.' in GPT

Initial word w='001122oo5354o1425oo133'

GPT(A,w), rows 0..59, columns -40..36 :

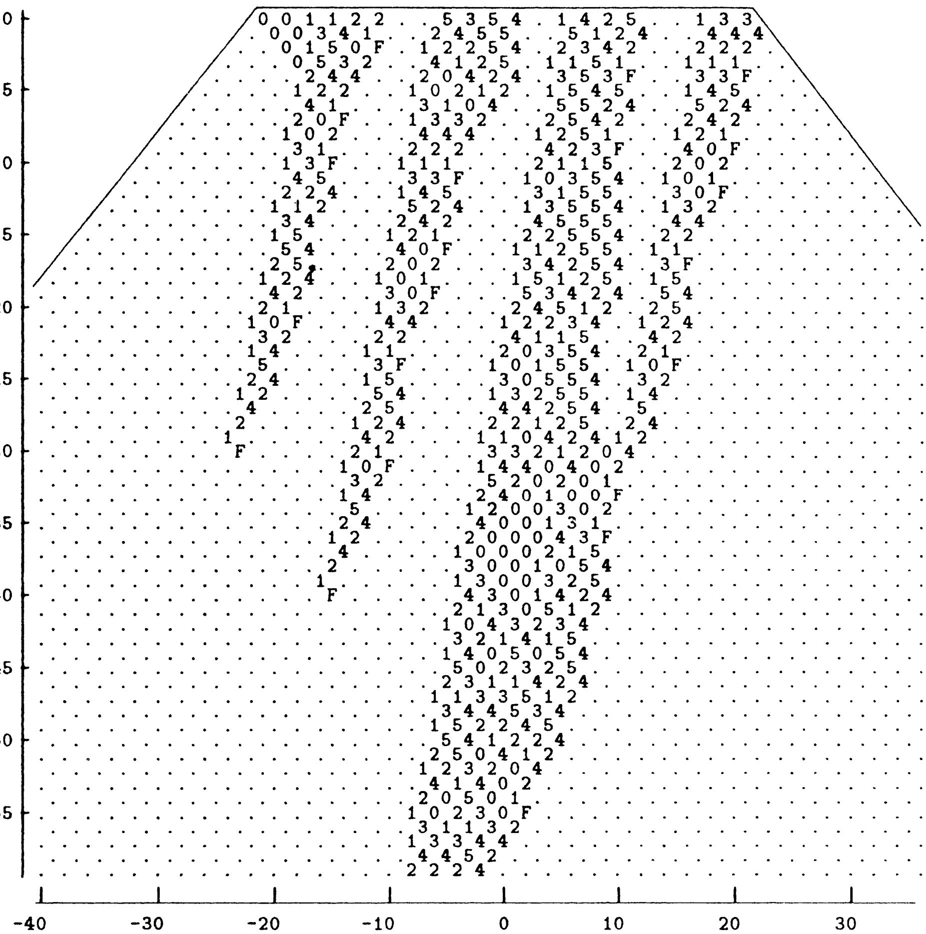

Figure 3.

In most situations the new element I can be replaced by 1 without influencing the next row. The only exception is $\mathrm{I} \oplus \mathrm{o}$, when o arises instead of 4. Together with $\mathrm{o} \oplus 2 = \mathrm{I}$ it causes the substantial part consisting of only digit 2 to vanish (i.e., become empty) in two steps. So periodical "tails" consisting of (repeated) $4, 2, 1$ in $\mathrm{GPT}(\mathcal{A}_1, w)$ are removed from $\mathrm{GPT}(\mathcal{A}_2, w)$. Therefore the following statements hold:

LEMMA 4.2. *For every* $w \in (\mathbf{A}_2 - \{\mathrm{o}\})^+$, $\mathrm{GPT}(\mathcal{A}_2, w)$ *is nilpotent if and only if there is* n *such that* $T^n(\mathrm{val}_6(w)) = 1$.

THEOREM 4.3. *The algebra* $\mathcal{A}_2$ *is nilpotent if and only if the* $3x+1$ *conjecture holds.*

Analogously as 1 was split into two elements 1, I in the above construction of $\mathcal{A}_2$, alternatively a similar splitting of the other elements of the periodical tails can be used. We shall formulate the result for the element 4, without proof, which is almost the same as above.

DEFINITION 4.4. *Let* $\mathcal{A}_3 = (\mathbf{A}_3; \otimes, \mathrm{o})$, *where* $\mathbf{A}_3 = \{\mathrm{o}, 0, 1, 2, 3, 4, 5, \mathrm{F}\}$ *and the operation* $\otimes$ *is defined by*

$$x \otimes y = \begin{cases} \mathrm{o} & \textit{if} \quad x = \mathrm{o},\ y = \mathrm{F}, \\ \mathrm{F} & \textit{if} \quad x = 1,\ y = \mathrm{o}, \\ x' * y' & \textit{otherwise}, \end{cases}$$

where $*$ *is as in* $\mathcal{A}_1$ *and* $x' = x$ *if* $x \neq \mathrm{F}$ *and* $\mathrm{F}' = 4$.

Notice only that the new element F arises usually at the right end of the substantial part of a row of $\mathrm{GPT}(\mathcal{A}_3, w)$. An example is displayed in Figure 3.

THEOREM 4.5. *The algebra* $\mathcal{A}_3$ *is nilpotent if and only if the* $3x+1$ *conjecture holds.*

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