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ON THE WARD THEOREM FOR \mathcal{P} -ADIC-PATH BASES
ASSOCIATED WITH A BOUNDED SEQUENCE

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Abstract. In this paper we prove that each differentiation basis associated with a \mathcal{P} -adic path system defined by a bounded sequence satisfies the Ward Theorem.

Keywords: \mathcal{P} -adic system, differentiation basis, variational measure, Ward Theorem

MSC 2000: 26A39, 26A42, 26A45, 28A12

1. INTRODUCTION

In this paper we prove that each \mathcal{P} -adic-path system associated with a bounded sequence \mathcal{P} defines a differentiation basis for which the Ward Theorem holds true. As an application of this result, we find a full descriptive characterization for the \mathcal{P} -adic-path integral defined by a bounded sequence \mathcal{P} .

2. PRELIMINARIES

We introduce some notation. If $E \subset \mathbb{R}$, then $|E|$ and $|E|_e$ denote respectively the Lebesgue measure and the outer Lebesgue measure of E . The terms “almost everywhere” (briefly a.e.) and “measurable” are always used in the sense of the Lebesgue measure. Let

$$(1) \quad \mathcal{P} = \{p_j\}_{j=0}^{\infty}$$

be a fixed sequence of integers with $p_j > 1$ for $j = 0, 1, \dots$

We set $m_0 = 1$, $m_k = \prod_{j=0}^{k-1} p_j$ for $k \geq 1$. We call the closed intervals

$$\left[\frac{r}{m_k}, \frac{r+1}{m_k} \right] = \Delta_r^{(k)}, \quad r = 0, 1, \dots, m_k - 1$$

for fixed $k = 0, 1, \dots$ the \mathcal{P} -adic intervals (or simply \mathcal{P} -intervals) of rank k . In what follows we denote by the symbol $\Delta^{(k)}$ any \mathcal{P} -adic interval of rank k .

Let $Q_{\mathcal{P}}$ be the set of all \mathcal{P} -adic rational points of $[0, 1]$, i.e. the points of the form $\frac{r}{m_k}$ with $0 \leq r \leq m_k$ and $k = 0, 1, \dots$. The complement in $[0, 1]$ is the set of all \mathcal{P} -adic irrational points in $[0, 1]$. For each \mathcal{P} -adic irrational point $x \in [0, 1]$ there exists only one \mathcal{P} -adic interval $\Delta^{(k)}(x) = [a_k(x), b_k(x)]$ of rank k containing x , so that $x \in \Delta^{(k)}(x)$. We call the sequence

$$\{[a_k(x), b_k(x)]\}_{k=0}^{\infty}$$

of nested intervals, the *basic sequence convergent to x* . With each \mathcal{P} -adic rational point x we can associate two decreasing sequences of \mathcal{P} -intervals for which x is the common end-point, starting with some k . So, for such a point we have two basic sequences convergent to x : the left one and the right one. Now we can define the \mathcal{P} -adic paths. If x is a \mathcal{P} -adic irrational point we set $\mathcal{P}_x^- = \{a_k(x)\}$ and $\mathcal{P}_x^+ = \{b_k(x)\}$. The set $\mathcal{P}_x = \mathcal{P}_x^+ \cup \mathcal{P}_x^- \cup \{x\}$ is the \mathcal{P} -adic path leading to x . If x is a \mathcal{P} -adic rational point we denote by \mathcal{P}_x^- and \mathcal{P}_x^+ respectively the sequences of left and right end-points of the intervals of the left and right basic sequence. The \mathcal{P} -adic-path system is the collection $\mathcal{P} = \{\mathcal{P}_x : x \in [0, 1]\}$.

We call \mathcal{P} -adic-path intervals of rank k attached to a point $x \in [0, 1]$ the intervals $[x, b_k(x)]$ or $[a_k(x), x]$, where $b_k(x) \in \mathcal{P}_x^+$ and $a_k(x) \in \mathcal{P}_x^-$, if $x \in (0, 1)$; the interval $[x, b_k(x)]$, if $x = 0$, and the interval $[a_k(x), x]$, if $x = 1$. We denote by the symbol $I^{(k)}(x)$ any \mathcal{P} -adic path interval of rank k attached to the point x .

Let $F: [0, 1] \rightarrow \mathbb{R}$ be a pointwise function. We also can view F as an additive interval-function if we write $F(I) = F(b) - F(a)$ for each subinterval $I = [a, b]$ of $[0, 1]$.

Given a function $F: [0, 1] \rightarrow \mathbb{R}$ and a point $x \in [0, 1]$, we say that F is \mathcal{P} -adic-path continuous at x if

$$\lim_{\substack{y \rightarrow x \\ y \in \mathcal{P}_x}} F(y) = F(x).$$

We say that F is \mathcal{P} -adic-path differentiable at x if

$$\lim_{\substack{y \rightarrow x \\ y \in \mathcal{P}_x}} \frac{F(y) - F(x)}{y - x} = \lim_{k \rightarrow \infty} \frac{F(I^{(k)}(x))}{|I^{(k)}(x)|}$$

exists and is finite. Then we write $F'_{\mathcal{P}}(x) = f(x)$. We also define the *lower* and the *upper \mathcal{P} -adic-path derivative* respectively as follows:

$$\begin{aligned} \underline{F}'_{\mathcal{P}}(x) &= \liminf_{\varrho \rightarrow 0} \left\{ \frac{F(y) - F(x)}{y - x} : 0 < |y - x| < \varrho, y \in \mathcal{P}_x \right\} \\ &= \liminf_{k \rightarrow \infty} \left\{ \frac{F(I^{(k)}(x))}{|I^{(k)}(x)|} \right\}, \\ \overline{F}'_{\mathcal{P}}(x) &= \limsup_{\varrho \rightarrow 0} \left\{ \frac{F(y) - F(x)}{y - x} : 0 < |y - x| < \varrho, y \in \mathcal{P}_x \right\} \\ &= \limsup_{k \rightarrow \infty} \left\{ \frac{F(I^{(k)}(x))}{|I^{(k)}(x)|} \right\}. \end{aligned}$$

3. THE WARD THEOREM FOR A \mathcal{P} -ADIC-PATH SYSTEM

We recall that a differentiation basis \mathcal{B} satisfies the Ward Theorem whenever each function is \mathcal{B} -differentiable almost everywhere on the set of all points at which at least one of its extreme \mathcal{B} -derivatives is finite.

In this section we will show that the Ward Theorem holds true for each differentiation basis associated with a \mathcal{P} -adic-path system defined by a bounded sequence.

We need the following lemma:

Lemma 3.1. *Let $G: [0, 1] \rightarrow \mathbb{R}$ be a function and E a subset of $[0, 1]$ with $|E|_e > 0$. If the sequence (1) is bounded by $p = \sup\{p_j\}$, and for some positive number $a > 0$ the inequality*

$$(2) \quad 0 < \underline{G}'_{\mathcal{P}}(x) < a$$

holds at every point x of E , then for each $\varepsilon > 0$ there exists a \mathcal{P} -adic interval $\Delta^{(k)}$ for which we have

$$(3) \quad |\Delta^{(k)}| < \varepsilon, |E \cap \Delta^{(k)}|_e > (1 - \varepsilon)|\Delta^{(k)}| \text{ and } G(\Delta^{(k)}) < a \cdot p|\Delta^{(k)}|.$$

Proof. By the definition of derivative and by condition (2), for each $x \in E$ there exists $\sigma(x) > 0$ such that $G(I) > 0$ for each \mathcal{P} -adic-path interval I attached to x with $|I| < \sigma(x)$.

Let $E_n = \{x \in E: \sigma(x) > \frac{1}{n}\}$. It is clear that $E = \bigcup_{n=1}^{\infty} E_n$ and that there exists $\bar{n} \in \mathbb{N}$ such that $|E_{\bar{n}}|_e > 0$. For a fixed $\varepsilon > 0$ we take $\bar{\sigma} \leq \min\{\frac{1}{\bar{n}}, \varepsilon, \frac{1}{p}\}$. So we have

$$(4) \quad G(I) > 0$$

whenever I is a \mathcal{P} -adic-path interval attached to $x \in E_{\bar{n}}$ with $|I| < \bar{\sigma}$.

Let $x_0 \in E_{\bar{\pi}}$ be a point of density for the set $E_{\bar{\pi}}$. We can assume that x_0 is a \mathcal{P} -adic irrational point.

By virtue of $\underline{G}'_{\mathcal{P}}(x) < a$ and by the density we can determine a \mathcal{P} -adic-path interval J attached to x_0 such that

$$(5) \quad |J| < \bar{\sigma}, |J \cap E_{\bar{\pi}}|_e > (1 - \bar{\sigma}^2)|J| \quad \text{and} \quad G(J) < a|J|.$$

It follows in particular that

$$(6) \quad |E_{\bar{\pi}} \cap I|_e > (1 - \bar{\sigma})|I|$$

for any interval $I \subset J$ such that

$$(7) \quad |I| > \bar{\sigma}|J|.$$

In fact, the inclusion

$$E_{\bar{\pi}} \cap J \subset (E_{\bar{\pi}} \cap I) \cup (J \setminus I)$$

and (5), (7) imply that

$$\begin{aligned} |E_{\bar{\pi}} \cap I|_e &\geq |E_{\bar{\pi}} \cap J|_e - |J| + |I| > (1 - \bar{\sigma}^2)|J| - |J| + |I| \\ &= |I| - \bar{\sigma}^2|J| > |I| - \bar{\sigma}|I| = |I|(1 - \bar{\sigma}). \end{aligned}$$

Let $\Delta_j^{(k)}$, $j = 1, 2, \dots, m$, be \mathcal{P} -adic intervals of minimal rank k contained in J and put

$$K = J \setminus \overline{\left(\bigcup_{j=1}^m \Delta_j^{(k)} \right)}.$$

We note that K is a \mathcal{P} -adic -path interval attached to x_0 .

For any \mathcal{P} -adic interval $\Delta_j^{(k)} \subset J$ we get

$$(8) \quad |\Delta_j^{(k)}| = \frac{1}{m_k} = \frac{1}{p_k} |\Delta^{(k-1)}| \geq \frac{1}{p_k} |J| \geq \frac{1}{p} |J| \geq \bar{\sigma} |J|,$$

where $\Delta^{(k-1)}$ is the \mathcal{P} -adic interval of rank $k-1$ with $\Delta^{(k-1)} \supset J$.

By (6) applied to $\Delta_j^{(k)}$ instead of I , we have

$$(9) \quad |E \cap \Delta_j^{(k)}|_e \geq |E_{\bar{\pi}} \cap \Delta_j^{(k)}|_e > (1 - \bar{\sigma})|\Delta_j^{(k)}| > (1 - \varepsilon)|\Delta_j^{(k)}|.$$

As $\Delta_j^{(k)} \subset J$ we note that $|\Delta_j^{(k)}| \leq |J| \leq \bar{\sigma} < \varepsilon$.

By (9) we deduce in particular that

$$(10) \quad E_{\bar{\pi}} \cap \Delta_j^{(k)} \neq \emptyset \text{ for each } j = 1, \dots, m.$$

Now we can write J as

$$(11) \quad J = K \cup \left(\bigcup_{j=1}^m \Delta_j^{(k)} \right)$$

Because of (10) we can represent $\Delta_j^{(k)} = I_j^{(k)-} \cup I_j^{(k)+}$, where $I_j^{(k)-}$ and $I_j^{(k)+}$ are the two \mathcal{P} -adic-path intervals of rank k attached to some $x_j \in \Delta_j^{(k)} \cap E_{\bar{\pi}}$.

By (4) applied to $I_j^{(k)-}$ and $I_j^{(k)+}$ and by the additivity of the interval-function G it follows that

$$(12) \quad G(\Delta_j^{(k)}) = G(I_j^{(k)-}) + G(I_j^{(k)+}) > 0.$$

Because $x_0 \in E_{\bar{\pi}} \cap K$, $K \subset J$, $|K| < |J| < \bar{\sigma}$, we can directly apply inequality (4) to K ; so we get

$$(13) \quad G(K) > 0.$$

Using (11) as a representation of J we have

$$(14) \quad G(J) = G(K) + \sum_{j=1}^m G(\Delta_j^{(k)}),$$

each of the terms of the sum on the right hand side being positive.

Then for any $j = 1, \dots, m$ we get by (13), (14), (5), (8)

$$(15) \quad G(\Delta_j^{(k)}) < G(J) < a|J| < a \cdot p|\Delta_j^{(k)}|.$$

So we can take any $\Delta_j^{(k)}$ as $\Delta^{(k)}$ in the claim of the lemma and this completes the proof. \square

Lemma 3.2. Let $G: [0, 1] \rightarrow \mathbb{R}$ be a function, $E \subset [0, 1]$, $\Delta^{(h)}$ a \mathcal{P} -adic interval of rank h , $\varepsilon > 0$ and b arbitrary fixed numbers. Suppose that

- (i) $|E \cap \Delta^{(h)}|_e > (1 - \varepsilon)|\Delta^{(h)}|$,
- (ii) $G(I) > 0$ for each \mathcal{P} -adic interval I such that $I \subset \Delta^{(h)}$ and $I \cap E \neq \emptyset$,
- (iii) $\overline{G'_p}(x) > b$ for each $x \in E$.

Then

$$G(\Delta^{(h)}) > \frac{b}{p}(1 - p\varepsilon)|\Delta^{(h)}|$$

where $p = \sup\{p_i\}$ of the sequence \mathcal{P} .

Proof. Since the proof is very similar to that of Lemma 11.8 in [7] where it was formulated for the ordinary interval basis, we omit it. We only observe that in our case, instead of the result 11.9 of [7] it needs to use the following lemma.

Lemma 3.3. Let G , E and $\Delta^{(h)}$ be as in Lemma 3.2. Given any $\eta > 0$, we can associate with any point $x \in E$ a \mathcal{P} -adic interval $\Delta^{(k)}(x) \subset \Delta^{(h)}$ of rank k such that

$$(16) \quad x \in \Delta^{(k)}(x), \quad G(\Delta^{(k)}(x)) > \frac{b}{p}|\Delta^{(k)}(x)|, \quad |\Delta^{(k)}(x)| < \eta.$$

Proof. Let x be a fixed point of E . By (iii) it follows that there exists at least one \mathcal{P} -adic-path interval J attached to x such that

$$(17) \quad G(J) > b|J| \quad \text{and} \quad |J| < \frac{\eta}{p}.$$

Let $k + 1$ be the minimal rank greater than h (i.e. $k + 1 > h$), such that $\Delta^{(k+1)} \subset J$. Then we take $\Delta^{(k)}(x) = \Delta^{(k)} \supset \Delta^{(k+1)}$. By the construction it follows that $x \in \Delta^{(k)}$. Since $k \geq h$ we have $\Delta^{(k)} \subset \Delta^{(h)}$. Also $|\Delta^{(k)}| = p_k|\Delta^{(k+1)}| \leq p|J| < p\frac{\eta}{p} = \eta$. We define $J' = \overline{\Delta^{(k)}} - J$ and get

$$x \in J', \quad J' \subset \Delta^{(h)} \quad \text{and} \quad x \in J' \cap E.$$

Applying (ii) we have

$$(18) \quad G(J') > 0.$$

Then by the additivity of G and by (18) we get

$$G(\Delta^{(k)}) = G(J') + G(J) \geq G(J) > b|J| \geq b|\Delta^{(k+1)}| = \frac{b|\Delta^{(k)}|}{p_k} \geq \frac{b}{p}|\Delta^{(k)}|,$$

and this completes the proof.

Theorem 3.1. *Let the sequence \mathcal{P} be bounded. Any function $F: [0, 1] \rightarrow \mathbb{R}$ is \mathcal{P} -adic-path derivable at almost all points x at which $\overline{F}'_{\mathcal{P}}(x) < +\infty$ or $\underline{F}'_{\mathcal{P}}(x) > -\infty$.*

Proof. First we define

$$D = \{x \in [0, 1]: \underline{F}'_{\mathcal{P}}(x) > -\infty\}$$

and a subset A of D ,

$$A = \{x \in D: \overline{F}'_{\mathcal{P}}(x) > \underline{F}'_{\mathcal{P}}(x)\}.$$

If we suppose that $|A|_e > 0$, then we can determine a number $a > 0$ and a set $B \subset A$, $|B|_e > 0$ such that $\underline{F}'_{\mathcal{P}}(x) \neq \infty$ and

$$(19) \quad \overline{F}'_{\mathcal{P}}(x) - \underline{F}'_{\mathcal{P}}(x) > a \text{ at each point } x \in B.$$

Given a positive number ε , we set

$$(20) \quad B_q = \{x \in B: q\varepsilon < \underline{F}'_{\mathcal{P}}(x) \leq (q+1)\varepsilon\}.$$

Let q_0 be an integer for which $|B_{q_0}|_e > 0$. We can determine a number $\sigma > 0$ and a set $E \subset B_{q_0}$, $|E|_e > 0$ such that $F(I) > q_0\varepsilon|I|$ for each \mathcal{P} -adic-path interval I attached to $x \in E$ such that $|I| < \sigma$ (so $E \cap I \neq \emptyset$).

Now we define an additive interval-function G by

$$G(I) = F(I) - q_0\varepsilon|I|$$

for any interval $I \subset [0, 1]$. Thus the function G fulfils

$$(21) \quad 0 < \underline{G}'_{\mathcal{P}}(x) < 2\varepsilon$$

and by (19) and (20)

$$(22) \quad \overline{G}'_{\mathcal{P}}(x) = \overline{F}'_{\mathcal{P}}(x) - q_0\varepsilon > \underline{F}'_{\mathcal{P}}(x) + a - q_0\varepsilon > a$$

at any point $x \in E$. Moreover,

$$(23) \quad G(I) > 0$$

for any \mathcal{P} -adic-path interval I attached to $x \in E$ such that $|I| < \sigma$.

We note that, splitting eventually the \mathcal{P} -adic interval, we can state that (23) is still true for a \mathcal{P} -adic interval I' such that $|I'| < \sigma$ and $I' \cap E \neq \emptyset$.

By Lemma 3.1 there exists a \mathcal{P} -adic interval $\Delta^{(h)}$ of rank h such that

$$(24) \quad |\Delta^{(h)}| < \varepsilon < \sigma, \quad |E \cap \Delta^{(h)}|_e > (1 - \varepsilon)|\Delta^{(h)}|$$

and

$$(25) \quad G(\Delta^{(h)}) < 2\varepsilon p |\Delta^{(h)}|.$$

From (24), (22) and (23) and using Lemma 3.2 we get

$$G(\Delta^{(h)}) > \frac{a}{p}(1 - p\varepsilon)|\Delta^{(h)}|.$$

Thus $\frac{a}{p}(1 - p\varepsilon) < 2\varepsilon p$ for each $\varepsilon > 0$ and this is impossible. Hence $|A|_e = 0$, i.e. $\overline{F'}_{\mathcal{P}}(x) = \underline{F'}_{\mathcal{P}}(x)$ for almost all x for which $\underline{F'}_{\mathcal{P}}(x) > -\infty$.

We have only to prove that the set

$$C = \{x \in [0, 1]: F'_{\mathcal{P}}(x) = +\infty\}$$

is of measure zero, i.e. $|C|_e = 0$.

If we suppose $|C|_e > 0$, then as in the proof of Lemma 3.1 there exists a number $\eta > 0$ such that $F(I) > 0$ whenever I is a \mathcal{P} -adic-path interval attached to a point x of C with $|I| < \eta$. Splitting, if necessary, the interval we can write the previous statement for a \mathcal{P} -adic interval. If we denote by R any \mathcal{P} -adic interval of rank h such that $|R \cap C| > (1 - \frac{1}{2p})|R|$ and $|R| < \eta$, and use a density argument, from Lemma 3.2 we get that $F(R) > \frac{b}{2p}|R|$ for every positive real number b , and this is a contradiction.

4. APPLICATION TO THE \mathcal{P} -ADIC-PATH INTEGRAL

In this section, as an application of the Ward Theorem, we find a full descriptive characterization of the \mathcal{P} -adic-path integral, in the case the sequence \mathcal{P} is bounded. We recall some definitions.

Given a positive function $\delta: [0, 1] \rightarrow \mathbb{R}$ we call the collection C_δ of interval-point pairs (I, x) with $x \in I \subset [0, 1]$ and $I = [y, z]$ where $y, z \in \mathcal{P}_x$, $y \leq x \leq z$ and $0 < z - y < \delta(x)$ the \mathcal{P} -adic-path-full cover of $[0, 1]$ associated to δ . If all (I, x) in the collection C_δ have the point $x \in E \subset [0, 1]$ then we will write $C_\delta(E)$.

A *partition* of $[0, 1]$ is a family of interval-point pairs $\{(I_j, x_j)\}_{j=1}^n$ for which $x_j \in I_j \subset [0, 1]$ and $\mathring{I}_j \cap \mathring{I}_i = \emptyset$ for $i \neq j$ and

$$\bigcup_{j=1}^n I_j = [0, 1].$$

Proposition 4.1. Let $\mathcal{P} = \{\mathcal{P}_x: x \in [0, 1]\}$ be the system of \mathcal{P} -adic paths. If C_δ is a \mathcal{P} -adic-path-full cover of the interval $[0, 1]$, then C_δ must contain a partition of every subinterval of $[0, 1]$.

(A version of this proposition is in Lemma 1.2.1 and Corollary 1.2.2 of [6].)

Definition 4.1. A function $f: [0, 1] \rightarrow \mathbb{R}$ is said to be \mathcal{P} -adic-path integrable (briefly $H_{\mathcal{P}}$ -integrable) on $[0, 1]$ to A , if for every $\varepsilon > 0$ there is a \mathcal{P} -adic-path-full cover C_δ of $[0, 1]$ such that for any partition $D = \{([u, v], x)\}$ from C_δ we have

$$\left| \sum f(x)(v - u) - A \right| < \varepsilon.$$

We denote the number A by the symbol $(H_{\mathcal{P}}) \int_0^1 f = A$.

The \mathcal{P} -adic-path integral has the following properties (see [4]):

(p₁) If f is \mathcal{P} -adic-path integrable on $[0, 1]$, then it is also \mathcal{P} -adic-path integrable on each subinterval of $[0, 1]$.

Therefore the indefinite \mathcal{P} -adic-path integral $F(x) = (H_{\mathcal{P}}) \int_0^x f$ is defined for any $x \in [0, 1]$.

(p₂) The \mathcal{P} -adic-path indefinite integral F of f is \mathcal{P} -adic-path continuous at each $x \in [0, 1]$, and it is \mathcal{P} -adic path differentiable a.e. with $F'_{\mathcal{P}}(x) = f(x)$ a.e. on $[0, 1]$.

In order to study the primitives of the \mathcal{P} -adic path integral it is useful to introduce the following notion of variational measure (see [1], [2] and [8]).

Given a function $F: [0, 1] \rightarrow \mathbb{R}$, a set $E \subset R$ and a \mathcal{P} -adic-path-full cover $C_\delta(E)$ on $[0, 1]$, we define the δ -variation of F on E by

$$\text{Var}(C_\delta(E), F) = \sup \sum_{(I,x) \in \pi} |F(I)|,$$

where the “sup” is taken over all π partitions of $[0, 1]$ from $C_\delta(E)$.

Then we define \mathcal{P} -adic-path variational measure by

$$V_F^{\mathcal{P}}(E) = \inf \text{Var}(C_\delta(E), F),$$

where the “inf” is taken over all \mathcal{P} -adic-path-full covers $C_\delta(E)$.

We observe that $V_F^{\mathcal{P}}$ is a metric outer measure on $[0, 1]$ (see [8]). So its restriction to Borel sets is a measure.

We recall that a measure μ is said to be *absolutely continuous* with respect to the Lebesgue measure if $|N| = 0$ implies $\mu(N) = 0$.

In [4] Theorem 4, using the equivalent definition of “strong Lusin condition” in the place of “variational measure absolutely continuous” the following property is proved:

(p₃) A function $F: [0, 1] \rightarrow \mathbb{R}$ is the indefinite \mathcal{P} -adic-path integral of a function f if and only if F generates a variational measure absolutely continuous with respect to the Lebesgue measure and F is \mathcal{P} -adic-path differentiable a.e. with $F'_{\mathcal{P}}(x) = f$ a.e. on $[0, 1]$.

The above property is also called a *partial descriptive characterization*. This means that we need the hypothesis of \mathcal{P} -adic-path differentiability of F a.e.

A descriptive characterization of the $H_{\mathcal{P}}$ -integral is called a *full descriptive characterization* if no differentiability assumption is supposed a priori.

We need the following results.

Theorem 4.1. *Let F be a function \mathcal{P} -adic-path continuous on $[0, 1]$ and let $E \subset [0, 1]$ be a closed set. If the variational measure $V_F^{\mathcal{P}}$ is σ -finite on all negligible Borel subsets of E then it is σ -finite on E .*

The proof follows as in [3] Theorem 4.3, with minor changes. Hence we omit it.

Corollary 4.2. *Let F be a function on $[0, 1]$ and let $E \subset [0, 1]$ be a closed set. If the variational measure $V_F^{\mathcal{P}}$ is absolutely continuous on E , then it is σ -finite.*

Proof. Since the measure $V_F^{\mathcal{P}}$ is absolutely continuous, hence F is \mathcal{P} -adic path continuous and we can apply Theorem 4.1.

Proposition 4.2. *Let F be a function on $[0, 1]$ and let $E \subset [0, 1]$ be a Borel subset of $[0, 1]$. If the variational measure $V_F^{\mathcal{P}}$ is σ -finite on E , then the extreme \mathcal{P} -adic-path derivative is finite almost everywhere on E .*

The proof follows as that of [1] p. 6 or [8] p. 850, where it is written for the ordinary extreme derivative.

Proposition 4.3. *Let the sequence \mathcal{P} be bounded, let $F: [0, 1] \rightarrow \mathbb{R}$ be a function. If the variational measure $V_F^{\mathcal{P}}$ is absolutely continuous on $[0, 1]$, then F is \mathcal{P} -adic-path differentiable a.e. on $[0, 1]$.*

Proof. Let $D_{\infty} = \{x \in [0, 1]: \overline{F'}_{\mathcal{P}}(x) = +\infty\} \cup \{x \in [0, 1]: \underline{F'}_{\mathcal{P}}(x) = -\infty\}$. First we will prove that $|D_{\infty}| = 0$. The absolute continuity of $V_F^{\mathcal{P}}$ implies the \mathcal{P} -adic-path continuity of F on $[0, 1]$. Then the function F is measurable and also the lower and upper \mathcal{P} -adic-path derivatives are measurable. Therefore the set D_{∞} is measurable. Let K be any closed subset of D_{∞} . By Corollary 4.2 we have that

the variational measure $V_F^{\mathcal{P}}$ is σ -finite on $[0, 1]$. Then $V_F^{\mathcal{P}}$ is σ -finite on K and by Proposition 4.2, $|K| = 0$. Since this is true for every closed subset of D_∞ , also $|D_\infty| = 0$. Now, by recalling that the sequence \mathcal{P} is bounded, as an application of Theorem 3.1 we get the result.

Theorem 4.3. *Let the sequence \mathcal{P} be bounded. A function $F: [0, 1] \rightarrow \mathbb{R}$ is the indefinite \mathcal{P} -adic-path integral of a function f if and only if the variational measure $V_F^{\mathcal{P}}$ is absolutely continuous on $[0, 1]$.*

P r o o f. It follows at once from the property (p_3) and from Proposition 4.3.

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