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Multivariate Models with Constraints Confidence Regions*

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Abstract

In multivariate linear statistical models with normally distributed observation matrix a structure of a covariance matrix plays an important role when confidence regions must be determined. In the paper it is assumed that the covariance matrix is a linear combination of known symmetric and positive semidefinite matrices and unknown parameters (variance components) which are unbiasedly estimable. Then insensitivity regions are found for them which enables us to decide whether plug-in approach can be used for confidence regions.

Key words: Multivariate model; constraints; variance components; plug-in estimator; insensitivity region.

2000 Mathematics Subject Classification: 62J05, 62H12

1 Introduction

Multivariate linear statistical models are analyzed in several monographs (cf. [1], [3], [5], etc.). Relatively small attention is given to problems of a determination of confidence regions. An attempt to contribute to a solution of the problem is the aim of the paper.

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An apriori information on a structure of the covariance matrix in multivariate linear statistical model can be of different forms. A determination of a confidence region for the mean value parameters of the observation matrix depends essentially on this structure.

In the paper it is assumed that the covariance matrix is a linear combination of known symmetric positive semidefinite matrices and unknown, however unbiasedly estimable, coefficients (variance components). Then the plug-in approach is used for a confidence regions. For a decision whether this approach is admisible, the insensitivity regions are determined.

In the following text the models

$$\text{vec}(\underline{\mathbf{Y}}) \sim N_{nm}[(\mathbf{I}\otimes\mathbf{X})\text{vec}(\mathbf{B}), \mathbf{\Sigma}\otimes\mathbf{I}], \quad \mathbf{H}_1\mathbf{B}\mathbf{H}_2+\mathbf{H}_0=\mathbf{0} \qquad (1)$$

$$\text{vec}(\underline{\mathbf{Y}}) \sim N_{nm}[(\mathbf{I}\otimes\mathbf{X})\text{vec}(\mathbf{B}), \mathbf{I}\otimes\mathbf{\Sigma}], \quad \mathbf{H}_1\mathbf{B}\mathbf{H}_2+\mathbf{H}_0=\mathbf{0} \qquad (2)$$

$$\text{vec}(\underline{\mathbf{Y}}) \sim N_{nr}[(\mathbf{Z}'\otimes\mathbf{X})\text{vec}(\mathbf{B}), \mathbf{\Sigma}\otimes\mathbf{I}], \quad \mathbf{H}_1\mathbf{B}\mathbf{H}_2+\mathbf{H}_0=\mathbf{0} \qquad (3)$$

$$\text{vec}(\underline{\mathbf{Y}}) \sim N_{nr}[(\mathbf{Z}'\otimes\mathbf{X})\text{vec}(\mathbf{B}), \mathbf{I}\otimes\mathbf{\Sigma}], \quad \mathbf{H}_1\mathbf{B}\mathbf{H}_2+\mathbf{H}_0=\mathbf{0}. \qquad (4)$$

will be considered.

Here $\underline{\mathbf{Y}}$ is an $n\times m$ and $n\times r$, respectively, matrix (observation matrix) normally distributed, vec($\underline{\mathbf{Y}}$) is the vector composed of the columns of the matrix $\underline{\mathbf{Y}}$, $\mathbf{Z},\mathbf{X},\mathbf{H}_1,\mathbf{H}_2,\mathbf{H}_0$ are known matrices of proper dimensions, $\mathbf{B}$ is a matrix of unknown parameters and $\mathbf{\Sigma}$ is a matrix of the structure $\mathbf{\Sigma}=\sum_{i=1}^{p}\vartheta_i\mathbf{V}_i$, $p\geq 2$. The notation $\text{vec}(\underline{\mathbf{Y}}) \sim_{nm} [(\mathbf{I}\otimes\mathbf{X})\,\text{vec}(\mathbf{B}), \mathbf{\Sigma}\otimes\mathbf{I}]$ means that the matrix need not be normally distributed. The matrices $\mathbf{V}_1,\dots,\mathbf{V}_p$, are given, symmetric and positive semidefinite, $\boldsymbol{\vartheta}=(\vartheta_1,\dots,\vartheta_p)'\in\underline{\vartheta}$, is unknown vector parameter, where $\underline{\vartheta}$ is an open set in R^p (p-dimensional Euclidean space). The matrix $\mathbf{H}_0$ must satisfy the condition

$$\text{vec}(\mathbf{H}_0)\in\mathcal{M}(\mathbf{H}_2'\otimes\mathbf{H}_1).$$

For the sake of simplicity either the matrix $\mathbf{H}_1$, or the matrix $\mathbf{H}_2$ will be considered to be the identity matrix $\mathbf{I}$.

Further symbols are of the following meaning.

$\mathcal{M}(\mathbf{A}_{m,n})=\{\mathbf{A}\mathbf{u}:\mathbf{u}\in R^n\}$ is the column subspace of the matrix $\mathbf{A}$, $\mathbf{P}_X=\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}$, $\mathbf{A}^-$ is a generalized inverse (g-inverse) of the matrix $\mathbf{A}$, i.e. $\mathbf{A}\mathbf{A}^-\mathbf{A}=\mathbf{A}$, $\mathbf{M}_X=\mathbf{I}-\mathbf{P}_X$, $\mathbf{A}^+$ is the Moore–Penrose g-inverse of the matrix $\mathbf{A}$, i.e. $\mathbf{A}\mathbf{A}^+\mathbf{A}=\mathbf{A}$, $\mathbf{A}^+\mathbf{A}\mathbf{A}^+=\mathbf{A}^+$, $\mathbf{A}\mathbf{A}^+=(\mathbf{A}\mathbf{A}^+)'$, $\mathbf{A}^+\mathbf{A}=(\mathbf{A}^+\mathbf{A})'$. Frequently used notation $(\mathbf{M}_X\mathbf{\Sigma}\mathbf{M}_X)^+$, means therefore

$$\begin{aligned}(\mathbf{M}_X\mathbf{\Sigma}\mathbf{M}_X)^+ &= \mathbf{\Sigma}^{-1}-\mathbf{\Sigma}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{\Sigma}^{-1}, \quad \mathbf{\Sigma} \quad \text{is p.d.},\\ &= \mathbf{\Sigma}^{+}-\mathbf{\Sigma}^{+}\mathbf{X}(\mathbf{X}\mathbf{\Sigma}^{-}\mathbf{X})^{-}\mathbf{X}\mathbf{\Sigma}^{+}, \quad \mathcal{M}(\mathbf{X})\subset\mathcal{M}(\mathbf{\Sigma}),\\ &= \mathbf{T}^{+}-\mathbf{T}^{+}\mathbf{X}(\mathbf{X}'\mathbf{T}^{-}\mathbf{X})^{-}\mathbf{X}'\mathbf{T}^{+}, \quad \mathbf{T}=\mathbf{\Sigma}+\mathbf{X}\mathbf{X}', \quad \text{otherwise.}\end{aligned}$$

If the matrix $\mathbf{B}$ is unbiasedly estimable, then the symbol $\widehat{\widehat{\mathbf{B}}}$ denotes the best linear unbiased estimator (BLUE) of the matrix $\mathbf{B}$. ($\widehat{\widehat{\ }}$ is used in order to emphasize that the estimator respects the constraints; $\widehat{\mathbf{B}}$ is the BLUE which does not respect the constraints). If the matrix $\mathbf{B}$ is not unbiasedly estimable, however the BLUE exists for the matrix $\mathbf{XB}$, then the symbol $\widehat{\widehat{\mathbf{XB}}}$ is used. The space of all $m \times n$ matrices is $\mathcal{M}_{m,n}$.

2 Estimation of the variance components

In the following text the structure of the matrix $\mathbf{\Sigma}$ is assumed to be $\mathbf{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$. In estimation of the variance components vector $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_p)'$, the following lemma will be used.

Lemma 2.1 *Let the univariate universal linear statistical model with constraints, i.e.*

$$\mathbf{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}, \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i), \quad \mathbf{h} + \mathbf{H}\boldsymbol{\beta} = \mathbf{0},$$

be considered. Here the $n \times k$ *matrix* $\mathbf{X}$, $n \times n$ *symmetric and p.s.d. matrices* $\mathbf{V}_1, \dots, \mathbf{V}_p$ *and the* $q \times k$ *matrix* $\mathbf{H}$ *are given. Also the* q*-dimensional vector* $\mathbf{h}$ *is given. Then the* $\boldsymbol{\vartheta}_0$*-MINQUE (minimum norm quadratic unbiased estimator) of the vector* $\boldsymbol{\vartheta}$ *is*

$$\widehat{\boldsymbol{\vartheta}} = \mathbf{S}^{-1}_{(M_{XM_{H'}} \Sigma_0 M_{XM_{H'}})^+} \boldsymbol{\gamma},$$

where

$$\begin{aligned}
\boldsymbol{\gamma} &= (\gamma_1, \dots, \gamma_p)', \\
\gamma_i &= [\mathbf{Y} + \mathbf{XH}'(\mathbf{HH}')^+\mathbf{h}]' \left(\mathbf{M}_{XM_{H'}} \mathbf{\Sigma}_0 \mathbf{M}_{XM_{H'}}\right)^+ \mathbf{V}_i \\
&\quad \times \left(\mathbf{M}_{XM_{H'}} \mathbf{\Sigma}_0 \mathbf{M}_{XM_{H'}}\right)^+ [\mathbf{Y} + \mathbf{XH}'(\mathbf{HH}')^+\mathbf{h}], \quad i = 1, \dots, p, \\
\mathbf{\Sigma}_0 &= \sum_{i=1}^{p} \vartheta_{0,i} \mathbf{V}_i, \\
&\left\{\mathbf{S}_{(M_{XM_{H'}} \Sigma_0 M_{XM_{H'}})^+}\right\}_{i,j} = \\
&= \mathrm{Tr}\left[\mathbf{V}_i \left(\mathbf{M}_{XM_{H'}} \mathbf{\Sigma}_0 \mathbf{M}_{XM_{H'}}\right)^+ \mathbf{V}_j \left(\mathbf{M}_{XM_{H'}} \mathbf{\Sigma}_0 \mathbf{M}_{XM_{H'}}\right)^+\right], \\
&\qquad i, j = 1, \dots, p,
\end{aligned}$$

and $\boldsymbol{\vartheta}_0 = (\vartheta_{0,1}, \dots, \vartheta_{0,p})'$ *is an approximate value of the vector* $\boldsymbol{\vartheta}$.

The $\boldsymbol{\vartheta}_0$*-MINQUE of the vector* $\boldsymbol{\vartheta}$ *exists iff the matrix* $\mathbf{S}_{(M_{XM_{H'}} \Sigma_0 M_{XM_{H'}})^+}$ *is regular.*

Proof cf. in [2], [4]. □

The formulae for the multivariate models with constraints can be now rewritten directly from this lemma.

Theorem 2.2 *(i) Let in the model (1) with $\mathbf{H}_2 = \mathbf{I}$, the matrix*

$$\mathbf{S}_{\left[M_{(I\otimes X)(I\otimes M_{H_1'})}(\Sigma_0\otimes I)M_{(I\otimes X)(I\otimes M_{H_1'})}\right]^+}$$

be regular. Then

$$\mathbf{S}_{\left[M_{(I\otimes X)(I\otimes M_{H_1'})}(\Sigma_0\otimes I)M_{(I\otimes X)(I\otimes M_{H_1'})}\right]^+} = \mathrm{Tr}\left(\mathbf{M}_{XM_{H_1'}}\right)\mathbf{S}_{\Sigma_0^+}$$

and the $\boldsymbol{\vartheta}_0$-MINQUE is

$$\begin{aligned}
\widehat{\boldsymbol{\vartheta}} &= \left[\mathrm{Tr}(\mathbf{M}_{XM_{H_1'}})\mathbf{S}_{\Sigma^+}\right]^{-1}\boldsymbol{\gamma},\\
\left\{\mathbf{S}_{\Sigma_0^+}\right\}_{i,j} &= \mathrm{Tr}(\mathbf{V}_i\boldsymbol{\Sigma}_0^+\mathbf{V}_j\boldsymbol{\Sigma}_0^+), \quad i,j=1,\dots,p,\\
\boldsymbol{\gamma} &= (\gamma_1,\dots,\gamma_p)'\\
\gamma_i &= \mathrm{Tr}\Big\{[\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_1'(\mathbf{H}_1\mathbf{H}_1')^+\mathbf{H}_0]'\mathbf{M}_{XM_{H_1'}}[\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_1'(\mathbf{H}_1\mathbf{H}_1')^+\mathbf{H}_0]\\
&\qquad \times \boldsymbol{\Sigma}_0^+\mathbf{V}_i\boldsymbol{\Sigma}_0^+\Big\}, \quad i=1,\dots,p.
\end{aligned}$$

(ii) Let in the model (1) with $\mathbf{H}_1 = \mathbf{I}$, the matrix

$$\mathbf{S}_{\left[M_{M_{H_2}\otimes X}(\Sigma_0\otimes I)M_{M_{H_2}\otimes X}\right]^+}$$

be regular. Then

$$\mathbf{S}_{\left[M_{M_{H_2}\otimes X}(\Sigma_0\otimes I)M_{M_{H_2}\otimes X}\right]^+} = [n - r(\mathbf{X})]\mathbf{S}_{\Sigma_0^+} + r(\mathbf{X})\mathbf{S}_{(P_{H_2}\Sigma_0 P_{H_2})^+}$$

and the $\boldsymbol{\vartheta}_0$-MINQUE is

$$\begin{aligned}
\widehat{\boldsymbol{\vartheta}} &= \left\{[n - r(\mathbf{X})]\mathbf{S}_{\Sigma_0^+} + r(\mathbf{X})\mathbf{S}_{(P_{H_2}\Sigma_0 P_{H_2})^+}\right\}^{-1}\boldsymbol{\gamma},\\
\boldsymbol{\gamma} &= (\gamma_1,\dots,\gamma_p)',\\
\gamma_i &= \mathrm{Tr}(\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\boldsymbol{\Sigma}_0^+\mathbf{V}_i\boldsymbol{\Sigma}_0^+) + \mathrm{Tr}\Big\{[\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_0(\mathbf{H}_2'\mathbf{H}_2)^+\mathbf{H}_2']'\mathbf{P}_X\\
&\qquad \times [\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_0(\mathbf{H}_2'\mathbf{H}_2)^+\mathbf{H}_2'](\mathbf{P}_{H_2}\boldsymbol{\Sigma}_0\mathbf{P}_{H_2})^+\Big\}, \quad i=1,\dots,p
\end{aligned}$$

and

$$\left\{\mathbf{S}_{P_{H_2}\Sigma_0 P_{H_2})^+}\right\}_{i,j} = \mathrm{Tr}\left[\mathbf{V}_i(\mathbf{P}_{H_2}\boldsymbol{\Sigma}_0\mathbf{P}_{H_2})^+\mathbf{V}_j(\mathbf{P}_{H_2}\boldsymbol{\Sigma}_0\mathbf{P}_{H_2})^+\right], \quad i,j=1,\dots,p.$$

Proof (i) It is implied by Lemma 2.1 and by the equality

$$\left[\mathbf{M}_{I\otimes(XM_{H_1'})}(\boldsymbol{\Sigma}_0\otimes\mathbf{I})\mathbf{M}_{I\otimes(XM_{H_1'})}\right]^+ = \boldsymbol{\Sigma}_0^+\otimes\mathbf{M}_{XM_{H_1'}}.$$

In (ii) the equality

$$\left[\mathbf{M}_{M_{H_2}\otimes X}(\boldsymbol{\Sigma}_0\otimes\mathbf{I})\mathbf{M}_{M_{H_2}\otimes X}\right]^+ = \boldsymbol{\Sigma}_0^+\otimes\mathbf{M}_X + (\mathbf{P}_{H_2}\boldsymbol{\Sigma}_0\mathbf{P}_{H_2})^+\otimes\mathbf{P}_X$$

must be used. □

Theorem 2.3 *(i) Let in the model (2) with* $\mathbf{H}_2 = \mathbf{I}$*, the matrix*

$$\mathbf{S}_{M_{[I\otimes(XM_{H_1'})]}(I\otimes\Sigma_0)M_{[I\otimes(XM_{H_1'})]^+}}$$

be regular. Then

$$\mathbf{S}_{M_{[I\otimes(XM_{H_1'})]}(I\otimes\Sigma_0)M_{[I\otimes(XM_{H_1'})]^+}} = m\mathbf{S}_{(M_{XM_{H_1'}}\Sigma_0 M_{XM_{H_1'}})^+}$$

and the $\boldsymbol{\vartheta}_0$*-MINQUE is*

$$\begin{aligned}
\widehat{\boldsymbol{\vartheta}} &= \left(m\mathbf{S}_{(M_{XM_{H_1'}}\Sigma_0 M_{XM_{H_1'}})^+}\right)^{-1}\boldsymbol{\gamma},\\
\boldsymbol{\gamma} &= (\gamma_1,\dots,\gamma_p)',\\
\gamma_i &= \mathrm{Tr}\Big\{[\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_1'(\mathbf{H}_1\mathbf{H}_1')^+\mathbf{H}_0]'\left(\mathbf{M}_{XM_{H_1'}}\boldsymbol{\Sigma}_0\mathbf{M}_{XM_{H_1'}}\right)^+\mathbf{V}_i\\
&\quad\times\left(\mathbf{M}_{XM_{H_1'}}\boldsymbol{\Sigma}_0\mathbf{M}_{XM_{H_1'}}\right)^+[\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_1'(\mathbf{H}_1\mathbf{H}_1')^+\mathbf{H}_0]\Big\},\\
&\quad i = 1,\dots,p.
\end{aligned}$$

(ii) Let in the model (2) with $\mathbf{H}_1 = \mathbf{I}$*, the matrix*

$$\mathbf{S}_{[M_{(M_{H_2}\otimes X)}(I\otimes\Sigma_0)M_{(M_{H_2}\otimes X)}]^+}$$

be regular. Then

$$\mathbf{S}_{[M_{(M_{H_2}\otimes X)}(I\otimes\Sigma_0)M_{(M_{H_2}\otimes X)}]^+} = [m - r(\mathbf{H}_2)]\mathbf{S}_{(M_X\Sigma_0 M_X)^+} + r(\mathbf{H}_2)\mathbf{S}_{\Sigma^+}$$

and the $\boldsymbol{\vartheta}_0$*-MINQUE is*

$$\begin{aligned}
\widehat{\boldsymbol{\vartheta}} &= \left\{[m - r(\mathbf{H}_2)]\mathbf{S}_{(M_X\Sigma_0 M_X)^+} + r(\mathbf{H}_2)\mathbf{S}_{\Sigma^+}\right\}^{-1}\boldsymbol{\gamma},\\
\boldsymbol{\gamma} &= (\gamma_1,\dots,\gamma_p)',\\
\gamma_i &= \mathrm{Tr}[\underline{\mathbf{Y}}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\underline{\mathbf{Y}}\mathbf{M}_{H_2}]\\
&\quad + \mathrm{Tr}\left\{[\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_0(\mathbf{H}_2'\mathbf{H}_2)^+\mathbf{H}_2']'\boldsymbol{\Sigma}_0^+\mathbf{V}_i\boldsymbol{\Sigma}_0^+[\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_0(\mathbf{H}_2'\mathbf{H}_2)^+\mathbf{H}_2']\mathbf{P}_{H_2}\right\}
\end{aligned}$$

Proof (i) The obvious equality

$$\left[\mathbf{M}_{[I\otimes(XM_{H_1'})}(I\otimes\Sigma_0)\mathbf{M}_{[I\otimes(XM_{H_1'})}\right]^+ = \mathbf{I}\otimes\left(\mathbf{M}_{XM_{H_1'}}\boldsymbol{\Sigma}_0\mathbf{M}_{XM_{H_1'}}\right)^+$$

must be taken into account.

(ii) The equality

$$\left[\mathbf{M}_{M_{H_2}\otimes X}(\mathbf{I}\otimes\boldsymbol{\Sigma}_0)\mathbf{M}_{M_{H_2}\otimes X}\right]^+ = \mathbf{M}_{H_2}\otimes(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+ + \mathbf{P}_{H_2}\otimes\boldsymbol{\Sigma}_0^+$$

must be taken into account. □

Theorem 2.4 *(i) Let in the model (3) with* $\mathbf{H}_2 = \mathbf{I}$ *the matrix*

$$\mathbf{S}_{[M_{(Z'\otimes X)(I\otimes M_{H_1'})}(\Sigma_0\otimes I)M_{(Z'\otimes X)(I\otimes M_{H_1'})}]^+}$$

be regular. Then

$$\begin{aligned}&\mathbf{S}_{[M_{(Z'\otimes X)(I\otimes M_{H_1'})}(\Sigma_0\otimes I)M_{(Z'\otimes X)(I\otimes M_{H_1'})}]^+} =\\ &= \operatorname{Tr}\left(\mathbf{M}_{XM_{H_1'}}\right)\mathbf{S}_{\Sigma_0^+} + \operatorname{Tr}\left(\mathbf{P}_{XM_{H_1'}}\right)\mathbf{S}_{(M_{Z'}\Sigma_0 M_{Z'})^+}\end{aligned}$$

and the $\boldsymbol{\vartheta}_0$*-MINQUE is*

$$\begin{aligned}\widehat{\boldsymbol{\vartheta}} &= \left[\operatorname{Tr}\left(\mathbf{M}_{XM_{H_1'}}\right)\mathbf{S}_{\Sigma_0^+} + \operatorname{Tr}\left(\mathbf{P}_{XM_{H_1'}}\right)\mathbf{S}_{(M_{Z'}\Sigma_0 M_{Z'})^+}\right]^{-1}\boldsymbol{\gamma},\\ \boldsymbol{\gamma} &= \gamma_1,\dots,\gamma_p)',\\ \gamma_i &= \operatorname{Tr}\Big\{[\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_1'(\mathbf{H}_1\mathbf{H}_1')^+\mathbf{H}_0\mathbf{Z}]'\mathbf{M}_{XM_{H_1'}}[\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_1'(\mathbf{H}_1\mathbf{H}_1')^+\mathbf{H}_0\mathbf{Z}]\\ &\quad\times \boldsymbol{\Sigma}_0^+\mathbf{V}_i\boldsymbol{\Sigma}_0^+\Big\} + \operatorname{Tr}\Big\{[\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_1'(\mathbf{H}_1\mathbf{H}_1')^+\mathbf{H}_0\mathbf{Z}]'\mathbf{P}_{XM_{H_1'}}\\ &\quad\times [\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_1'(\mathbf{H}_1\mathbf{H}_1')^+\mathbf{H}_0\mathbf{Z}]\left(\mathbf{M}_{Z'}\boldsymbol{\Sigma}_0\mathbf{M}_{Z'}\right)^+\mathbf{V}_i\left(\mathbf{M}_{Z'}\boldsymbol{\Sigma}_0\mathbf{M}_{Z'}\right)^+\Big\},\\ &\quad i = 1,\dots,p.\end{aligned}$$

(ii) Let in the model (3) with $\mathbf{H}_1 = \mathbf{I}$ *the matrix*

$$\mathbf{S}_{[M_{(Z'\otimes X)(M_{H_2}\otimes I)}(\Sigma_0\otimes I)M_{(Z'\otimes X)(M_{H_2}\otimes I)}]^+}$$

be regular. Then

$$\begin{aligned}&\mathbf{S}_{[M_{(Z'\otimes X)(M_{H_2}\otimes I)}(\Sigma_0\otimes I)M_{(Z'\otimes X)(M_{H_2}\otimes I)}]^+} =\\ &\quad= [n - r(\mathbf{X})]\mathbf{S}_{\Sigma_0^+} + r(\mathbf{X})\mathbf{S}_{(M_{Z'M_{H_2}}\Sigma_0 M_{Z'M_{H_2}})^+}\end{aligned}$$

and the $\boldsymbol{\vartheta}_0$*-MINQUE is*

$$\begin{aligned}\widehat{\boldsymbol{\vartheta}} &= \left\{[n - r(\mathbf{X})]\mathbf{S}_{\Sigma_0^+} + r(\mathbf{X})\mathbf{S}_{(M_{Z'M_{H_2}}\Sigma_0 M_{Z'M_{H_2}})^+}\right\}^{-1}\boldsymbol{\gamma},\\ \boldsymbol{\gamma} &= (\gamma_1,\dots,\gamma_p)',\\ \gamma_i &= \operatorname{Tr}(\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\boldsymbol{\Sigma}_0^+\mathbf{V}_i\boldsymbol{\Sigma}_0^+) + \operatorname{Tr}\Big\{[\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_0(\mathbf{H}_2'\mathbf{H}_2)^+\mathbf{H}_2'\mathbf{Z}]'\mathbf{P}_X\\ &\quad\times [\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_0(\mathbf{H}_2'\mathbf{H}_2)^+\mathbf{H}_2'\mathbf{Z}]\left(\mathbf{M}_{Z'M_{H_2}}\boldsymbol{\Sigma}_0\mathbf{M}_{Z'M_{H_2}}\right)^+\mathbf{V}_i\\ &\quad\times \left(\mathbf{M}_{Z'M_{H_2}}\boldsymbol{\Sigma}_0\mathbf{M}_{Z'M_{H_2}}\right)^+\Big\}, \quad i = 1,\dots,p.\end{aligned}$$

Proof (i) It is necessary to take into account the equality

$$\begin{aligned}&\left[\mathbf{M}_{(Z'\otimes X)(I\otimes M_{H_1'})}(\boldsymbol{\Sigma}_0\otimes\mathbf{I})\mathbf{M}_{(Z'\otimes X)(I\otimes M_{H_1'})}\right]^+ =\\ &\quad= \boldsymbol{\Sigma}_0^+\otimes\mathbf{M}_{XM_{H_1'}} + (\mathbf{M}_{Z'}\boldsymbol{\Sigma}_0\mathbf{M}_{Z'})^+\otimes\mathbf{P}_{XM_{H_1'}}.\end{aligned}$$

(ii) The equality

$$\left[\mathbf{M}_{(Z'\otimes X)(M_{H_2}\otimes I)}(\boldsymbol{\Sigma}_0\otimes\mathbf{I})\mathbf{M}_{(Z'\otimes X)(M_{H_2}\otimes I)}\right]^+ = \\ = \boldsymbol{\Sigma}_0^+\otimes\mathbf{M}_X + (\mathbf{M}_{Z'M_{H_2}}\boldsymbol{\Sigma}_0\mathbf{M}_{Z'M_{H_2}})^+\otimes\mathbf{P}_X$$

must be utilized. □

Theorem 2.5 *(i) Let in the model (4) with* $\mathbf{H}_2 = \mathbf{I}$ *the matrix*

$$\mathbf{S}_{[M_{(Z'\otimes X)(I\otimes M_{H_1'})}(I\otimes\Sigma_0)M_{(Z'\otimes X)(I\otimes M_{H_1'})}]^+}$$

be regular. Then

$$\begin{aligned}
\widehat{\boldsymbol{\vartheta}} &= \mathbf{S}^{-1}_{[M_{(Z'\otimes X)(I\otimes M_{H_1'})}(I\otimes\Sigma_0)M_{(Z'\otimes X)(I\otimes M_{H_1'})}]^+}\boldsymbol{\gamma}, \qquad \boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_p)',\\
\gamma_i &= \mathrm{Tr}\left\{\left[\underline{\mathbf{Y}} + \mathbf{H}_1'(\mathbf{H}_1\mathbf{H}_1')^+\mathbf{H}_0\right]'\boldsymbol{\Sigma}_0^+\mathbf{V}_i\boldsymbol{\Sigma}_0^+\left[\underline{\mathbf{Y}} + \mathbf{H}_1'(\mathbf{H}_1\mathbf{H}_1')^+\mathbf{H}_0\right]\mathbf{M}_{Z'}\right\}\\
&\quad + \mathrm{Tr}\left\{\left[\underline{\mathbf{Y}} + \mathbf{H}_1'(\mathbf{H}_1\mathbf{H}_1')^+\mathbf{H}_0\right]'\left(\mathbf{M}_{XM_{H_1'}}\boldsymbol{\Sigma}_0\mathbf{M}_{XM_{H_1'}}\right)^+\mathbf{V}_i\right.\\
&\quad \left.\times\left(\mathbf{M}_{XM_{H_1'}}\boldsymbol{\Sigma}_0\mathbf{M}_{XM_{H_1'}}\right)^+\left[\underline{\mathbf{Y}} + \mathbf{H}_1'(\mathbf{H}_1\mathbf{H}_1')^+\mathbf{H}_0\right]\mathbf{P}_{Z'}\right\}, \quad i = 1,\dots,p
\end{aligned}$$

and

$$\mathbf{S}_{[M_{(Z'\otimes X)(I\otimes M_{H_1'})}(I\otimes\Sigma_0)M_{(Z'\otimes X)(I\otimes M_{H_1'})}]^+} = \\ = \mathrm{Tr}(\mathbf{M}_{Z'})\mathbf{S}_{\Sigma_0^+} + \mathrm{Tr}(\mathbf{P}_{Z'})\mathbf{S}_{(M_{XM_{H_1'}}\Sigma_0 M_{XM_{H_1'}})^+}.$$

(ii) If in the model (4) $\mathbf{H}_1 = \mathbf{I}$ *and the matrix*

$$\mathbf{S}_{[M_{(Z'M_{H_2})\otimes X}(I\otimes\Sigma_0)M_{(Z'M_{H_2})\otimes X}]^+}$$

is regular, then

$$\widehat{\boldsymbol{\vartheta}} = \mathbf{S}^{-1}_{[M_{(Z'M_{H_2})\otimes X}(I\otimes\Sigma_0)M_{(Z'M_{H_2})\otimes X}]^+}\boldsymbol{\gamma}, \qquad \boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_p)',$$

$$\gamma_i = \mathrm{Tr}\left\{\left[\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_0(\mathbf{H}_2'\mathbf{H}_2)^+\mathbf{H}_2'\right]'\boldsymbol{\Sigma}_0^+\mathbf{V}_i\boldsymbol{\Sigma}_0^+\left[\underline{\mathbf{Y}} + \mathbf{X}\mathbf{H}_0(\mathbf{H}_2'\mathbf{H}_2)^+\mathbf{H}_2'\right]\mathbf{M}_{Z'M_{H_2}}\right\}$$
$$+\mathrm{Tr}\left\{\left[\underline{\mathbf{Y}}+\mathbf{X}\mathbf{H}_0(\mathbf{H}_2'\mathbf{H}_2)^+\mathbf{H}_2'\right]'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\left[\underline{\mathbf{Y}}+\mathbf{X}\mathbf{H}_0(\mathbf{H}_2'\mathbf{H}_2)^+\mathbf{H}_2'\right]\mathbf{P}_{Z'M_{H_2}}\right\}$$

and

$$\mathbf{S}_{[M_{(Z'M_{H_2})\otimes X}(I\otimes\Sigma_0)M_{(Z'M_{H_2})\otimes X}]^+} = \mathrm{Tr}(\mathbf{M}_{Z'M_{H_2}})\mathbf{S}_{\Sigma_0^+} + \mathrm{Tr}(\mathbf{P}_{Z'M_{H_2}})\mathbf{S}_{(M_X\Sigma_0 M_X)^+}.$$

Proof In (i) the equality

$$\Big[\mathbf{M}_{(Z'\otimes X)(I\otimes M_{H_1'})}(\mathbf{I}\otimes\mathbf{\Sigma}_0)\mathbf{M}_{(Z'\otimes X)(I\otimes M_{H_1'})}\Big]^+ = \\ = \mathbf{M}_{Z'}\otimes\mathbf{\Sigma}_0+\mathbf{P}_{Z'}\otimes(\mathbf{M}_{XM_{H_1'}}\mathbf{\Sigma}_0\mathbf{M}_{XM_{H_1'}})^+$$

must be used.

In (ii) the equality

$$\Big[\mathbf{M}_{[(Z'M_{H_2})\otimes X]}(\mathbf{I}\otimes\mathbf{\Sigma}_0)\mathbf{M}_{[(Z'M_{H_2})\otimes X]}\Big]^+ = \\ = \mathbf{M}_{Z'M_{H_2}}\otimes\mathbf{\Sigma}_0+\mathbf{P}_{Z'M_{H_2}}\otimes(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+$$

must be used. □

3 Confidence regions

3.1 The matrix Σ is given

In this section the observation matrix is assumed to be normally distributed. Since confidence regions for multivariate models can be directly rewritten from the formulae for univariate models, the following lemmas are given without proofs.

Lemma 3.1 *(i) Let in the model (1) with* $\mathbf{H}_2=\mathbf{I}$ *the* $s\times k$ *matrix* $\mathbf{G}_1$ *and the* $m\times t$ *matrix* $\mathbf{G}_2$ *be given. Let* $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ *be unbiasedly estimable, i.e.*

$$\mathcal{M}(\mathbf{G}_2\otimes\mathbf{G}_1')\subset\mathcal{M}[\mathbf{I}\otimes(\mathbf{X}',\mathbf{H}_1')].$$

Then the $(1-\alpha)$*-confidence region is*

$$\mathcal{E} = \Big\{\mathbf{U}:\mathbf{U}\in\mathcal{M}_{s,t},\,\mathrm{Tr}\Big((\mathbf{U}-\widehat{\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2}})'[\mathbf{G}_1(\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{X}\mathbf{M}_{H_1'})^+\mathbf{G}_1']^+ \\ \times(\mathbf{U}-\widehat{\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2}})(\mathbf{G}_2'\mathbf{\Sigma}\mathbf{G}_2)^+\Big)\le\chi^2_f(0,1-\alpha)\Big\},$$

$$f = r\{\mathrm{Var}[\mathrm{vec}(\widehat{\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2}})]\}=r(\mathbf{G}_2'\mathbf{\Sigma}\mathbf{G}_2)r[\mathbf{G}_1(\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{X}\mathbf{M}_{H_1'})^+\mathbf{G}_1'].$$

Here $\widehat{\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2}}$ *is the BLUE of the matrix* $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$*.*

(ii) Let in the model (1) with $\mathbf{H}_1=\mathbf{I}$ *the matrix* $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ *be unbiasedly estimable, i.e.* $\mathcal{M}(\mathbf{G}_2\otimes\mathbf{G}_1')\subset\mathcal{M}(\mathbf{I}\otimes\mathbf{X}',\mathbf{H}_2\otimes\mathbf{I})$*. Then the* $(1-\alpha)$*-confidence region is*

$$\mathcal{E} = \Big\{\mathbf{U}:\mathbf{U}\in\mathcal{M}_{s,t},\,\mathrm{Tr}\Big[(\mathbf{U}-\widehat{\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2}})'[\mathbf{G}_1(\mathbf{X}'\mathbf{X})^+\mathbf{G}_1']^+(\mathbf{U}-\widehat{\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2}}) \\ \times\Big(\mathbf{G}_2'\{[\mathbf{M}_{H_2}(\mathbf{\Sigma}+\mathbf{M}_{H_2})^+\mathbf{M}_{H_2}]^+-\mathbf{M}_{H_2}\}\mathbf{G}_2\Big)^+\Big]\le\chi^2_f(0;1-\alpha)\Big\},$$

$$f = r\Big\{\mathrm{Var}\left[\mathrm{vec}(\widehat{\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2}})\right]\Big\} \\ = r\Big(\mathbf{G}_2'\{[\mathbf{M}_{H_2}(\mathbf{\Sigma}+\mathbf{M}_{H_2})^+\mathbf{M}_{H_2}]^+-\mathbf{M}_{H_2}\}\mathbf{G}_2\Big)\,r[\mathbf{G}_1(\mathbf{X}'\mathbf{X})^+\mathbf{G}_1'].$$

Lemma 3.2 *(i) Let in the model (2) with* $\mathbf{H}_2 = \mathbf{I}$ *the* $s \times k$ *matrix* $\mathbf{G}_1$ *and the* $m \times t$ *matrix* $\mathbf{G}_2$ *be given and let* $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ *be unbiasedly estimable. Then the* $(1-\alpha)$*-confidence region is*

$$\mathcal{E} = \Big\{\mathbf{U} : \mathbf{U} \in \mathcal{M}_{s,t} \operatorname{Tr}\Big[(\mathbf{U} - \widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2})'\Big(\mathbf{G}_1\big\{[\mathbf{M}_{H_1'}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{H_1'}\mathbf{X}')^+ \times \mathbf{X}\mathbf{M}_{H_1'}]^+ - \mathbf{M}_{H_1'}\big\}\mathbf{G}_1'\Big)^+(\mathbf{U} - \widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2})(\mathbf{G}_2'\mathbf{G}_2)^+\Big] \leq \chi_f^2(0; 1-\alpha)\Big\},$$

$$f = r\Big\{\operatorname{Var}\big[\operatorname{vec}(\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2})\big]\Big\} = r(\mathbf{G}_2)r\Big(\mathbf{G}_1\big\{[\mathbf{M}_{H_1'}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{H_1'}\mathbf{X}')^+\mathbf{X}\mathbf{M}_{H_1'}]^+ - \mathbf{M}_{H_1'}\big\}\mathbf{G}_1'\Big).$$

(ii) Let in the model (2) with $\mathbf{H}_1 = \mathbf{I}$ *the* $s \times k$ *matrix* $\mathbf{G}_1$ *and the* $m \times t$ *matrix* $\mathbf{G}_2$ *be given and let* $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ *be unbiasedly estimable. Then the* $(1-\alpha)$*-confidence region is*

$$\mathcal{E} = \Bigg\{\mathbf{U} : \mathbf{U} \in \mathcal{M}_{s,t}, \operatorname{Tr}\Big((\mathbf{U} - \widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2})'\big\{\mathbf{G}_1[(\mathbf{X}'\mathbf{T}^+\mathbf{X})^+ - \mathbf{I}]\mathbf{G}_1'\big\}^+ \times (\mathbf{U} - \widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2})(\mathbf{G}_2'\mathbf{M}_{H_2}\mathbf{G}_2)^+\Big) \leq \chi_f^2(0; 1-\alpha)\Bigg\},$$

$$f = r\Big\{\operatorname{Var}\big[\operatorname{vec}(\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2})\big]\Big\} = r\big\{\mathbf{G}_1[(\mathbf{X}'\mathbf{T}^+\mathbf{X})^+ - \mathbf{I}]\mathbf{G}_1'\big\}r(\mathbf{G}_2'\mathbf{M}_{H_2}\mathbf{G}_2).$$

Lemma 3.3 *(i) Let in the model (3) with* $\mathbf{H}_2 = \mathbf{I}$ *the* $s \times k$ *matrix* $\mathbf{G}_1$ *and the* $m \times t$ *matrix* $\mathbf{G}_2$ *be given and let* $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ *be unbiasedly estimable, i.e.* $\mathcal{M}(\mathbf{G}_2 \otimes \mathbf{G}_1') \subset \mathcal{M}(\mathbf{Z} \otimes \mathbf{X}', \mathbf{I} \otimes \mathbf{H}_1')$*. Then the* $(1-\alpha)$*-confidence region is*

$$\mathcal{E} = \Big\{\mathbf{U} : \mathbf{U} \in \mathcal{M}_{s,t}, \operatorname{Tr}\Big((\mathbf{U} - \widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2})'\Big[\mathbf{G}_1(\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{X}\mathbf{M}_{H_1'})^+\mathbf{G}_1'\Big]^+ \times (\mathbf{U} - \widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2})\Big\{\mathbf{G}_2'\big[(\mathbf{Z}\mathbf{U}^+\mathbf{Z}')^+ - \mathbf{I}\big]\mathbf{G}_2\Big\}\Big) \leq \chi_f^2(0; 1-\alpha)\Big\},$$

$$f = r\Big\{\operatorname{Var}\big[\operatorname{vec}(\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2})\big]\Big\} = r\Big[\mathbf{G}_1(\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{X}\mathbf{M}_{H_1'})^+\mathbf{G}_1'\Big]\, r\Big\{\mathbf{G}_2'\big[(\mathbf{Z}\mathbf{U}^+\mathbf{Z}')^+ - \mathbf{I}\big]\mathbf{G}_2\Big\}.$$

(ii) Let in the model (3) with $\mathbf{H}_1 = \mathbf{I}$ *the* $s \times k$ *matrix* $\mathbf{G}_1$ *and the* $m \times t$ *materix* $\mathbf{G}_2$ *be given and let* $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ *be unbiasedly estimable, i.e.* $\mathcal{M}(\mathbf{G}_2 \otimes \mathbf{G}_1') \subset \mathcal{M}(\mathbf{Z} \otimes \mathbf{X}', \mathbf{H}_2 \otimes \mathbf{I})$*. Then the* $(1-\alpha)$*-confidence region is*

$$\mathcal{E} = \Big\{\mathbf{U} : \mathbf{U} \in \mathcal{M}_{s,t}, \operatorname{Tr}\Big[(\mathbf{U} - \widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2})'[\mathbf{G}_1(\mathbf{X}'\mathbf{X})^+\mathbf{G}_1']^+(\mathbf{U} - \widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2}) \times\Big(\mathbf{G}_2'\Big\{\big[\mathbf{M}_{H_2}\mathbf{Z}(\boldsymbol{\Sigma} + \mathbf{Z}'\mathbf{M}_{H_2}\mathbf{Z})^+\mathbf{Z}'\mathbf{M}_{H_2}\big]^+ - \mathbf{M}_{H_2}\Big\}\mathbf{G}_2\Big)^+\Big] \leq \chi_f^2(0; 1-\alpha)\Big\},$$

$$f = r\Big\{\operatorname{Var}\big[\operatorname{vec}(\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}_2})\big]\Big\} = r[\mathbf{G}_1(\mathbf{X}'\mathbf{X})^+\mathbf{G}_1']\, r\Big(\mathbf{G}_2'\Big\{\big[\mathbf{M}_{H_2}\mathbf{Z}(\boldsymbol{\Sigma} + \mathbf{Z}'\mathbf{M}_{H_2}\mathbf{Z})^+\mathbf{Z}'\mathbf{M}_{H_2}\big]^+ - \mathbf{M}_{H_2}\Big\}\mathbf{G}_2\Big).$$

Lemma 3.4 *(i) Let in the model (4) with $\mathbf{H}_2 = \mathbf{I}$ the $s \times k$ matrix $\mathbf{G}_1$ and $m \times t$ matrix $\mathbf{G}_2$ be given and let the matrix $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ be unbiasedly estimable, i.e. $\mathcal{M}(\mathbf{G}_2 \otimes \mathbf{G}_1') \subset \mathcal{M}(\mathbf{Z} \otimes \mathbf{X}', \mathbf{I} \otimes \mathbf{H}_1')$. Then the $(1-\alpha)$-confidence region is*

$$\mathcal{E} = \Big\{\mathbf{U} : \mathbf{U} \in \mathcal{M}_{s,t}, \mathrm{Tr}\Big[(\mathbf{U} - \widehat{\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}}}_2)'\Big(\mathbf{G}_1\{[\mathbf{M}_{H_1'}\mathbf{X}'(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{H_1'}\mathbf{X})^+ \times \mathbf{X}\mathbf{M}_{H_1'}]^+ - \mathbf{M}_{H_1'}\}\mathbf{G}_1'\Big)^+(\mathbf{U} - \widehat{\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}}}_2)[\mathbf{G}_2'(\mathbf{Z}'\mathbf{Z})^+\mathbf{G}_2]^+\Big] \le \chi_f^2(0; 1-\alpha)\Big\},$$

$$f = r\{\mathrm{Var}[\mathrm{vec}(\widehat{\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}}}_2)]\} = r\Big(\mathbf{G}_1\{[\mathbf{M}_{H_1'}\mathbf{X}'(\mathbf{\Sigma} + \mathbf{X}\mathbf{M}_{H_1'}\mathbf{X})^+\mathbf{X}\mathbf{M}_{H_1'}]^+ - \mathbf{M}_{H_1'}\}\mathbf{G}_1'\Big)r[\mathbf{G}_2'(\mathbf{Z}'\mathbf{Z})^+\mathbf{G}_2].$$

(ii) Let in the model (4) with $\mathbf{H}_1 = \mathbf{I}$ the $s \times k$ matrix $\mathbf{G}_1$ and the $m \times t$ matrix $\mathbf{G}_2$ be given and let $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ be unbiasedly estimable, i.e. $\mathcal{M}(\mathbf{G}_2 \otimes \mathbf{G}_1') \subset \mathcal{M}(\mathbf{Z} \otimes \mathbf{X}', \mathbf{H}_2 \otimes \mathbf{I})$. Then the $(1-\alpha)$-confidence region is

$$\mathcal{E} = \Big\{\mathbf{U} : \mathbf{U} \in \mathcal{M}_{s,t}, \mathrm{Tr}\Big((\mathbf{U} - \widehat{\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}}}_2)'\Big\{\mathbf{G}_1\big[(\mathbf{X}'\mathbf{T}^+\mathbf{X})^+ - \mathbf{I}\big]\mathbf{G}_1'\Big\}^+ \times (\mathbf{U} - \widehat{\widehat{\mathbf{G}_1\mathbf{B}\mathbf{G}}}_2)\big[\mathbf{G}_2'(\mathbf{M}_{H_2}\mathbf{Z}\mathbf{Z}'\mathbf{M}_{H_2})^+\mathbf{G}_2\big]^+\Big) \le \chi_f^2(0; 1-\alpha)\Big\},$$

$$f = r\Big\{\mathbf{G}_1\big[(\mathbf{X}'\mathbf{T}^+\mathbf{X})^+ - \mathbf{I}\big]\mathbf{G}_1'\Big\}\, r\big[\mathbf{G}_2'(\mathbf{M}_{H_2}\mathbf{Z}\mathbf{Z}'\mathbf{M}_{H_2})^+\mathbf{G}_2\big].$$

3.2 The matrix $\mathbf{\Sigma}$ is of the form $\sum_{i=1}^p \vartheta_i \mathbf{V}_i$

If the estimators of the variance components $\vartheta_1, \dots, \vartheta_p$, are sufficiently accurate, then confidence regions cover the functions of parameter matrix with probability sufficiently near to prescribed confidence level $1-\alpha$. How rigorous conditions on the accuracy is, the nonsensitivity region can show.

In the first step let an univariate universal linear statistical model with constraints be considered, i.e.

$$\mathbf{Y} \sim N_n\left(\mathbf{X}\boldsymbol{\beta}, \sum_{i=1}^p \vartheta_i \mathbf{V}_i\right), \quad \mathbf{H}_{q,k}\boldsymbol{\beta} + \mathbf{h}_{q,1} = \mathbf{0}.$$

The $(1-\alpha)$-confidence region for the function $\mathbf{G}_{r,k}\boldsymbol{\beta}$, $\mathbf{H}\boldsymbol{\beta} + \mathbf{h} = \mathbf{0}$, is

$$\mathcal{C}_G = \left\{\mathbf{u} : \mathbf{u} \in R^k, (\mathbf{u} - \widehat{\widehat{\mathbf{G}\boldsymbol{\beta}}})'[\mathrm{Var}(\widehat{\widehat{\mathbf{G}\boldsymbol{\beta}}})]^-(\mathbf{u} - \widehat{\widehat{\mathbf{G}\boldsymbol{\beta}}}) \le \chi_f^2(0; 1-\alpha)\right\},$$

where

$$\widehat{\widehat{\mathbf{G}\boldsymbol{\beta}}} = \mathbf{G}\Big(\big[(\mathbf{M}_{H'}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{Y} - \Big\{\mathbf{I} - \big[(\mathbf{M}_{H'}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{X}\Big\}\mathbf{H}'(\mathbf{H}\mathbf{H}')^+\mathbf{h},$$

$$f = r[\mathrm{Var}(\widehat{\widehat{\mathbf{G}\boldsymbol{\beta}}})],$$

$$\mathrm{Var}(\widehat{\widehat{\mathbf{G}\boldsymbol{\beta}}})] = \mathbf{V}_G = \mathbf{G}\Big(\big[(\mathbf{M}_{H'}\mathbf{X}')^-_{m(\Sigma)})\big]'\mathbf{\Sigma}(\mathbf{M}_{H'}\mathbf{X}')^-_{m(\Sigma)})\mathbf{G}',$$

$$\mathbf{\Sigma} = \sum_{i=1}^p \vartheta_i \mathbf{V}_i.$$

Lemma 3.5 *Let*

$$T(\boldsymbol{\vartheta}) = (\widehat{\mathbf{G}\boldsymbol{\beta}} - \mathbf{G}\boldsymbol{\beta})'[\mathrm{Var}(\widehat{\mathbf{G}\boldsymbol{\beta}})]^{-}(\widehat{\mathbf{G}\boldsymbol{\beta}} - \mathbf{G}\boldsymbol{\beta}).$$

Then

$$\begin{aligned}
\frac{\partial T(\boldsymbol{\vartheta})}{\partial \vartheta_i} &= -2\mathbf{v}'(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{H'}\mathbf{X}')^{+}\mathbf{V}_i\mathbf{T}_G'\mathbf{V}_G^{+}(\widehat{\mathbf{G}\boldsymbol{\beta}} - \mathbf{G}\boldsymbol{\beta}) \\
&\quad -(\widehat{\mathbf{G}\boldsymbol{\beta}} - \mathbf{G}\boldsymbol{\beta})'\mathbf{V}_G^{+}\mathbf{T}_G\mathbf{V}_i\mathbf{T}_G'\mathbf{V}_G^{+}(\widehat{\mathbf{G}\boldsymbol{\beta}} - \mathbf{G}\boldsymbol{\beta}), \\
\mathbf{V}_G &= \mathrm{Var}(\widehat{\mathbf{G}\boldsymbol{\beta}}) = \mathbf{G}\Big\{\big[\mathbf{M}_{H'}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{H'}\mathbf{X}')^{+}\mathbf{X}\mathbf{M}_{H'}\big]^{+} - \mathbf{M}_{H'}\Big\}\mathbf{G}', \\
\mathbf{T}_G &= \mathbf{G}\big[\mathbf{M}_{H'}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{H'}\mathbf{X}')^{+}\mathbf{X}\mathbf{M}_{H'}\big]^{+}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{H'}\mathbf{X}')^{+}, \\
\mathbf{v} &= \mathbf{Y} - \widehat{\mathbf{X}\boldsymbol{\beta}}, \\
E\left(\frac{\partial T(\boldsymbol{\vartheta})}{\partial \vartheta_i}\right) &= -\,\mathrm{Tr}(\mathbf{V}_i\mathbf{T}_G'\mathbf{V}_G^{+}\mathbf{T}_G) = -a_i.
\end{aligned}$$

Further

$$\begin{aligned}
\mathrm{cov}\left(\frac{\partial T(\boldsymbol{\vartheta})}{\partial \vartheta_i}, \frac{\partial T(\boldsymbol{\vartheta})}{\partial \vartheta_j}\right) &= 4\,\mathrm{Tr}\left[\mathbf{V}_i\mathbf{T}_G'\mathbf{V}_G^{+}\mathbf{T}_G\mathbf{V}_j\big(\mathbf{M}_{XM_{H'}}\boldsymbol{\Sigma}\mathbf{M}_{XM_{H'}}\big)^{+}\right] \\
&\quad + 2\,\mathrm{Tr}(\mathbf{V}_i\mathbf{T}_G'\mathbf{V}_G^{+}\mathbf{T}_G\mathbf{V}_j\mathbf{T}_G'\mathbf{V}_G^{+}\mathbf{T}_G) = \{\mathbf{A}\}_{i,j}.
\end{aligned}$$

Theorem 3.6 *Let* $\mathbf{a} = (a_1, \ldots, a_p)'$ *be the vector given by the preceding lemma and* $\mathbf{A}$ *be the matrix with the* (i,j) *entry equal to* $\{\mathbf{A}\}_{i,j}$ *given also by the preceding lemma. Then the nonsensitivity region for the confidence region* $\mathcal{C}_G$ *is*

$$\begin{aligned}
\mathcal{N}_G &= \Bigg\{\delta\boldsymbol{\vartheta} : \big[\delta\boldsymbol{\vartheta} - \delta_{\max}(t^2\mathbf{A} - \mathbf{a}\mathbf{a}')^{+}\mathbf{a}\big]'(t^2\mathbf{A} - \mathbf{a}\mathbf{a}') \\
&\qquad \big[\delta\boldsymbol{\vartheta} - \delta_{\max}(t^2\mathbf{A} - \mathbf{a}\mathbf{a}')^{+}\mathbf{a}\big] \le \delta_{\max}^2\frac{\mathbf{a}'\mathbf{A}^{+}\mathbf{a}}{t^2 - \mathbf{a}'\mathbf{A}^{+}\mathbf{a}}\Bigg\}, \\
\delta_{\max} &= \chi_f^2(0; 1-\alpha) - \chi_f^2(0; 1-\alpha-\varepsilon)
\end{aligned}$$

and $t > 0$ *is sufficiently large real number. It is valid that*

$$\delta\boldsymbol{\vartheta} \in \mathcal{N}_G \Rightarrow P\{\mathbf{G}\boldsymbol{\beta} \in \mathcal{C}_B\} \ge 1 - \alpha - \varepsilon.$$

Proof Let t be sufficiently large, such that

$$\begin{aligned}
\sum_{i=1}^{p}\frac{\partial T(\boldsymbol{\vartheta})}{\partial \vartheta_i}\delta\vartheta_i &< E\left(\sum_{i=1}^{p}\frac{\partial T(\boldsymbol{\vartheta})}{\partial \vartheta_i}\delta\vartheta_i\right) + t\sqrt{Var\left(\sum_{i=1}^{p}\frac{\partial T(\boldsymbol{\vartheta})}{\partial \vartheta_i}\delta\vartheta_i\right)} \\
&= -\mathbf{a}'\delta\boldsymbol{\vartheta} + t\sqrt{\delta\boldsymbol{\vartheta}'\mathbf{A}\delta\boldsymbol{\vartheta}}.
\end{aligned}$$

If

$$-\mathbf{a}'\delta\boldsymbol{\vartheta} + t\sqrt{\delta\boldsymbol{\vartheta}'\mathbf{A}\delta\boldsymbol{\vartheta}} \le \delta_{\max},$$

where $\delta_{\max} = \chi^2_f(0; 1-\alpha) - \chi^2_f(0; 1-\alpha-\varepsilon)$, then

$$P\{T(\boldsymbol{\vartheta} + \delta\boldsymbol{\vartheta}) \le \chi^2_f(0; 1-\alpha)\} \ge 1-\alpha-\varepsilon.$$

Thus

$$t^2\delta\boldsymbol{\vartheta}'\mathbf{A}\delta\boldsymbol{\vartheta} \le (\delta_{\max} + \mathbf{a}'\delta\boldsymbol{\vartheta})^2 \Leftrightarrow t^2\delta\boldsymbol{\vartheta}'\mathbf{A}\delta\boldsymbol{\vartheta} - \delta\boldsymbol{\vartheta}'\mathbf{a}\mathbf{a}'\delta\boldsymbol{\vartheta} - 2\delta_{\max}\mathbf{a}'\delta\boldsymbol{\vartheta} \le \delta^2_{\max}.$$

If $\mathbf{a} \in \mathcal{M}(t^2\mathbf{A} - \mathbf{a}\mathbf{a}')$, then the last inequality can be rearranged as

$$\begin{aligned}\left[\delta\boldsymbol{\vartheta} - \delta_{\max}(t^2\mathbf{A} - \mathbf{a}\mathbf{a}')^+\mathbf{a}\right]'(t^2\mathbf{A} - \mathbf{a}\mathbf{a}')\left[\delta\boldsymbol{\vartheta} - \delta_{\max}(t^2\mathbf{A} - \mathbf{a}\mathbf{a}')^+\mathbf{a}\right]\\ \le \delta^2_{\max}\frac{\mathbf{a}'\mathbf{A}^+\mathbf{a}}{t^2 - \mathbf{a}'\mathbf{A}^+\mathbf{a}}.\end{aligned}$$

Here the equality

$$\mathbf{a}'(t^2\mathbf{A} - \mathbf{a}\mathbf{a}')^+\mathbf{a} = \frac{\mathbf{a}'\mathbf{A}^+\mathbf{a}}{t^2 - \mathbf{a}'\mathbf{A}^+\mathbf{a}}$$

is used. It is valid that $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$, where

$$\begin{aligned}\{\mathbf{A}_1\}_{i,j} &= 4\,\mathrm{Tr}\left[\mathbf{V}_i\mathbf{T}'_G\mathbf{V}^+_G\mathbf{T}_G\mathbf{V}_j\left(\mathbf{M}_{XM_{H'}}\boldsymbol{\Sigma}\mathbf{M}_{XM_{H'}}\right)^+\right],\\ \{\mathbf{A}_2\}_{i,j} &= 2\,\mathrm{Tr}(\mathbf{V}_i\mathbf{T}'_G\mathbf{V}^+_G\mathbf{T}_G\mathbf{V}_j\mathbf{T}'_G\mathbf{V}^+_G\mathbf{T}_G) = \{\mathbf{A}\}_{i,j}.\end{aligned}$$

Both matrices are p.s.d., i.e. $\mathcal{M}(\mathbf{A}_1 + \mathbf{A}_2) = \mathcal{M}(\mathbf{A}_1, \mathbf{A}_2)$. Since $\mathbf{a} \in \mathcal{M}(\mathbf{A}_2)$ because of the relationship

$$\begin{pmatrix}\mathrm{Tr}(\mathbf{U}\mathbf{V}_1)\\ \mathrm{Tr}(\mathbf{U}\mathbf{V}_2)\\ \vdots\\ \mathrm{Tr}(\mathbf{U}\mathbf{V}_p)\end{pmatrix} \in \mathcal{M}\begin{pmatrix}\mathrm{Tr}(\mathbf{U}\mathbf{V}_1\mathbf{U}\mathbf{V}_1), & \mathrm{Tr}(\mathbf{U}\mathbf{V}_1\mathbf{U}\mathbf{V}_2, & \dots, & \mathrm{Tr}(\mathbf{U}\mathbf{V}_1\mathbf{U}\mathbf{V}_p)\\ \mathrm{Tr}(\mathbf{U}\mathbf{V}_2\mathbf{U}\mathbf{V}_1), & \mathrm{Tr}(\mathbf{U}\mathbf{V}_2\mathbf{U}\mathbf{V}_2), & \dots, & \mathrm{Tr}(\mathbf{U}\mathbf{V}_2\mathbf{U}\mathbf{V}_p)\\ \dots & \dots & \dots & \dots\\ \mathrm{Tr}(\mathbf{U}\mathbf{V}_p\mathbf{U}\mathbf{V}_1), & \mathrm{Tr}(\mathbf{U}\mathbf{V}_p\mathbf{U}\mathbf{V}_2), & \dots, & \mathrm{Tr}(\mathbf{U}\mathbf{V}_p\mathbf{U}\mathbf{V}_p)\end{pmatrix},$$

where $\mathbf{U} = \mathbf{T}'_G\mathbf{V}^+_G\mathbf{T}_G$, what can be easily proved, and the number t can be suitably chosen, the assumption $\mathbf{a} \in \mathcal{M}(t^2\mathbf{A} - \mathbf{a}\mathbf{a}')$ can be ensured. □

A construction of the nonsensitivity region for different multivariate linear statistical models with constraints can be derived from the last theorem. It is sufficient to find the vector **a** and the matrix **A** for different situations.

Theorem 3.7 *Let in the model (1) with* $\mathbf{H}_2 = \mathbf{I}$ *the matrix* $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ *be unbiasedly estimable, i.e.* $\mathcal{M}(\mathbf{G}_2 \otimes \mathbf{G}'_1) \subset \mathcal{M}[\mathbf{I} \otimes (\mathbf{X}', \mathbf{H}'_1)]$. *Then*

$$\begin{aligned}\mathbf{a} &= (a_1, \dots, a_p)',\\ a_i &= r[\mathbf{G}_1(\mathbf{M}_{H'_1}\mathbf{X}'\mathbf{X})^+\mathbf{G}'_1]\,\mathrm{Tr}[\mathbf{V}_i\mathbf{G}_2(\mathbf{G}'_2\boldsymbol{\Sigma}\mathbf{G}_2)^+\mathbf{G}'_2], \quad i = 1, \dots, p,\\ \mathbf{A} &= r[\mathbf{G}_1(\mathbf{M}_{H'_1}\mathbf{X}'\mathbf{X})^+\mathbf{G}'_1]\mathbf{S}_{G_2(G'_2\Sigma G_2)^+G'_2},\end{aligned}$$

where

$$\left\{\mathbf{S}_{G_2(G'_2\Sigma G_2)^+G'_2}\right\}_{i,j} = \mathrm{Tr}[\mathbf{V}_i\mathbf{G}_2(\mathbf{G}'_2\boldsymbol{\Sigma}\mathbf{G}_2)^+\mathbf{G}'_2\mathbf{V}_j\mathbf{G}_2(\mathbf{G}'_2\boldsymbol{\Sigma}\mathbf{G}_2)^+\mathbf{G}'_2].$$

Proof With respect to Lemma 3.5 the following scheme

$$\mathbf{V}_i \to \mathbf{V}_i \otimes \mathbf{I}, \quad \mathbf{T}_G \to \mathbf{T}_{G_2' \otimes G_1}, \quad \mathbf{V}_G \to \mathbf{V}_{G_2' \otimes G_1},$$

will be used. Here

$$\begin{aligned}
\mathbf{T}_{G_2' \otimes G_1} &= \mathbf{T}'_{G_2} \otimes \mathbf{T}_{G_1} = \mathbf{G}_2' \otimes \mathbf{G}_1[(\mathbf{M}_{H_1'}\mathbf{X}')^-_{m(I)}]',\\
\mathbf{T}_{G_2} &= \mathbf{G}_2, \quad \mathbf{T}_{G_1} = \mathbf{G}_1[(\mathbf{M}_{H_1'}\mathbf{X}')^-_{m(I)}]',\\
\mathbf{V}_{G_2' \otimes G_1} &= \mathbf{V}_{G_2} \otimes \mathbf{V}_{G_1},\\
\mathbf{V}_{G_1} &= \mathbf{G}_1(\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{X}\mathbf{M}_{H_1'})^+\mathbf{G}_1',\\
\mathbf{V}_{G_2} &= \mathbf{G}_2'\mathbf{\Sigma}\mathbf{G}_2.
\end{aligned}$$

Now we use the formulae from Lemma 3.5 and thus we obtain

$$\begin{aligned}
a_i &= \mathrm{Tr}[(\mathbf{V}_1 \otimes \mathbf{I})(\mathbf{T}_{G_2} \otimes \mathbf{T}'_{G_1})(\mathbf{V}^+_{G_2} \otimes \mathbf{V}^+_{G_1})(\mathbf{T}'_{G_2} \otimes \mathbf{T}_{G_1})]\\
&= r[\mathbf{G}_1(\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{X})^+\mathbf{M}_{H_1'})^+\mathbf{G}_1'] \,\mathrm{Tr}[\mathbf{V}_i\mathbf{G}_2(\mathbf{G}_2'\mathbf{\Sigma}\mathbf{G}_2)^+\mathbf{G}_2],
\end{aligned}$$

since

$$\begin{aligned}
&\mathrm{Tr}(\mathbf{T}'_{G_1}\mathbf{V}^+_{G_1}\mathbf{T}_{G_1}) =\\
&= \mathrm{Tr}\left\{(\mathbf{M}_{H_1'}\mathbf{X}')^-_{m(I)}\mathbf{G}_1'[\mathbf{G}_1(\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{X})^+\mathbf{G}_1']^+\mathbf{G}_1[(\mathbf{M}_{H_1'}\mathbf{X}')^-_{m(I)}]'\right\}\\
&= \mathrm{Tr}\left\{\mathbf{G}_1[(\mathbf{M}_{H_1'}\mathbf{X}')^-_{m(I)}]'(\mathbf{M}_{H_1'}\mathbf{X}')^-_{m(I)}\mathbf{G}_1'[\mathbf{G}_1(\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{X})^+\mathbf{G}_1']^+\right\}\\
&= \mathrm{Tr}\left\{[\mathbf{G}_1(\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{X})^+\mathbf{G}_1'][\mathbf{G}_1(\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{X})^+\mathbf{G}_1']^+\right\}\\
&= r[\mathbf{G}_1(\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{X})^+\mathbf{G}_1'].
\end{aligned}$$

As far as the matrix $\mathbf{A}$ be concerned, it is valid that

$$\begin{aligned}
\{\mathbf{A}\}_{i,j} &= 4\,\mathrm{Tr}\Big\{(\mathbf{V}_i \otimes \mathbf{I})(\mathbf{T}_{G_2} \otimes \mathbf{T}'_{G_1})(\mathbf{V}^+_{G_2} \otimes \mathbf{V}^+_{G_1})(\mathbf{T}'_{G_2} \otimes \mathbf{T}_{G_1})(\mathbf{V}_j \otimes \mathbf{I})\\
&\quad \times [\mathbf{M}_{I\otimes(XM_{H_1'})}(\mathbf{\Sigma} \otimes \mathbf{I})\mathbf{M}_{I\otimes(XM_{H_1'})}]^+\Big\}\\
&\quad + 2\,\mathrm{Tr}\Big\{(\mathbf{V}_i \otimes \mathbf{I})(\mathbf{T}_{G_2} \otimes \mathbf{T}'_{G_1})(\mathbf{V}^+_{G_2} \otimes \mathbf{V}^+_{G_1})(\mathbf{T}'_{G_2} \otimes \mathbf{T}_{G_1})(\mathbf{V}_j \otimes \mathbf{I})\\
&\quad \times (\mathbf{T}_{G_2} \otimes \mathbf{T}'_{G_1})(\mathbf{V}^+_{G_2} \otimes \mathbf{V}^+_{G_1})(\mathbf{T}'_{G_2} \otimes \mathbf{T}_{G_1})(\mathbf{V}_j \otimes \mathbf{I})\Big\}\\
&= r[\mathbf{G}_1(\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{X})^+\mathbf{G}_1']\\
&\quad \times \mathrm{Tr}\left\{\mathbf{V}_i\mathbf{G}_2(\mathbf{G}_2'\mathbf{\Sigma}\mathbf{G}_2)^+\mathbf{G}_2'\mathbf{V}_j\mathbf{G}_2(\mathbf{G}_2'\mathbf{\Sigma}\mathbf{G}_2)^+\mathbf{G}_2'\right\},
\end{aligned}$$

since

$$\begin{aligned}
[\mathbf{M}_{I\otimes(XM_{H_1'})}(\mathbf{\Sigma} \otimes \mathbf{I})\mathbf{M}_{I\otimes(XM_{H_1'})}]^+ &= [(\mathbf{I} \otimes \mathbf{M}_{XM_{H_1'}})(\mathbf{\Sigma} \otimes \mathbf{I})(\mathbf{I} \otimes \mathbf{M}_{XM_{H_1'}})]^+\\
&= \mathbf{\Sigma}^+ \otimes \mathbf{M}_{XM_{H_1'}}.
\end{aligned}$$

Now it is easy to finish the proof. □

Theorem 3.8 *Let in the model (1) with* $\mathbf{H}_1 = \mathbf{I}$ *the matrix* $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ *be unbiasedly estimable, i.e.* $\mathcal{M}(\mathbf{G}_2 \otimes \mathbf{G}_1') \subset \mathcal{M}(\mathbf{I} \otimes \mathbf{X}', \mathbf{H}_2 \otimes \mathbf{I})$. *Then*

$$\begin{aligned}
\mathbf{a} &= (a_1, \dots, a_p)', \\
a_i &= r[\mathbf{G}_1(\mathbf{X}'\mathbf{X})^+\mathbf{G}_1']\,\mathrm{Tr}(\mathbf{V}_i\mathbf{T}_{G_2}\mathbf{V}_{G_2}^+\mathbf{T}_{G_2}'), \\
\mathbf{A} &= 4r[\mathbf{G}_1(\mathbf{X}'\mathbf{X})^+\mathbf{G}_1']\mathbf{C}_{T_{G_2}V_{G_2}^+T_{G_2}',(P_{H_2}\Sigma P_{H_2})^+} + 2r[\mathbf{G}_1(\mathbf{X}'\mathbf{X})^+\mathbf{G}_1']\mathbf{S}_{T_{G_2}V_{G_2}^+T_{G_2}'},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{T}_{G_2'\otimes G_1} &= \mathbf{T}_{G_2}' \otimes \mathbf{T}_{G_1}, \quad \mathbf{T}_{G_1} = \mathbf{G}_1(\mathbf{X}'\mathbf{X})^+\mathbf{X}', \quad \mathbf{T}_{G_2} = (\mathbf{M}_{H_2})_{m(\Sigma)}^-\mathbf{G}_2, \\
\mathbf{V}_{G_2'\otimes G_1} &= \mathbf{V}_{G_2} \otimes \mathbf{V}_{G_1}, \\
\mathbf{V}_{G_1} &= \mathbf{G}_1(\mathbf{X}'\mathbf{X})^+\mathbf{G}_1', \quad \mathbf{V}_{G_2} = \mathbf{G}_2'\Big\{[\mathbf{M}_{H_2}(\boldsymbol{\Sigma} + \mathbf{M}_{H_2})^+\mathbf{M}_{H_2}]^+ - \mathbf{M}_{H_2}\Big\}
\end{aligned}$$

and

$$\begin{aligned}
\Big\{\mathbf{C}_{T_{G_2}V_{G_2}^+T_{G_2}',(P_{H_2}\Sigma P_{H_2})^+}\Big\}_{i,j} &= \mathrm{Tr}[\mathbf{V}_i\mathbf{T}_{G_2}\mathbf{V}_{G_2}^+\mathbf{T}_{G_2}'\mathbf{V}_j(\mathbf{P}_{H_2}\boldsymbol{\Sigma}\mathbf{P}_{H_2})^+], \\
\Big\{\mathbf{S}_{T_{G_2}V_{G_2}^+T_{G_2}'}\Big\}_{i,j} &= \mathrm{Tr}[\mathbf{V}_i\mathbf{T}_{G_2}\mathbf{V}_{G_2}^+\mathbf{T}_{G_2}'\mathbf{V}_j\mathbf{T}_{G_2}\mathbf{V}_{G_2}^+\mathbf{T}_{G_2}'].
\end{aligned}$$

Proof It is analogous as in preceding theorem. The formulae from Lemma 3.5 must be used. The equality

$$\big[\mathbf{M}_{M_{H_2}\otimes X}(\boldsymbol{\Sigma} \otimes \mathbf{I})\mathbf{M}_{M_{H_2}\otimes X}\big] = \boldsymbol{\Sigma}^+ \otimes \mathbf{M}_X + (\mathbf{P}_{H_2}\boldsymbol{\Sigma}\mathbf{P}_{H_2})^+ \otimes \mathbf{P}_X$$

must be taken into account. □

Theorem 3.9 *Let in the model (2) with* $\mathbf{H}_2 = \mathbf{I}$ *the matrix* $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ *be unbiasedly estimable, i.e.* $\mathcal{M}(\mathbf{G}_2' \otimes \mathbf{G}_1) \subset \mathcal{M}[\mathbf{I} \otimes (\mathbf{X}', \mathbf{H}_1')]$. *Then*

$$\begin{aligned}
\mathbf{a} &= (a_1, \dots, a_p)', \\
a_i &= r(\mathbf{G}_2)\,\mathrm{Tr}(\mathbf{V}_i\mathbf{T}_{G_1}'\mathbf{V}_{G_1}^+\mathbf{T}_{G_1}), \quad i = 1, \dots, p, \\
\mathbf{A} &= 4r(\mathbf{G}_2)\mathbf{C}_{T_{G_1}'V_{G_1}^+T_{G_1},(M_{XM_{H_1'}}\Sigma M_{XM_{H_1'}})^+} + 2r(\mathbf{G}_2)\mathbf{S}_{T_{G_1}'V_{G_1}^+T_{G_1}},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{T}_{G_2'\otimes G_1} &= \mathbf{T}_{G_2}' \otimes \mathbf{T}_{G_1}, \\
\mathbf{T}_{G_1} &= \mathbf{G}_1[\mathbf{M}_{H_1'}\mathbf{X}')_{m(\Sigma)}^-]' \\
&= \mathbf{G}_1[\mathbf{M}_{H_1'}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{H_1'}\mathbf{X}')^+\mathbf{X}\mathbf{M}_{H_1'}]^+\mathbf{M}_{H_1'}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{H_1'}\mathbf{X}')^+, \\
\mathbf{T}_{G_2} &= \mathbf{G}_2, \\
\mathbf{V}_{G_1} &= \mathbf{G}_1\big\{[\mathbf{M}_{H_1'}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{H_1'}\mathbf{X}')^+\mathbf{X}\mathbf{M}_{H_1'}]^+ - \mathbf{M}_{H_1'}\big\}\mathbf{G}_1', \\
\mathbf{V}_{G_2} &= \mathbf{G}_2'\mathbf{G}_2
\end{aligned}$$

and

$$\left\{\mathbf{C}_{T'_{G_1}V^+_{G_1}T_{G_1},(M_{XM_{H'_1}}\Sigma M_{XM_{H'_1}})^+}\right\}_{i,j} =$$
$$= \mathrm{Tr}[\mathbf{V}_i\mathbf{T}'_{G_1}\mathbf{V}^+_{G_1}\mathbf{T}_{G_1}\mathbf{V}_j(\mathbf{M}_{XM_{H'_1}}\mathbf{\Sigma}\mathbf{M}_{XM_{H'_1}})^+],$$
$$\left\{\mathbf{S}_{T'_{G_1}V^+_{G_1}T_{G_1}}\right\}_{i,j} = \mathrm{Tr}(\mathbf{V}_i\mathbf{T}'_{G_1}\mathbf{V}^+_{G_1}\mathbf{T}_{G_1}\mathbf{V}_j\mathbf{T}'_{G_1}\mathbf{V}^+_{G_1}\mathbf{T}_{G_1}).$$

Proof The obvious equality

$$\left[\mathbf{M}_{I\otimes(XM_{H'_1})}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{M}_{I\otimes(XM_{H'_1})}\right] = \mathbf{I}\otimes\left(\mathbf{M}_{XM_{H'_1}}\mathbf{\Sigma}\mathbf{M}_{XM_{H'_1}}\right)^+$$

and Lemma 3.5 must be used. □

Theorem 3.10 *Let in the model (2) with* $\mathbf{H}_1 = \mathbf{I}$ *the matrix* $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ *be unbiasedly estimable, i.e.* $\mathcal{M}(\mathbf{G}_2\otimes\mathbf{G}'_1)\subset\mathcal{M}(\mathbf{I}\otimes\mathbf{X}',\mathbf{H}_2\otimes\mathbf{I})$*. Then*

$$\begin{aligned}
\mathbf{a} &= (a_1,\dots,a_p)',\\
a_i &= r(\mathbf{M}_{H_2}\mathbf{G}_2)\,\mathrm{Tr}(\mathbf{V}_i\mathbf{T}'_{G_1}\mathbf{V}^+_{G_1}\mathbf{T}_{G_1}),\quad i=1,\dots,p,\\
\mathbf{A} &= 4r(\mathbf{M}_{H_2}\mathbf{G}_2)\mathbf{C}_{T'_{G_1}V^+_{G_1}T_{G_1},(M_X\Sigma M_X)^+} + 2r(\mathbf{M}_{H_2}\mathbf{G}_2)\mathbf{S}_{T'_{G_1}V^+_{G_1}T_{G_1}},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{T}_{G'_2\otimes G_1} &= \mathbf{T}'_{G_2}\otimes\mathbf{T}_{G_1},\quad \mathbf{T}_{G_1} = \mathbf{G}_1\left[(\mathbf{X}')^-_{M(\Sigma)}\right]' = \mathbf{G}_1\mathbf{X}'\mathbf{T}^+\mathbf{X})^+\mathbf{X}'\mathbf{T}^+,\\
\mathbf{T} &= \mathbf{\Sigma}+\mathbf{X}\mathbf{X},\quad \mathbf{T}_{G_2} = \mathbf{M}_{H_2}\mathbf{G}_2,\\
\mathbf{V}_{G_1} &= \mathbf{G}_1[\mathbf{X}'\mathbf{T}^+\mathbf{X})^+ - \mathbf{I}]\mathbf{G}'_1,\quad \mathbf{V}_{G_2} = \mathbf{G}'_2\mathbf{M}_{H_2}\mathbf{G}_2
\end{aligned}$$

and

$$\begin{aligned}
\left\{\mathbf{C}_{T'_{G_1}V^+_{G_1}T_{G_1},(M_X\Sigma M_X)^+}\right\}_{i,j} &= \mathrm{Tr}[\mathbf{V}_i\mathbf{T}'_{G_1}\mathbf{V}^+_{G_1}\mathbf{T}_{G_1}\mathbf{V}_j(\mathbf{M}_X\mathbf{\Sigma}\mathbf{M}_X)^+],\\
\left\{\mathbf{S}_{T'_{G_1}V^+_{G_1}T_{G_1}}\right\}_{i,j} &= \mathrm{Tr}(\mathbf{V}_i\mathbf{T}'_{G_1}\mathbf{V}^+_{G_1}\mathbf{T}_{G_1}).
\end{aligned}$$

Proof The equalities

$$\left[\mathbf{M}_{M_{H_2}\otimes X}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{M}_{M_{H_2}\otimes X}\right] = \mathbf{M}_{H_2}\otimes(\mathbf{M}_X\mathbf{\Sigma}\mathbf{M}_X)^+ + \mathbf{P}_{H_2}\otimes\mathbf{\Sigma}^+,$$
$$\mathbf{T}'_{G_2}\mathbf{P}_{H_2} = \mathbf{0}$$

and Lemma 3.5 must be taken into account. □

Theorem 3.11 *Let in the model (3) with* $\mathbf{H}_2 = \mathbf{I}$ *the matrix* $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ *be unbiasedly estimable, i.e.* $\mathcal{M}(\mathbf{G}_2\otimes\mathbf{G}'_1)\subset\mathcal{M}(\mathbf{Z}\otimes\mathbf{X}',\mathbf{I}\otimes\mathbf{H}'_1)$*. Then*

$$\begin{aligned}
\mathbf{a} &= (a_1,\dots,a_p)',\\
a_i &= r[\mathbf{V}^+_{G_1}\mathbf{G}_1(\mathbf{M}_{H'_1}\mathbf{X}'\mathbf{X}\mathbf{M}_{H'_1})^+\mathbf{G}'_1]\,\mathrm{Tr}(\mathbf{V}_i\mathbf{T}_{G_2}\mathbf{V}^+_{G_2}\mathbf{T}'_{G_2}),\quad i=1,\dots,p,\\
\mathbf{A} &= 4r[\mathbf{V}^+_{G_1}\mathbf{G}_1(\mathbf{M}_{H'_1}\mathbf{X}'\mathbf{X}\mathbf{M}_{H'_1})^+\mathbf{G}'_1]\mathbf{C}_{T_{G_2}V^+_{G_2}T'_{G_2},(M_{Z'}\Sigma M_{Z'})^+}\\
&\quad + 2r[\mathbf{V}^+_{G_1}\mathbf{G}_1(\mathbf{M}_{H'_1}\mathbf{X}'\mathbf{X}\mathbf{M}_{H'_1})^+\mathbf{G}'_1]\mathbf{S}_{T_{G_2}V^+_{G_2}T'_{G_2}},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{T}_{G_2'\otimes G_1} &= \mathbf{T}_{G_2}' \otimes \mathbf{T}_{G_1}, \quad \mathbf{T}_{G_1} = \mathbf{G}_1 \left(\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{X}\mathbf{M}_{H_1'}\right)^+ \mathbf{M}_{H_1'}\mathbf{X}',\\
\mathbf{T}_{G_2} &= = \mathbf{Z}^-_{m(\Sigma)}\mathbf{G}_2 = \mathbf{U}^+\mathbf{Z}'(\mathbf{Z}\mathbf{U}^+\mathbf{Z}')^+, \quad \mathbf{U} = \mathbf{\Sigma} + \mathbf{Z}'\mathbf{Z},\\
\mathbf{V}_{G_1} &= \mathbf{G}_1(\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{X}\mathbf{M}_{H_1'})^+\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{\Sigma}\mathbf{X}\mathbf{M}_{H_1'}(\mathbf{M}_{H_1'}\mathbf{X}'\mathbf{X}\mathbf{M}_{H_1'})^+\mathbf{G}_1',\\
\mathbf{V}_{G_2} &= \mathbf{G}_2'[(\mathbf{Z}\mathbf{U}^+\mathbf{Z}')^+ - \mathbf{I}]\mathbf{G}_2
\end{aligned}$$

and

$$\begin{aligned}
\left\{\mathbf{C}_{T_{G_2}V^+_{G_2}T'_{G_2},(M_{Z'}\Sigma M_{Z'})^+}\right\}_{i,j} &= \mathrm{Tr}[\mathbf{V}_i\mathbf{T}_{G_2}\mathbf{V}^+_{G_2}\mathbf{T}'_{G_2}\mathbf{V}_j(\mathbf{M}_{Z'}\mathbf{\Sigma}\mathbf{M}_{Z'})^+],\\
\left\{\mathbf{S}_{T_{G_2}V^+_{G_2}T'_{G_2}}\right\}_{i,j} &= \mathrm{Tr}(\mathbf{V}_i\mathbf{T}_{G_2}\mathbf{V}^+_{G_2}\mathbf{T}'_{G_2}\mathbf{V}_j\mathbf{T}_{G_2}\mathbf{V}^+_{G_2}\mathbf{T}'_{G_2}).
\end{aligned}$$

Proof Lemma 3.5 and the equalities

$$\left[\mathbf{M}_{Z'\otimes(XM_{H_1'})}(\mathbf{\Sigma}\otimes\mathbf{I})\mathbf{M}_{Z'\otimes(XM_{H_1'})}\right] = \mathbf{\Sigma}^+\otimes\mathbf{M}_{XM_{H_1'}} + (\mathbf{M}_{Z'}\mathbf{\Sigma}\mathbf{M}_{Z'})^+\otimes\mathbf{P}_{XM_{H_1'}},$$

$$\mathbf{T}_{G_1}\mathbf{M}_{XM_{H_1'}} = \mathbf{0}$$

must be used. □

Theorem 3.12 *Let in the model (3) with* $\mathbf{H}_1 = \mathbf{I}$ *the matrix* $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ *be unbiasedly estimable, i.e.* $\mathcal{M}(\mathbf{G}_2\otimes\mathbf{G}_1') \subset \mathcal{M}(\mathbf{Z}\otimes\mathbf{X}', \mathbf{H}_2\otimes\mathbf{I})$. *Then*

$$\begin{aligned}
\mathbf{a} &= (a_1,\dots,a_p)',\\
a_i &= r[\mathbf{G}_1(\mathbf{X}'\mathbf{X})^+\mathbf{G}_1']\,\mathrm{Tr}(\mathbf{V}_i\mathbf{T}_{G_2}\mathbf{V}^+_{G_2}\mathbf{T}'_{G_2}), \quad i = 1,\dots,p,\\
\mathbf{A} &= 4r[\mathbf{G}_1(\mathbf{X}'\mathbf{X})^+\mathbf{G}_1']\mathbf{C}_{T_{G_2}V^+_{G_2}T'_{G_2},(M_{Z'M_{H_2}}\Sigma M_{Z'M_{H_2}})^+}\\
&\quad + 2r[\mathbf{G}_1(\mathbf{X}'\mathbf{X})^+\mathbf{G}_1']\mathbf{S}_{T_{G_2}V^+_{G_2}T'_{G_2}},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{T}_{G_2'\otimes G_1} &= \mathbf{T}'_{G_2}\otimes\mathbf{T}_{G_1}, \quad \mathbf{T}_{G_1} = \mathbf{G}_1(\mathbf{X}'\mathbf{X})^+\mathbf{X}',\\
\mathbf{T}_{G_2} &= (\mathbf{M}_{H_2}\mathbf{Z})^-_{m(\Sigma)}\mathbf{G}_2\\
&= (\mathbf{\Sigma} + \mathbf{Z}'\mathbf{M}_{H_2}\mathbf{Z}')^+\mathbf{Z}'\mathbf{M}_{H_2}[\mathbf{M}_{H_2}\mathbf{Z}(\mathbf{\Sigma}+\mathbf{Z}'\mathbf{M}_{H_2}\mathbf{Z})^+\mathbf{Z}'\mathbf{M}_{H_2}]^+\mathbf{G}_2,\\
\mathbf{V}_{G_1} &= \mathbf{G}_1(\mathbf{X}'\mathbf{X})^+\mathbf{G}_1',\\
\mathbf{V}_{G_2} &= \mathbf{G}_2'\big\{[\mathbf{M}_{H_2}\mathbf{Z}'(\mathbf{\Sigma}+\mathbf{Z}'\mathbf{M}_{H_2}\mathbf{Z})^+\mathbf{Z}'\mathbf{M}_{H_2}]^+ - \mathbf{M}_{H_2}\big\}\mathbf{G}_2
\end{aligned}$$

and

$$\begin{aligned}
\left\{\mathbf{C}_{T_{G_2}V^+_{G_2}T'_{G_2},(M_{Z'M_{H_2}}\Sigma M_{Z'M_{H_2}})^+}\right\}_{i,j} &=\\
&= \mathrm{Tr}[\mathbf{V}_i\mathbf{T}_{G_2}\mathbf{V}^+_{G_2}\mathbf{T}'_{G_2}\mathbf{V}_j(\mathbf{M}_{Z'M_{H_2}}\mathbf{\Sigma}\mathbf{M}_{Z'M_{H_2}})^+],\\
\left\{\mathbf{S}_{T_{G_2}V^+_{G_2}T'_{G_2}}\right\} &= \mathrm{Tr}(\mathbf{V}_i\mathbf{T}_{G_2}\mathbf{V}^+_{G_2}\mathbf{T}'_{G_2}\mathbf{V}_j\mathbf{T}_{G_2}\mathbf{V}^+_{G_2}\mathbf{T}'_{G_2}).
\end{aligned}$$

Proof The equalities

$$\left[\mathbf{M}_{(Z'M_{H_2})\otimes X}(\boldsymbol{\Sigma}\otimes\mathbf{I})\mathbf{M}_{(Z'M_{H_2})\otimes X}\right] = \boldsymbol{\Sigma}^{+}\otimes\mathbf{M}_X + \left(\mathbf{M}_{Z'M_{H_2}}\boldsymbol{\Sigma}\mathbf{M}_{Z'M_{H_2}}\right)^{+}\otimes\mathbf{P}_X,$$

$$\mathbf{T}_{G_1}\mathbf{M}_X = \mathbf{0}$$

and Lemma 3.5 must be taken into account. □

Theorem 3.13 *Let in the model (4) with* $\mathbf{H}_2 = \mathbf{I}$ *the matrix* $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ *be unbiasedly estimable, i.e.* $\mathcal{M}(\mathbf{G}_2\otimes\mathbf{G}_1') \subset \mathcal{M}(\mathbf{Z}\otimes\mathbf{X}', \mathbf{I}\otimes\mathbf{H}_1')$. *Then*

$$\begin{aligned}
\mathbf{a} &= (a_1,\ldots,a_p)',\\
a_i &= r[\mathbf{G}_2'(\mathbf{Z}\mathbf{Z}')^{+}\mathbf{G}_2]\,\mathrm{Tr}(\mathbf{V}_i\mathbf{T}_{G_1}'\mathbf{V}_{G_1}^{+}\mathbf{T}_{G_1}), \quad i=1,\ldots,p,\\
\mathbf{A} &= 4r[\mathbf{G}_2'(\mathbf{Z}\mathbf{Z}')^{+}\mathbf{G}_2]\mathbf{C}_{T_{G_1}'V_{G_1}^{+}T_{G_1}',(M_{XM_{H_1'}}\Sigma M_{XM_{H_1'}})^{+}}\\
&\quad + 2r[\mathbf{G}_2'(\mathbf{Z}\mathbf{Z}')^{+}\mathbf{G}_2]\mathbf{S}_{T_{G_1}'V_{G_1}^{+}T_{G_1}'},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{T}_{G_2'\otimes G_1} &= \mathbf{T}_{G_2}'\otimes\mathbf{T}_{G_1},\\
\mathbf{T}_{G_1} &= \mathbf{G}_1\left[(\mathbf{M}_{H_1'}\mathbf{X}')^{-}_{m(\Sigma)}\right]'\\
&= \mathbf{G}_1[\mathbf{M}_{H_1'}\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{H_1'}\mathbf{X}')^{+}\mathbf{X}\mathbf{M}_{H_1'}^{+}\mathbf{M}_{H_1'}\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{H_1'}\mathbf{X}')^{+},\\
\mathbf{T}_{G_2} &= \mathbf{Z}'(\mathbf{Z}\mathbf{Z})^{+}\mathbf{G}_2,\\
\mathbf{V}_{G_1} &= \mathbf{G}_1\{[\mathbf{M}_{H_1'}\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{X}\mathbf{M}_{H_1'}\mathbf{X}')^{+}\mathbf{X}\mathbf{M}_{H_1'}]^{+}-\mathbf{M}_{H_1'}\}\mathbf{G}_1',\\
\mathbf{V}_{G_2} &= \mathbf{G}_2'(\mathbf{Z}\mathbf{Z}')^{+}\mathbf{G}_2
\end{aligned}$$

and

$$\begin{aligned}
\left\{\mathbf{C}_{T_{G_1}'V_{G_1}^{+}T_{G_1}',(M_{XM_{H_1'}}\Sigma M_{XM_{H_1'}})^{+}}\right\}_{i,j} &=\\
&= \mathrm{Tr}[\mathbf{V}_i\mathbf{T}_{G_1}'\mathbf{V}_{G_1}^{+}\mathbf{T}_{G_1 2}\mathbf{V}_j(\mathbf{M}_{XM_{H_1'}}\boldsymbol{\Sigma}\mathbf{M}_{XM_{H_1'}})^{+}],\\
\left\{\mathbf{S}_{T_{G_1}'V_{G_1}^{+}T_{G_1}}\right\} &= \mathrm{Tr}(\mathbf{V}_i\mathbf{T}_{G_1}'\mathbf{V}_{G_1}^{+}\mathbf{T}_{G_1}\mathbf{V}_j\mathbf{T}_{G_1}'\mathbf{V}_{G_1}^{+}\mathbf{T}_{G_1}).
\end{aligned}$$

Proof The equalities

$$\left[\mathbf{M}_{Z'\otimes(XM_{H_1'}}(\mathbf{I}\otimes\boldsymbol{\Sigma})\mathbf{M}_{Z'\otimes(XM_{H_1'}}\right]^{+} = \mathbf{M}_{Z'}\otimes\boldsymbol{\Sigma}^{+} + \mathbf{P}_{Z'}\otimes(\mathbf{M}_{XM_{H_1'}}\boldsymbol{\Sigma}\mathbf{M}_{XM_{H_1'}})^{+},$$

$$\mathbf{T}_{G_2}'\mathbf{M}_{Z'} = \mathbf{0}$$

and Lemma 3.5 must be utilized. □

Theorem 3.14 *Let in the model (4) with* $\mathbf{H}_1 = \mathbf{I}$ *the matrix* $\mathbf{G}_1\mathbf{B}\mathbf{G}_2$ *be unbiasedly estimable, i.e.* $\mathcal{M}(\mathbf{G}_2 \otimes \mathbf{G}_1') \subset \mathcal{M}(\mathbf{Z} \otimes \mathbf{X}', \mathbf{H}_2 \otimes \mathbf{I})$. *Then*

$$\begin{aligned}
\mathbf{a} &= (a_1, \dots, a_p)', \\
a_i &= r[\mathbf{G}_2'(\mathbf{M}_{H_2}\mathbf{Z}\mathbf{Z}'\mathbf{M}_{H_2})^+\mathbf{G}_2]\,\mathrm{Tr}(\mathbf{V}_i\mathbf{T}_{G_1}'\mathbf{V}_{G_1}^+\mathbf{T}_{G_1}'), \quad i = 1, \dots, p, \\
\mathbf{A} &= 4r[\mathbf{G}_2'(\mathbf{M}_{H_2}\mathbf{Z}\mathbf{Z}'\mathbf{M}_{H_2})^+\mathbf{G}_2]\mathbf{C}_{T_{G_1}'V_{G_1}^+T_{G_1},(M_X\Sigma M_X)^+} \\
&\quad + 2r[\mathbf{G}_2'(\mathbf{M}_{H_2}\mathbf{Z}\mathbf{Z}'\mathbf{M}_{H_2})^+\mathbf{G}_2]\mathbf{S}_{T_{G_1}'V_{G_1}^+T_{G_1}},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{T}_{G_2'\otimes G_1} &= \mathbf{T}_{G_2}' \otimes \mathbf{T}_{G_1}, \\
\mathbf{T}_{G_1} &= \mathbf{G}_1[(\mathbf{X}')^-_{m(\Sigma)}]' = \mathbf{G}_1(\mathbf{X}'\mathbf{T}^+\mathbf{X})^+\mathbf{X}'\mathbf{T}^+, \quad \mathbf{T} = \mathbf{\Sigma} + \mathbf{X}\mathbf{X}', \\
\mathbf{T}_{G_2} &= \mathbf{Z}'\mathbf{M}_{H_2}(\mathbf{M}_{H_2}\mathbf{Z}\mathbf{Z}'\mathbf{M}_{H_2})^+\mathbf{G}_2, \\
\mathbf{V}_{G_1} &= \mathbf{G}_1[(\mathbf{X}'\mathbf{T}^+\mathbf{X})^+ - \mathbf{I}]\mathbf{G}_1', \\
\mathbf{V}_{G_2} &= \mathbf{G}_2'(\mathbf{M}_{H_2}Z\mathbf{Z}\mathbf{Z}'\mathbf{M}_{H_2})^+\mathbf{G}_2
\end{aligned}$$

and

$$\begin{aligned}
\left\{\mathbf{C}_{T_{G_1}'V_{G_1}^+T_{G_1},(M_X\Sigma M_X)^+}\right\}_{i,j} &= \mathrm{Tr}[\mathbf{V}_i\mathbf{T}_{G_1}'\mathbf{V}_{G_1}^+\mathbf{T}_{G_1}\mathbf{V}_j(\mathbf{M}_X\mathbf{\Sigma}\mathbf{M}_X)^+], \\
\left\{\mathbf{S}_{T_{G_1}'V_{G_1}^+T_{G_1}}\right\}_{i,j} &= \mathrm{Tr}(\mathbf{V}_i\mathbf{T}_{G_1}'\mathbf{V}_{G_1}^+\mathbf{T}_{G_1}\mathbf{V}_j\mathbf{T}_{G_1}'\mathbf{V}_{G_1}^+\mathbf{T}_{G_1}.
\end{aligned}$$

Proof The equalities

$$\left[\mathbf{M}_{(Z'M_{H_2})\otimes X}(\mathbf{I}\otimes\mathbf{\Sigma})\mathbf{M}_{(Z'M_{H_2})\otimes X}\right]^+ = \mathbf{M}_{Z'M_{H_2}}\otimes\mathbf{\Sigma}^+ + \mathbf{P}_{Z'M_{H_2}}\otimes(\mathbf{M}_X\mathbf{\Sigma}\mathbf{M}_X)^+,$$

$$\mathbf{T}_{G_2}'\mathbf{M}_{Z'M_{H_2}} = \mathbf{0}$$

and Lemma 3.5 must be used. □

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