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QUASI-MODAL ALGEBRAS

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Abstract. In this paper we introduce the class of Boolean algebras with an operator between the algebra and the set of ideals of the algebra. This is a generalization of the Boolean algebras with operators. We prove that there exists a duality between these algebras and the Boolean spaces with a certain relation. We also give some applications of this duality.

Keywords: Boolean algebras, modal algebras, Boolean spaces with relations

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1. Introduction

A modal algebra is a Boolean algebra A with an operator $\Box\colon A \to A$ such that $\Box(a \wedge b) = \Box a \wedge \Box b$ and $\Box 1 = 1$. This variety of algebras plays a key role in modal logic, and has very important applications in theoretical computer science (see for instance [1] and [2]). Modal algebras are dual objects of descriptive general frames, i.e., Boolean spaces with a relation verifying certain conditions (see [3], [5] and [7]). The aim of this paper is to define and study a notion weaker than modal algebras. We will define Boolean algebras with a map between the algebra and the set of ideals of the algebra. This map is not an operation in the Boolean algebra, but has some similar properties to modal operators. In particular, these structures have a nice duality theory.

In Section 2 we will recall some notions on Boolean duality. Section 3 is devoted to the definition of the quasi-modal algebras, and we will prove that there exists a duality between quasi-modal algebras and Boolean spaces with a relation. These spaces are called quasi-modal spaces. As an application of this duality we shall prove that there exists a bijective correspondence between certain filters of a quasi-modal algebra and certain closed subsets of the associated dual space. In Section 4 we introduce and study the classes of compacts, R-compacts, and semi-replete quasi-modal spaces. The definition of these classes of quasi-modal spaces is motivated by similar notions used in the context of classical modal logic (see [2] and [4]). In Section 5 we will study some correspondences between algebraic conditions valid in a quasi-modal algebra and first-order conditions defined in the associated quasi-modal space.

2. Preliminaries

We shall recall some concepts of the topological duality for Boolean algebras. Some familiarity with topology, in particular with Boolean spaces (see [6]) and modal spaces, is assumed (see, for example, [2] and [7]).

A topological space is a pair $\langle X, \mathcal{O}(X)\rangle$, or X for short, where $\mathcal{O}(X)$ is a subset of $\mathcal{P}(X)$ that is closed under finite intersections and arbitrary unions. The set $\mathcal{O}(X)$ is called the set of open subsets of the topological space. The collection of all closed subsets of a topological space $\langle X, \mathcal{O}(X)\rangle$ is denoted by $\mathcal{C}(X)$. The set $\mathrm{Clop}(X)$ is the set of closed and open sets of $\langle X, \mathcal{O}(X)\rangle$. A *Boolean space* $\langle X, \mathcal{O}(X)\rangle$ is a topological space that is *compact and totally disconnected*, i.e., given distinct points $x, y \in X$, there is a clopen subset U of X such that $x \in U$ and $y \notin U$. If $\langle X, \mathcal{O}(X)\rangle$ is a Boolean space, then $\mathrm{Clop}(X)$ is a basis for X and it is a Boolean algebra under set-theoretical complement and intersection. Also, the application $\varepsilon\colon X \to \mathrm{Ul}\,(\mathrm{Clop}(X))$ given by $\varepsilon(x) = \{U \in \mathrm{Clop}\,(X)\colon x \in U\}$ is a bijective and continuous function. With each Boolean algebra A we can associate a Boolean space whose points are the elements of $\mathrm{Ul}(A)$ with the topology determined by the clopen basis $\beta(A) = \{\beta(a)\colon a \in A\}$, where $\beta(a) = \{P \in \mathrm{Ul}\,(A)\colon a \in P\}$. By the above considerations we have that, if X is a Boolean space, then $X \cong \mathrm{Ul}\,(\mathrm{Clop}(X))$, and if A is a Boolean algebra, then $A \cong \mathrm{Clop}\,(\mathrm{Ul}(A))$.

It is known that if A is a Boolean algebra and $\mathrm{Ul}(A)$ is the associated Boolean space, then there exists a duality between ideals (filters) of A and open (closed) sets. More precisely, if $\beta\colon A \to \mathcal{P}\,(\mathrm{Ul}(A))$ is the map given by $\beta(a) = \{P \in \mathrm{Ul}\,(A)\colon a \in P\}$, then for $I \in \mathrm{Id}(A)$ and $F \in \mathrm{Fi}(A)$ we have that

$$\beta(I) = \{P \in \mathrm{Ul}(A)\colon I \cap P \neq \emptyset\} \in \mathcal{O}\,(\mathrm{Ul}(A))$$

defines an isomorphism between $\mathrm{Id}(A)$ and $\mathcal{O}\,(\mathrm{Ul}(A))$, and

$$\beta(F) = \{P \in \mathrm{Ul}(A)\colon F \subseteq P\} \in \mathcal{C}\,(\mathrm{Ul}(A))\,,$$

defines a dual-isomorphism between $\mathrm{Fi}(A)$ and $\mathcal{C}\,(\mathrm{Ul}(A))$.

Let A be a Boolean algebra. The filter (ideal) generated by a subset $Y \subseteq A$ is denoted by $F(Y)$ $(I(Y))$. The set complement of a subset $Y \subseteq A$ will be denoted by Y^c or $A - Y$.

3. Quasi-modal algebras

Definition 1. A quasi-modal algebra, or qm-algebra, is an algebra $\langle A, \vee, \wedge, \neg, \Delta, 0, 1\rangle$ where $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra and Δ is a function

$$\Delta\colon\ A \to \mathrm{Id}(A)$$

such that it verifies the following conditions:

1. $\Delta(a \wedge b) = \Delta a \cap \Delta b$,
2. $\Delta 1 = A$.

The class of qm-algebras is denoted by $\mathcal{QMA}$. Note that $\mathcal{QMA}$ is not a variety, because Δ is not an operation on A.

Example 2. Let $\langle A, \vee, \wedge, \neg, \Box, 0, 1\rangle$ be a modal algebra. The operator $\Box$ can be extended to a map $\Delta\colon\ A \to \mathrm{Id}(A)$ of the following form. Put $\Delta(a) = I(\Box\, a)$. It is clear that Δ verifies the equalities $\Delta(a \wedge b) = \Delta(a) \cap \Delta(b)$ and $\Delta(1) = A$. Then $\langle A, \vee, \wedge, \neg, \Delta, 0, 1\rangle$ is a qm-algebra.

Example 3. Let $\langle A, \vee, \wedge, \neg, \Delta, 0, 1\rangle$ be a qm-algebra such that for all $a \in A$ the ideal Δa is principal. Let $\Delta a = I(a')$. Then $\Box\, a = a'$ defines a modal operator on A such that $\langle A, \vee, \wedge, \neg, \Box, 0, 1\rangle$ is a modal algebra.

Let $A \in \mathcal{QMA}$. We define the dual operator

$$\nabla\colon\ A \to \mathrm{Fi}(A)$$

by $\nabla a = \neg\Delta\neg a$, where $\neg\Delta x = \{\neg y\colon\ y \in \Delta x\}$. It is easy to see that the operator ∇ verifies the following conditions:

Q3 $\quad \nabla(a \vee b) = \nabla a \cap \nabla b$,

Q4 $\quad \nabla 0 = A$.

Let A be a qm-algebra. For each $P \in \mathrm{Ul}(A)$ we define the set

$$\Delta^{-1}(P) = \{a \in A\colon\ \Delta a \cap P \neq \emptyset\}\,.$$

Dually, we can define the set

$$\nabla^{-1}(P) = \{a \in A\colon\ \nabla a \subseteq P\}\,.$$

Lemma 4. *Let $A \in \mathcal{QMA}$. Then for each $P \in \mathrm{Ul}(A)$*

1. $\Delta^{-1}(P) \in \mathrm{Fi}(A)$,
2. $\nabla^{-1}(P)^c \in \mathrm{Id}(A)$.

P r o o f. We prove only the assertion 1. The proof of 2 is similar and it is left to the reader.

Since $\Delta 1 = A$, hence $\Delta 1 \cap P \neq \emptyset$, i.e., $1 \in \Delta^{-1}(P)$. Let $x, y \in A$ be such that $x \leqslant y$ and $\Delta x \cap P \neq \emptyset$. Since $\Delta x \subseteq \Delta y$, hence $\Delta y \cap P \neq \emptyset$.

Let $x, y \in \Delta^{-1}(P)$. So, $\Delta x \cap P \neq \emptyset$ and $\Delta y \cap P \neq \emptyset$. Then there are elements $a, b \in A$ such that $a \in \Delta x \cap P$ and $b \in \Delta y \cap P$. Since $\Delta x, \Delta y \in \mathrm{Id}(A)$, we have $a \wedge b \in \Delta x \cap \Delta y$, and since P is a filter, we have $a \wedge b \in P$. So, $a \wedge b \in \Delta x \cap \Delta y \cap P$. Therefore, $\Delta^{-1}(P) \in \mathrm{Fi}(A)$. □

Our next objective is to give a representation theorem for the quasi-modal algebras based on relational structures.

Let us consider a structure $\langle X, D\rangle$ where D is a boolean subalgebra of $\mathcal{P}(X)$. Let us consider the topology $\mathcal{O}(X)$ defined on X by taking the set D as the basis of $\mathcal{O}(X)$.

Definition 5. Let us consider a relational structure $\mathcal{F}_g = \langle X, R, D\rangle$ where D is a boolean subalgebra of $\mathcal{P}(X)$ and R is a binary relation defined on X. We say that $\mathcal{F}_g$ is a quasi-modal space, or qm-modal space for short, if

$$\Delta_R(O) = \{x \in X\colon\ R(x) \subseteq O\} \in \mathcal{O}(X)$$

for each $O \in D$.

Lemma 6. *Let $\mathcal{F}_g = \langle X, R, D\rangle$ be a quasi-modal space. Let us consider a function*

$$\overline{\Delta}\colon\ D \to \mathrm{Id}(D)$$

given by

$$\overline{\Delta}(O) = I_D\left(\Delta_R(O)\right) = \{U \in D\colon\ U \subseteq \Delta_R(O)\}\,.$$

Then $\mathcal{A}\left(\mathcal{F}_g\right) = \left\langle D, \cup, {}^c, \overline{\Delta}, \emptyset\right\rangle \in \mathcal{QMA}$.

P r o o f. Since for any $O_1, O_2 \in D$, $\Delta_R(O_1) \cap \Delta_R(O_2) = \Delta_R(O_1 \cap O_2)$ and $\Delta_R(X) = X$, we conclude that $\overline{\Delta}(O_1 \cap O_2) = \overline{\Delta}(O_1) \cap \overline{\Delta}(O_2)$ and $\overline{\Delta}(X) = D$. □

If $\mathcal{F}_g = \langle X, R, D\rangle$ is a quasi-modal space where $\langle X, D\rangle$ is a Boolean space, then $\mathrm{Id}(D) \cong \mathcal{O}(X)$. Thus, in this case, we can identify Δ_R with $\overline{\Delta}$.

Let A be a qm-algebra. We define on $\mathrm{Ul}(A)$ a relation R_A by

$$\begin{aligned}(P,Q) \in R_A &\Leftrightarrow \forall a \in A\colon \text{ if } \Delta a \cap P \neq \emptyset \text{ then } a \in Q\\ &\Leftrightarrow \Delta^{-1}(P) \subseteq Q.\end{aligned}$$

We note that the relation R_A can be defined using the operator ∇ as follows:

$$(P,Q) \in R_A \Leftrightarrow Q \subseteq \nabla^{-1}(P).$$

Indeed, let $\Delta^{-1}(P) \subseteq Q$. Suppose that $q \in Q$ and $\nabla q \subsetneq P$. Then there exists $y \in \nabla q$ such that $\neg y \in P$. Since $\nabla q = \neg\Delta\neg q$, hence $\neg y \in \Delta\neg q \cap P$, and this implies that $\neg q \in Q$, which is a contradiction. Thus, $Q \subseteq \nabla^{-1}(P)$. The proof in the other direction is similar.

Theorem 7. *Let $A \in \mathcal{QMA}$. Let $a \in A$ and $P \in \mathrm{Ul}(A)$. Then*

1. $a \in \Delta^{-1}(P) \Leftrightarrow \forall Q \in \mathrm{Ul}(A)\colon\ \Delta^{-1}(P) \subseteq Q$ *then* $a \in Q$;
2. $a \in \nabla^{-1}(P) \Leftrightarrow \exists Q \in \mathrm{Ul}(A)\colon\ Q \subseteq \nabla^{-1}(P)$ *and* $a \in Q$.

Proof. We prove only 1. The proof of 2 is similar and it is left to the reader. Let us suppose that $a \in \Delta^{-1}(P)$ and let $Q \in \mathrm{Ul}(A)$ be such that $\Delta^{-1}(P) \subseteq Q$. Then $a \in Q$.

Let us assume that $a \notin \Delta^{-1}(P)$. Let us consider the filter

$$F = F\left(\Delta^{-1}(P) \cup \{\neg a\}\right).$$

Suppose that F is not proper. Then $p \wedge \neg a = 0$ for some $p \in \Delta^{-1}(P)$. It follows that $p \leqslant a$, and thus $\Delta p \subseteq \Delta a$. This implies that $\Delta a \cap P \neq \emptyset$, which is a contradiction. Then, as F is proper, there exists an ultrafilter Q such that $\Delta^{-1}(P) \subseteq Q$ and $a \notin Q$. $\square$

Definition 8. Let A_1 and A_2 be qm-algebras. A function $h\colon A_1 \to A_2$ is a homomorphism of quasi-modal algebras, or a q-homomorphism, if

1. h is a homomorphism of Boolean algebras, and
2. for any $a \in A_1$, $I\left(h(\Delta_1 a)\right) = \Delta_2\left(h(a)\right)$.

A *quasi-isomorphism* is a Boolean isomorphism that is a q-homomorphism.

Theorem 9. *Let $A \in \mathcal{QMA}$. Then the structure $\mathcal{F}_g(A) = \langle \mathrm{Ul}(A), R_A, \beta(A)\rangle$ is a quasi-modal space such that $\mathcal{A}\left(\mathcal{F}_g(A)\right) \cong A$.*

Proof. We have to prove that the Stone isomorphism

$$\beta\colon A \to \beta(A)$$

is an isomorphism of quasi-modal algebras, i.e., we have to prove that $\beta(\Delta a) = \Delta_R\left(\beta(a)\right)$. But this follows by the assertion 1 of Theorem 7. $\square$

Now we shall define the quasi-modal spaces that are dual to quasi-modal algebras.

Definition 10. A descriptive quasi-modal space, or a q-descriptive space for short, is a quasi-modal space $\mathcal{F}_g = \langle X, R, D \rangle$ such that

1. $\langle X, D \rangle$ is a Boolean space,
2. $R(x) \in \mathcal{C}(X)$ for any $x \in X$.

A descriptive frame is a q-descriptive frame $\mathcal{F}_g = \langle X, R, D \rangle$ such that $\Delta_R(O) \in D$ for any $O \in D$.

We note that descriptive frames are the dual spaces of modal algebras (see [2], [3] and [7]).

Let $\mathcal{F}_g = \langle X, R, D \rangle$ be a quasi-modal space. Since D is a quasi-modal algebra, we can define in the set $\mathrm{Ul}(D)$ a relation $R_D \subseteq \mathrm{Ul}(D)^2$ by

$$(P, Q) \in R_D \Leftrightarrow \Delta_R^{-1}(P) \subseteq Q,$$

where

$$\Delta_R^{-1}(P) = \{O \in D\colon\ I_D(\Delta_R O) \cap P \neq \emptyset\} = \{O \in D\colon\ \exists U \in P \quad U \subseteq \Delta_R(O)\}\,.$$

If the space $\langle X, D \rangle$ is compact, then for all $P \in \mathrm{Ul}(D)$ there exists $x \in X$ such that $\varepsilon(x) = P$. So,

$$\Delta_R^{-1}(\varepsilon(x)) = \{O \in D\colon\ \exists U \in \varepsilon(x) \quad U \subseteq \Delta_R(O)\} = \{O \in D\colon\ R(x) \subseteq O\}\,,$$

and therefore

$$(\varepsilon(x), \varepsilon(y)) \in R_D \Leftrightarrow \Delta_R^{-1}(\varepsilon(x)) \subseteq \varepsilon(y).$$

Definition 11. Let $\mathcal{F}_{g_1} = \langle X_1, R_1, D_1 \rangle$ and $\mathcal{F}_{g_2} = \langle X_2, R_2, D_2 \rangle$ be two quasi-modal spaces. A function $f\colon \mathcal{F}_{g_1} \to \mathcal{F}_{g_2}$ is a q-morphism provided the following implications hold:

1. If $(x, y) \in R_1$, then $(f(x), f(y)) \in R_2$.
2. If $(f(x), y) \in R_2$, then there exists $y' \in X_2$ such that $(x, y') \in R_1$ and $f(y') = y$.
3. For any $O \in D_2$, $f^{-1}(O) \in D_1$.

Theorem 12. *Let A_1 and A_2 be two quasi-modal algebras. A Boolean homomorphism $h\colon A_1 \to A_2$ is a quasi-homomorphism if and only if the map $\mathcal{F}_g(h)\colon \mathcal{F}_g(A_2) \to \mathcal{F}_g(A_1)$ defined by $\mathcal{F}_g(h)(P) = h^{-1}(P)$ for each $P \in \mathrm{Ul}(A_2)$, is a q-morphism.*

P r o o f. $\Rightarrow$ We prove 1 and 2 of Definition 11. The assertion 3 is easy and it is left to the reader.

1. Let $P, Q \in \mathrm{Ul}(A_2)$ be such that $\Delta^{-1}(P) \subseteq Q$. Let $a \in \Delta^{-1}\left(h^{-1}(P)\right)$. Then $\Delta a \cap h^{-1}(P) \neq \emptyset$. This implies that $I\left(h(\Delta a)\right) \cap P \neq \emptyset$. Since h is a q-homomorphism, $\Delta h(a) \cap P \neq \emptyset$. Thus, $h(a) \in Q$, i.e., $a \in h^{-1}(Q)$.

2. Let $\Delta^{-1}\left(h^{-1}(P)\right) \subseteq Q'$. Let us consider the filter $\Delta^{-1}(P)$ and the filter $h(Q')$. We prove that the filter $F\left(\Delta^{-1}(P) \cup h(Q')\right)$ is proper. Suppose that there exist $a \in \Delta^{-1}(P)$ and $b \in Q'$ such that $a \wedge h(b) = 0$. Then $a \leqslant \neg h(b) = h(\neg b)$. It follows that $\Delta a \subseteq \Delta h(\neg b)$. Then $\Delta h(\neg b) \cap P \neq \emptyset$. This implies that $\neg b \in Q'$, which is absurd, because Q' is a ultrafilter. Since $F\left(\Delta^{-1}(P) \cup h(Q')\right)$ is proper, there exists $Q \in \mathrm{Ul}(A)$ such that $\Delta^{-1}(P) \cup h(Q') \subseteq Q$. Therefore, $\Delta^{-1}(P) \subseteq Q$ and $h^{-1}(Q) = Q'$.

$\Leftarrow$ Let $a \in A$. We have to prove that $I\left(h(\Delta a)\right) = \Delta h(a)$. By Boolean duality, this is equivalent to

$$\beta\left(I(h(\Delta a))\right) = \beta\left(\Delta h(a)\right).$$

Let $P \in \mathrm{Ul}(A)$ be such that $I\left(h(\Delta a)\right) \cap P \neq \emptyset$. It is easy to see that this implies that $h^{-1}(P) \cap \Delta a \neq \emptyset$. Let us suppose that $\Delta h(a) \cap P = \emptyset$. Then there exists $Q \in \mathrm{Ul}(A)$ such that $\Delta^{-1}(P) \subseteq Q$ and $h(a) \notin Q$. By condition 1 of Definition 11, we have that $\Delta^{-1}\left(h^{-1}(P)\right) \subseteq h^{-1}(Q)$. However, as $h^{-1}(P) \cap \Delta a \neq \emptyset$, we get $a \in h^{-1}(Q)$, which is absurd.

Let us assume that $P \cap \Delta\left(h(a)\right) \neq \emptyset$. We prove that

$$h^{-1}(P) \cap \Delta a \neq \emptyset.$$

Suppose the contrary. Then there exists $Q \in \mathrm{Ul}(A)$ such that $\Delta^{-1}\left(h^{-1}(P)\right) \subseteq Q$ and $a \notin Q$. By condition 2 of Definition 11 we can assert that there exists $D \in \mathrm{Ul}(A)$ such that $\Delta^{-1}(P) \subseteq D$ and $h^{-1}(D) = Q$. However, since $P \cap \Delta\left(h(a)\right) \neq \emptyset$, we have $h(a) \in D$, which is a contradiction. □

Now we shall give an auxiliary result.

Lemma 13. *Let $\mathcal{F}_{g_1} = \langle X_1, R_1, D_1\rangle$ and $\mathcal{F}_{g_2} = \langle X_2, R_2, D_2\rangle$ be two quasi-modal spaces. Let $f\colon \mathcal{F}_{g_1} \to \mathcal{F}_{g_2}$ be a function such that $f^{-1}(O) \in D_1$ for any $O \in D_2$. Then the following conditions are equivalent for any $O \in D_2$:*

1. $I_D\left(f^{-1}(\overline{\Delta} O)\right) = \overline{\Delta}\left(f^{-1}(O)\right)$;
2. $\Delta_{R_1}\left(f^{-1}(O)\right) = f^{-1}\left(\Delta_{R_2}(O)\right)$.

P r o o f. 1. $\Rightarrow$ 2. We prove the inclusion $\Delta_{R_1}\left(f^{-1}(O)\right) \subseteq f^{-1}\left(\Delta_{R_2}(O)\right)$. The proof of the other inclusion is similar. Let us assume that $x \in \Delta_R\left(f^{-1}(O)\right)$. Since

$\Delta_R\left(f^{-1}(O)\right) \in \mathcal{O}(X)$, there exists $U \in D$ such that $x \in U$ and $U \subseteq \Delta_{R_1}\left(f^{-1}(O)\right)$. This implies that $U \in \overline{\Delta}\left(f^{-1}(O)\right) = I_D\left(f^{-1}\left(\overline{\Delta}(O)\right)\right)$. Then there exists $Z \in D$ such that $U \subseteq Z$ and $Z \in f^{-1}\left(\overline{\Delta}(O)\right) = \left\{f^{-1}(W)\colon\ W \in \overline{\Delta}(O)\right\}$. Thus, there exists $W \in D$ such that $Z = f^{-1}(W)$ and $W \subseteq \Delta_{R_2}(O)$. It follows that $f(x) \in W$, and consequently $R_2\left(f(x)\right) \subseteq O$, i.e., $x \in f^{-1}\left(\Delta_R(O)\right)$.

2. $\Rightarrow$ 1. We prove only the inclusion $I_D\left(f^{-1}\left(\overline{\Delta}O\right)\right) \subseteq \overline{\Delta}\left(f^{-1}\left(O\right)\right)$. Let $U \in I_D\left(f^{-1}\left(\overline{\Delta}O\right)\right)$. Then there exists $Z \in D$ such that $U \subseteq Z$ and $Z \in f^{-1}\left(\overline{\Delta}O\right)$. Then there exists $W \in D$ such that $W \subseteq \Delta_{R_2}\left(O\right)$ and $Z = f^{-1}\left(W\right)$. Thus, $U \subseteq f^{-1}\left(W\right)$. Let $x \in U$. Then $f\left(x\right) \in W \subseteq \Delta_{R_2}\left(O\right)$. Therefore, $U \subseteq f^{-1}\left(\Delta_{R_2}\left(O\right)\right)$, and as by assumption $f^{-1}\left(\Delta_{R_2}\left(O\right)\right) = \Delta_{R_1}\left(f^{-1}\left(O\right)\right)$, we get $U \in \overline{\Delta}f^{-1}\left(O\right)$. □

Theorem 14. *Let $\mathcal{F}_{g_1} = \langle X_1, R_1, D_1\rangle$ and $\mathcal{F}_{g_2} = \langle X_2, R_2, D_2\rangle$ be two quasi-modal spaces. If $f\colon \mathcal{F}_{g_1} \to \mathcal{F}_{g_2}$ is a q-morphism, then the function*

$$\mathcal{A}(f)\colon\ D_2 \to D_1$$

given by

$$\mathcal{A}\left(f\right)\left(O\right) = f^{-1}\left(O\right)$$

for each $O \in D_2$ is a q-homomorphism.

P r o o f. By the above lemma, it is enough to prove that $\Delta_{R_2}\left(f^{-1}\left(O\right)\right) = f^{-1}\left(\Delta_{R_1}\left(O\right)\right)$ for each $O \in D_2$. Let $R_1\left(x\right) \subseteq f^{-1}\left(O\right)$. We prove that $R_2\left(f\left(x\right)\right) \subseteq O$. Let $\left(f\left(x\right), y\right) \in R_2$. By condition 2 of Definition 11, there exists $z \in X_1$ such that $\left(x, z\right) \in R_1$ and $f\left(z\right) = y$. By assumption, $z \in f^{-1}\left(O\right)$. Thus, $y \in O$.

By condition 1 of Definition 11 the other inclusion follows. □

Theorem 15. *Let $\mathcal{F}_{g_1} = \langle X_1, R_1, D_1\rangle$ and $\mathcal{F}_{g_2} = \langle X_2, R_2, D_2\rangle$ be two q-descriptive spaces. Then $f\colon \mathcal{F}_{g_1} \to \mathcal{F}_{g_2}$ is a q-morphism if and only if the function $\mathcal{A}(f)\colon D_2 \to D_1$ given by $\mathcal{A}\left(f\right)\left(O\right) = f^{-1}\left(O\right)$ for each $O \in D_2$ is a q-homomorphism.*

P r o o f. The proof of the implication $\Rightarrow$ follows by the above theorem.

$\Leftarrow$: Let $x, y \in X_1$ be such that $\left(x, y\right) \in R_1$. Suppose that $\left(f\left(x\right), f\left(y\right)\right) \notin R_2$. Since $R_2\left(f\left(x\right)\right) \in \mathcal{C}\left(X\right)$ and X_2 is a Boolean space, there exists $O \in D_2$ such that $R_2\left(f\left(x\right)\right) \subseteq O$ and $y \notin f^{-1}\left(O\right)$. It follows that $x \in f^{-1}\left(\Delta_{R_2}\left(O\right)\right) = \Delta_{R_1}\left(f^{-1}\left(O\right)\right)$, i.e., $R_1\left(x\right) \subseteq f^{-1}\left(O\right)$, which is a contradiction.

Let $\left(f\left(x\right), y\right) \in R_2$ and suppose that $f\left(z\right) \neq y$ for any $z \in R_1\left(x\right)$. Then for each $z \in R_1\left(x\right)$ there exists $O_z \in D_1$ such that $z \in f^{-1}\left(O_z\right)$ and $y \notin O_z$. So, $R_1\left(x\right) \subseteq \bigcup f^{-1}\left(O_z\right)$. Since $R_1\left(x\right)$ is closed and X_1 is compact, hence $R_1\left(x\right)$ is compact. Then for some finite family $O_{z_1}, \ldots, O_{z_n}$ we have

$$R_1\left(x\right) \subseteq f^{-1}\left(O_{z_1}\right) \cup \ldots \cup f^{-1}\left(O_{z_n}\right) = f^{-1}\left(O_{z_1} \cup \ldots \cup O_{z_n}\right) = f^{-1}\left(O\right).$$

This implies that $x \in \Delta_{R_1}\left(f^{-1}(O)\right) = f^{-1}\left(\Delta_{R_2}(O)\right)$, i.e., $y \in O$, which is impossible. $\square$

By the above results, we can say that there exists a duality between the class of quasi-modal algebras with a q-homomorphism and the quasi-modal spaces with a q-morphism.

Now, we shall give an application of the above duality. Let us recall that if A is a modal algebra, then there exists a duality between the filters of A closed under the modal operator $\Box$, and the closed subsets Y of the Boolean space $\mathrm{Ul}(A)$ such that $R(x) \subseteq Y$ for each $x \in Y$ (see [7]). We give a similar result for quasi-modal algebras.

Definition 16. Let A be a quasi-modal algebra. A filter F of A is called a Δ-filter, if $\Delta a \cap F \neq \emptyset$, provided $a \in F$.

Definition 17. Let $\mathcal{F}_g = \langle X, R, D\rangle$ be a quasi-modal space. A closed subset $Y \subseteq X$ is called an R-subset, if $R(x) \subseteq Y$ for each $x \in Y$.

Theorem 18. *Let A be a quasi-modal algebra. Then the lattice of Δ-filters of A is anti-isomorphic to the lattice of R-subsets of $\mathcal{F}_g(A) = \langle \mathrm{Ul}(A), R_A, \beta(A)\rangle$.*

Proof. By the Boolean duality the map $F \to \beta(F) = \{P \in \mathrm{Ul}(A)\colon\ F \subseteq P\}$ gives an anti-isomorphism between filters and closed subsets. Let F be a Δ-filter of A. Let $P, Q \in \mathrm{Ul}(A)$ be such that $F \subseteq P$ and $\Delta^{-1}(P) \subseteq Q$. Let $a \in F$ and suppose that $a \notin Q$. Then $\Delta a \cap P = \emptyset$. However, since $a \in F$, we get $\Delta a \cap F \neq \emptyset$. It follows that $\Delta a \cap P \neq \emptyset$, which is absurd. Then, $a \in Q$. Thus, $\beta(F)$ is R_A-closed.

Assume that $\beta(F)$ is R_A-closed. Let $a \in F$. If $\Delta a \cap F = \emptyset$, then there exists $P \in \mathrm{Ul}(A)$ such that $\Delta a \cap P = \emptyset$ and $F \subseteq P$. Then, by Theorem 7, there exists $Q \in \mathrm{Ul}(A)$ such that $\Delta^{-1}(P) \subseteq Q$ and $a \notin Q$. Since $\beta(F)$ is R_A-closed, we have $F \subseteq Q$, which is a contradiction. $\square$

4. Some special classes of quasi-modal spaces

In classical modal logics, there exist some classes of general frames which have important applications in the study of canonical modal logics (see [3]), and the study of the Hennessey-Milner class of models (see, for instance, [2], [4]). Our aim in this section is to introduce some similar classes of quasi-modal spaces, and to give a characterization of descriptive quasi-modal spaces in terms of these notions.

Definition 19. Let $\mathcal{F}_g = \langle X, R, D\rangle$ be a quasi-modal space. We shall say that:

1. $\mathcal{F}_g$ is compact, if $\langle X, D\rangle$ is compact topological space;
2. $\mathcal{F}_g$ is R-compact if for each $x \in X$, the set $R(x)$ is a compact set;
3. $\mathcal{F}_g$ is semi-replete, if for any $x, y \in X$ such that $(\varepsilon(x), \varepsilon(y)) \in R_D$, there exists $z \in X$ such that $(x, z) \in R$ and $\varepsilon(y) = \varepsilon(z)$.

Proposition 20. *Let $\mathcal{F}_g = \langle X, R, D\rangle$ be a quasi-modal space. Then the following conditions are equivalent:*

1. *$\mathcal{F}_g$ is R-compact and compact.*
2. *$\mathcal{F}_g$ is semi-replete and compact.*

Proof. 1. $\Rightarrow$ 2. Let $x, y \in X$ be such that $\varepsilon(y) \subseteq \nabla_R^{-1}(\varepsilon(x))$. It is clear that for each $O \in \varepsilon(y)$ we get $R(x) \cap O \neq \emptyset$. Since $R(x)$ is a compact set, we have $R(x) \cap \bigcap_{O \in \varepsilon(y)} O \neq \emptyset$. Thus there exists a $z \in X$ such that $(x, z) \in R$ and $\varepsilon(z) = \varepsilon(y)$.

2. $\Rightarrow$ 1. Let $A \subseteq D$. Assume that for any finite subset A_0 of A we have

$$R(x) \cap \bigcap A_0 \neq \emptyset.$$

Let us consider the filter $F(A)$. Then

$$F(A) \cap \left(\nabla_R^{-1}(\varepsilon(x))\right)^c = \emptyset,$$

because if not there would exist $U_1, \ldots, U_n \in A$ and $V \notin \nabla_R^{-1}(\varepsilon(x))$ such that

$$U_1 \cap \ldots \cap U_n \subseteq V.$$

By assumption, $R(x) \cap U_1 \cap \ldots \cap U_n \neq \emptyset$. So, there exists $z \in R(x)$ and $z \in U_1 \cap \ldots \cap U_n$. It follows that $z \in V$. But this implies that $V \in \nabla_R^{-1}(\varepsilon(x))$, which is a contradiction. Thus there exists $P \in \mathrm{Ul}(D)$ such that $F(A) \subseteq P$ and $P \subseteq \nabla_R^{-1}(\varepsilon(x))$. Since $\mathcal{F}_g$ is compact, there exists $z \in X$ such that $P = \varepsilon(z)$. So,

$$\varepsilon(z) \subseteq \nabla_R^{-1}(\varepsilon(x)),$$

and since $\mathcal{F}_g$ is semi-replete, there exists $k \in X$ such that $k \in R(x)$ and $\varepsilon(k) = \varepsilon(z)$. Thus, $R(x) \cap \bigcap A \neq \emptyset$. □

Now we give the above mentioned characterization of descriptive quasi-modal spaces.

Theorem 21. *Let $\mathcal{F}_g = \langle X, R, D\rangle$ be a semi-modal space. Suppose that $\langle X, D\rangle$ is a Boolean space. Then the following conditions are equivalent:*

1. *$\mathcal{F}_g$ is a R-compact.*
2. *For any $x \in X$, $R(x) \in \mathcal{C}(X)$.*
3. *$\mathcal{F}_g$ is semi-replete.*
4. *For any $x, y \in X$, if $(\varepsilon(x), \varepsilon(y)) \in R_D$, then $(x, y) \in R$.*

Proof. $1 \Rightarrow 2$. Let $y \in \overline{R(x)}$ (closure of $R(x)$) and suppose that $y \notin R(x)$. Since $\langle X, D\rangle$ is a Boolean space, we have $R(x) = \bigcap\{U_i \in \mathcal{O}(X)\colon R(x) \subseteq U_i\}$. So there exists an open U_0 such that $R(x) \subseteq U_0$ and $y \notin U_0$. Since $\mathcal{F}_g$ is R-compact, hence

$$\Delta_R(U_0) = \Delta_R\big(\bigcup\{O_i \in D\colon O_i \subseteq U_0\}\big) = \bigcup\{\Delta_R(O_i) : O_i \subseteq U_0\},$$

and since $x \in \Delta_R(U_0)$, we have $x \in \Delta_R(O_i)$ for some $O_i \subseteq U_0$. So $y \notin O_i$ and $R(x) \cap O_i^c = \emptyset$, which is a contradiction. Thus, $y \in R(x)$.

2. $\Rightarrow$ 3. Let $x, y \in X$ be such that $(\varepsilon(x), \varepsilon(y)) \in R_D$. Assume that $\varepsilon(z) \neq \varepsilon(y)$ for any $z \in R(x)$. Then for each $z \in R(x)$ there exists $O_z \in D$ such that $z \in O_z$ and $y \notin O_z$. Then $R(x) \subseteq \bigcup_{z \in R(x)} O_z$ and $y \notin \bigcup_{z \in R(x)} O_z$. Since $\langle X, D\rangle$ is compact and $R(x)$ is closed, we have $R(x) \subseteq O_{z_1} \cup \ldots \cup O_{z_n} = O_y$. It follows that $x \in \Delta_R(O_y)$ and by assumption, $y \in O_y$, which is absurd. Thus, $\mathcal{F}_g$ is semi-replete.

The proof of $3 \Rightarrow 4$ is immediate if we take into account that $\langle X, D\rangle$ is a Boolean space and thus the map $\varepsilon\colon X \to \mathrm{Ul}(D)$ is injective.

4. $\Rightarrow$ 1. Let $A \subseteq D$ be such that for any finite subset A_0 of A we have

$$R(x) \cap \bigcap_{O \in A_0} O \neq \emptyset. \tag{1}$$

We prove that $R(x) \cap \bigcap_{O \in A} O \neq \emptyset$. Let us consider the filter $F = F\left(\Delta_R^{-1}(\varepsilon(x)) \cup A\right)$. This filter is proper, because if $\emptyset \in F$ then there exists $U \in \Delta_R^{-1}(\varepsilon(x))$ and there exists $A_0 = \{O_1, \ldots, O_n\} \subseteq A$ such that $U \cap O_1 \cap \ldots \cap O_n = \emptyset$. So, $x \in \Delta_R(U) \subseteq \Delta_R(O_1^c \cup \ldots \cup O_n^c)$, and this implies that $R(x) \cap O_1 \cap \ldots \cap O_n = \emptyset$, but by (1) this is a contradiction. Thus, there exists $P \in \mathrm{Ul}(D)$ such that $\Delta_R^{-1}(\varepsilon(x)) \subseteq P$ and $A \subseteq P$. Since $\langle X, D\rangle$ is compact, we have $P = \varepsilon(y)$ for some $y \in X$. Thus, $y \in R(x) \cap \bigcap_{O \in A} O$. □

We conclude this section by proving a result that shows that the semi-modal spaces that are compacts, R-compacts, and semi-repletes, are preserved by surjective quasi-morphisms.

Proposition 22. *Let $f\colon \mathcal{F}_{g_1} \to \mathcal{F}_{g_2}$ be a surjective q-morphism. Then:*

1. *If $\mathcal{F}_{g_1}$ is compact, then $\mathcal{F}_{g_2}$ is compact.*
2. *If $\mathcal{F}_{g_1}$ is R-compact, then $\mathcal{F}_{g_2}$ is R-compact.*
3. *If $\mathcal{F}_{g_1}$ is semi-replete, then $\mathcal{F}_{g_2}$ is semi-replete.*

The proof of 1 is easy and we leave it to the reader. We prove 2. Let $y \in X_2$ and let $A \subseteq D_2$. Suppose that

$$R_2(y) \cap \bigcap A_0 \neq \emptyset \tag{2}$$

for every finite subset A_0 of A. Let $A' = \left\{f^{-1}(U)\colon U \in A\right\}$. Since f is surjective, $f(x) = y$ for some $x \in X_1$. We prove that

$$R_1(x) \cap \bigcap_{U \in A_0} f^{-1}(U) \neq \emptyset.$$

By 2, there exists $z \in X_2$ such that $(y, z) \in R_2$ and $z \in \bigcap A_0$. Since f is a q-morphism, there exists $k \in X_1$ such that $(x, k) \in R_1$ and $f(k) = z$. Thus, $k \in R_1(x) \cap \bigcap_{U \in A_0} f^{-1}(U)$. Since $\mathcal{F}_{g_1}$ is R-compact, there exists $w \in X_1$ such that

$$w \in R_1(x) \cap \bigcap_{U \in A} f^{-1}(U).$$

It is easy to see that

$$f(w) \in R_2(y) \cap \bigcap A.$$

Thus, $\mathcal{F}_{g_2}$ is R-compact.

3. Let $x, y \in X_2$ be such that $\varepsilon_2(y) \subseteq \nabla_{R_2}^{-1}(\varepsilon_2(x))$. Since f is surjective, $f(a) = x$ and $f(b) = y$ for some pair $a, b \in X_1$. It is easy to show that

$$\varepsilon_1(b) \subseteq \nabla_{R_1}^{-1}(\varepsilon_1(a)).$$

Since $\mathcal{F}_{g_1}$ is semi-replete, there exists $c \in X_1$ such that

$$(a, c) \in R_1 \text{ and } \varepsilon_1(c) = \varepsilon_1(b).$$

It is easy to prove that

$$(x, f(c)) \in R_2 \text{ and } \varepsilon_2(f(c)) = \varepsilon_2(y).$$

Thus, $\mathcal{F}_{g_2}$ is semi-replete. □

In the literature on the modal logic there exist many classes of modal algebras that are obtained by adding new axioms. For example, the topological boolean algebras are modal algebras $\langle A, \Box \rangle$ with axioms $\Box a \leqslant a$ and $\Box a \leqslant \Box\Box a$. Some of these classes of modal algebras can be characterized by means of first-order conditions defined in the associated modal space. For instance, a modal algebra $(A, \Box)$ is a topological modal algebra if and only if in the descriptive modal space $(\mathrm{Ul}(A), R_A)$ the relation R_A is reflexive and transitive. The aim of this section is to give some similar results.

Let $A \in \mathcal{QMA}$. Let $X \subseteq A$. Define an ideal ΔX and a filter ∇X as follows:

$$\Delta X = I\Big(\bigcup_{x\in X} \Delta x\Big),$$
$$\nabla X = F\Big(\bigcup_{x\in X} \nabla x\Big).$$

Lemma 23. *Let $A \in \mathcal{QMA}$. Then for each $a \in A$*

1. $\Delta a = \Delta I(a)$
2. $\nabla a = \nabla F(a)$.

Proof. We prove 1. The proof of 2 is similar and it is left to the reader.

We prove that $\Delta a = I\Big(\bigcup_{x\leqslant a} \Delta x\Big)$. Since $a \leqslant a$, we have $\Delta a \subseteq I\Big(\bigcup_{x\leqslant a} \Delta x\Big)$. Let $y \in I\Big(\bigcup_{x\leqslant a} \Delta x\Big)$. Then there exist $x_i \leqslant a$ and $z_i \in \Delta x_i$ for $i = 1, \ldots, n$ such that $y \leqslant z_1 \vee \ldots \vee z_n$. Since $x_i \leqslant a$, we have $\Delta x_i \subseteq \Delta a$. Then $z_1 \vee \ldots \vee z_n \in \Delta a$, because Δa is an ideal. So, $y \in \Delta a$. $\Box$

Lemma 24. *Let $A \in \mathcal{QMA}$. Let $P \in \mathrm{Ul}(A)$ and $I \in \mathrm{Id}(A)$. Then*

$$\Delta I \cap P = \emptyset \Leftrightarrow \exists Q \in \mathrm{Ul}(A)\left[\Delta^{-1}(P) \subseteq Q \ \text{ and } \ I \cap Q = \emptyset\right].$$

Proof. Assume that $\Delta I \cap P = \emptyset$. Then $\Delta^{-1}(P) \cap I = \emptyset$, because if there existed $p \in \Delta^{-1}(P)$ and $p \in I$, then $\Delta p \cap P \neq \emptyset$ and $\Delta p \subseteq \Delta I$, which is absurd. Then, as $\Delta^{-1}(P) \in \mathrm{Fi}(A)$, there exists $Q \in \mathrm{Ul}(A)$ such that $\Delta^{-1}(P) \subseteq Q$ and $I \cap Q = \emptyset$.

Assume that there exists $Q \in \mathrm{Ul}(A)$ be such that $\Delta^{-1}(P) \subseteq Q$ and $I \cap Q = \emptyset$. This implies that $\Delta^{-1}(P) \cap I = \emptyset$, and thus $\Delta I \cap P = \emptyset$. $\Box$

Let $A \in \mathcal{QMA}$. For $a \in A$ we define recursively

$$\Delta^0 a = I(a),$$
$$\Delta^{n+1} a = \Delta(\Delta^n a).$$

Proposition 25. *Let $A \in \mathcal{QMA}$. Let $P \in \mathrm{Ul}(A)$. Then for all $a \in A$ and for $n \geqslant 1$,*

$$\Delta^n a \cap P \neq \emptyset \Leftrightarrow \forall Q \in \mathrm{Ul}(A)\,[(P,Q) \in R_A^n \text{ implies that } a \in Q]. \tag{3}$$

P r o o f. The proof proceeds by induction on n. The case $n = 1$ is Theorem 7. Assume that (3) is valid for n.

Suppose that there exists $a \in A$ such that

$$\Delta^{n+1} a \cap P = \emptyset.$$

Since $\Delta^{n+1} a$ is an ideal, then by Lemma 24 there exists $Q \in \mathrm{Ul}(A)$ such that

$$(P,Q) \in R_A \text{ and } \Delta^n a \cap Q = \emptyset.$$

By the induction hypothesis, there exists $D \in \mathrm{Ul}(A)$ such that

$$(Q,D) \in R_A^n \text{ and } a \notin D.$$

Thus there exists $D \in \mathrm{Ul}(D)$ such that $(P,D) \in R_A^{n+1}$ and $a \notin D$.

Assume that $(P,Q) \in R_A^{n+1}$. Then $(P,D) \in R_A$ and $(D,Q) \in R_A^n$ for some $D \in \mathrm{Ul}(A)$. Let $a \in A$ be such that $\Delta^{n+1} a \cap P \neq \emptyset$. Since $\Delta(\Delta^n a) \cap P \neq \emptyset$ and $(P,D) \in R_A$, we get by Lemma 24 that $\Delta^n a \cap D \neq \emptyset$. Thus, by the induction hypothesis, $a \in Q$. □

Theorem 26. *Let $A \in \mathcal{QMA}$. Then for all $a \in A$, the following equivalences hold:*

1. $\Delta a \subseteq I(a) \Leftrightarrow R_A$ *is reflexive.*
2. $\Delta^n a \subseteq \Delta^{n+1} a \Leftrightarrow R_A^{n+1} \subseteq R_A^n$, $n \geqslant 1$.
3. $I(a) \cap \Delta a \cap \ldots \cap \Delta^n a \subseteq \Delta^{n+1} a \Leftrightarrow \forall P,Q \in \mathrm{Ul}(A)$ *if* $(P,Q) \in R_A^{n+1}$, *then there exists* $j \leqslant n$ *such that* $(P,Q) \in R_A^j$.
4. $I(a) \subseteq \Delta\nabla a = \bigcap_{x \in \nabla y} \Delta y \Leftrightarrow R_A$ *is symmetrical.*
5. $\Delta 0 = \{0\} \Leftrightarrow R_A$ *is serial.*

P r o o f. 1. $\Rightarrow$ Let $P \in \mathrm{Ul}(A)$ and $\Delta a \cap P \neq \emptyset$. Then $I(a) \cap P \neq \emptyset$. It follows that $a \in P$. Thus, $\Delta^{-1}(P) \subseteq P$.

$\Leftarrow$ Suppose that $\Delta a \subsetneq I(a)$. Then there exists $x \in \Delta a$ such that $x \nleq a$. Then $x \in P$ and $a \notin P$ for some $P \in \mathrm{Ul}(A)$. Since $x \in \Delta a$, we get $\Delta a \cap P \neq \emptyset$, and as $\Delta^{-1}(P) \subseteq P$, we have $a \in P$, which is a contradiction. Thus, $\Delta a \subseteq I(a)$.

2. $\Rightarrow$ Let us suppose that $\Delta^n a \subsetneq \Delta^{n+1} a$. Then there exists $x \in A$ such that $x \in \Delta^n a$ and $x \notin \Delta^{n+1} a$. Since $\Delta^{n+1} a$ is an ideal and $\Delta^{n+1} a \cap F(x) = \emptyset$, we have $\Delta^{n+1} a \cap P = \emptyset$ and $x \in P$ for some $P \in \mathrm{Ul}(A)$. By Proposition 25, there exists $Q \in \mathrm{Ul}(A)$ such that $(P, Q) \in R_A^{n+1}$ and $a \notin Q$. By hypothesis $(P, Q) \in R_A^n$, but as $\Delta^n a \cap P \neq \emptyset$, we get $a \in Q$, which is a contradiction.

$\Leftarrow$ It is immediate.

3. $\Rightarrow$ Let $P, Q \in \mathrm{Ul}(A)$. Assume that $(P, Q) \in R_A^{n+1}$ and $(P, Q) \notin R_A^j$ for all $j \leqslant n$. By Proposition 25, there exist $p_0, \ldots, p_n \in A$ such that $p_0 \in P$, $p_0 \notin Q$, $\Delta p_1 \cap P \neq \emptyset$, $p_1 \notin Q$, $\ldots$, $\Delta^n p_n \cap P \neq \emptyset$, and $p_n \notin Q$. It is easy to see that for all $j \leqslant n$

$$I(p_0) \cap \Delta p_1 \cap \ldots \cap \Delta^n p_n \subseteq \Delta^j (p_0 \vee p_1 \vee \ldots \vee p_n).$$

Thus,

$$I(p_0) \cap \Delta p_1 \cap \ldots \cap \Delta^n p_n \subseteq \Delta^{n+1} (p_0 \vee p_1 \vee \ldots \vee p_n).$$

Then $\Delta^{n+1}(p_0 \vee p_1 \vee \ldots \vee p_n) \cap P \neq \emptyset$, and this implies that $p_0 \vee p_1 \vee \ldots \vee p_n \in Q$, which is a contradiction. Thus, $(P, Q) \in R_A^j$ for some $j \leqslant n$.

$\Leftarrow$ Suppose that there exists $a \in A$ such that $I(a) \cap \Delta a \cap \ldots \cap \Delta^n a \subsetneq \Delta^{n+1} a$. Then there exists $x \in I(a) \cap \Delta a \cap \ldots \cap \Delta^n a$ and $x \notin \Delta^{n+1} a$. Since $F(x) \cap \Delta^{n+1} a = \emptyset$, we have $x \in P$ and $\Delta^{n+1} a \cap P = \emptyset$ for some $P \in \mathrm{Ul}(A)$. By Proposition 25 there exists $Q \in \mathrm{Ul}(A)$ such that $(P, Q) \in R_A^{n+1}$ and $a \notin Q$. By assumption, $(P, Q) \in R_A^j$ for some $j \leqslant n$. Since $\Delta^j a \cap P \neq \emptyset$, hence $a \in Q$, which is absurd. Therefore, $I(a) \cap \Delta a \cap \ldots \cap \Delta^n a \subseteq \Delta^{n+1} a$.

4. $\Rightarrow$ Assume that $I(a) \subseteq \Delta \nabla a = \bigcap_{x \in \nabla a} \Delta x$ and let $P, Q \in \mathrm{Ul}(A)$ be such that $\Delta^{-1}(P) \subseteq Q$. We prove that $P \subseteq \nabla^{-1}(Q)$. Let $a \in P$. Then $I(a) \cap P \neq \emptyset$. It follows that $\bigcap_{x \in \nabla a} \Delta x \cap P \neq \emptyset$, and this implies that $\Delta x \cap P \neq \emptyset$ for any $x \in \nabla a$. Since $\Delta^{-1}(P) \subseteq Q$, we have $x \in Q$ for any $x \in \nabla a$, i.e., $\nabla a \subseteq Q$. Thus, $(Q, P) \in R_A$.

$\Leftarrow$ Suppose that there exists $y \leqslant a$ such that $y \notin \bigcap_{x \in \nabla a} \Delta x$. Then $y \notin \Delta x$ for some $x \in \nabla a$. Since $\Delta x \in \mathrm{Id}(A)$, there exists $P \in \mathrm{Ul}(A)$ such that $y \in P$ and $\Delta x \cap P = \emptyset$. Then there exists $Q \in \mathrm{Ul}(A)$ such that $(P, Q) \in R_A$ and $x \notin Q$. Since R_A is symmetrical, $P \subseteq \nabla^{-1}(Q)$, and thus $\nabla a \subseteq Q$. But this implies that $x \in Q$, which is absurd. Thus, $I(a) \subseteq \Delta \nabla a$.

5. It is easy and we leave it to the reader. $\square$

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