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EXISTENCE OF NONOSCILLATORY SOLUTIONS OF A CLASS OF
NONLINEAR DIFFERENCE EQUATIONS WITH A FORCED TERM

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Abstract. In this paper, necessary and sufficient conditions for the existence of nonoscillatory solutions of the forced nonlinear difference equation

$$\Delta(x_n - p_n x_{\tau(n)}) + f(n, x_{\sigma(n)}) = q_n$$

are obtained. Examples are included to illustrate the results.

Keywords: difference equations, nonlinear, forced term, nonoscillation

MSC 2000: 39A10

1. INTRODUCTION

In this paper, we consider the nonlinear difference equation with a forced term

$$(1) \quad \Delta(x_n - p_n x_{\tau(n)}) + f(n, x_{\sigma(n)}) = q_n, \quad n = 1, 2, 3, \dots,$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $f: \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and $\tau, \sigma: \mathbb{N} \rightarrow \mathbb{N}$ with $\lim_{n \rightarrow \infty} \sigma(n) = +\infty$, $\lim_{n \rightarrow \infty} \tau(n) = +\infty$; $\{p_n\}, \{q_n\}$ are real sequences. A solution of (1) is a real sequence x_n defined for all $n \geq \min\{N_0, \min_{n \geq N_0} \sigma(n), \min_{n \geq N_0} \tau(n)\}$ and satisfying (1) for all $n \geq N_0$. A nontrivial solution $\{x_n\}$ of (1) is said to be *oscillatory* if for any $N \geq N_0$ there exists $n > N$ such that $x_{n+1}x_n \leq 0$. Otherwise, the solution is said to be *nonoscillatory*.

Difference equations of neutral type have been studied by a number of authors in recent years, for example, see [2–11,13] and the references contained therein. Various

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authors have obtained results guaranteeing the oscillation of equation (1), and we cite the papers [2, 6–8]. In this paper we are interested in obtaining necessary and sufficient conditions for the existence of nonoscillatory solutions of (1).

2. MAIN RESULTS

Let X denote the Banach space l_∞^N of all bounded real sequences $x = \{x_n\}$, $n \geq N$, with the norm $\|x\| = \sup_{n \geq N} |x_n|$. We will use the following assumptions:

- (i) $|f(n, x)| \leq |f(n, y)|$, provided $|x| \leq |y|$;
- (ii) for each closed interval $L = [d_1, d_2]$ ($0 < d_1 < d_2$), there exists $L(n)$ such that

$$|f(n, x) - f(n, y)| \leq L(n)|x - y|, \quad x, y \in L,$$

and $\sum_{i=N}^{\infty} L(i) < \infty$;

- (iii) $xf(n, x) \geq 0$ ($x \neq 0$);
- (iv) $\sum_{i=N}^{\infty} |q_i| < \infty$;
- (v) there exists $r \in (0, 1)$ such that

$$0 \leq p_n \leq 1 - r, \quad n \geq N;$$

- (vi) there exists $r \in (0, 1)$ such that

$$r - 1 \leq p_n \leq 0, \quad n \geq N;$$

- (vii) $|p_n| \leq 1 - r$, $n \geq N$, $r \in (\frac{1}{2}, 1)$;
- (viii) $p_n \equiv 1$.

Theorem 1. *Suppose that (i), (ii) and (iv) hold. Further suppose that either (v) or (vi) holds. If*

$$(2) \quad \sum_{n=N}^{\infty} |f(n, d)| < \infty \text{ for some } d \neq 0,$$

then Eq. (1) has a bounded nonoscillatory solution $\{x_n\}$ such that $\liminf_{n \rightarrow \infty} |x_n| > 0$.

P r o o f. Define a subset Ω of X as follows:

$$\Omega = \{ \{x_n\} \subset X : d_1 \leq x_n \leq |d|, \quad n \geq N \}$$

and an operator T on Ω :

$$Tx_n = \begin{cases} c_1 + p_n x_{\tau(n)} + \sum_{i=n}^{\infty} f(i, x_{\sigma(i)}) - \sum_{i=n}^{\infty} q_i, & n \geq N_1, \\ Tx_{N_1}, & N \leq n < N_1, \end{cases}$$

where $0 < d_1 < r|d|$, c_1 and N_1 satisfy the following conditions: If (v) holds, $d_1 < c_1 < r|d|$ and N_1 is sufficiently large such that $\tau(n) \geq N$, $\sigma(n) \geq N$ as $n \geq N_1 \geq N$ and

$$\sum_{n=N_1}^{\infty} f(n, d) + \sum_{n=N_1}^{\infty} |q_n| \leq \min\{c_1 - d_1, r|d| - c_1\}$$

and

$$\sum_{i=N_1}^{\infty} L(i) \leq \frac{r}{2}.$$

If (vi) holds, $d_1 + (1-r)|d| < c_1 < \frac{1}{2}(d_1 + (2-r)|d|)$, N_1 is sufficiently large such that $\tau(n) \geq N$, $\sigma(n) \geq N$ as $n \geq N_1$ and

$$\sum_{n=N_1}^{\infty} |f(n, d)| + \sum_{n=N_1}^{\infty} |q_n| \leq c_1 - d_1 - (1-r)|d|.$$

First, we claim that $T\Omega \subset \Omega$.

If (v) holds, then for any $x \in \Omega$, $n \geq N_1$ we have

$$\begin{aligned} Tx_n &= c_1 + p_n x_{\tau(n)} + \sum_{i=n}^{\infty} f(i, x_{\sigma(i)}) - \sum_{i=n}^{\infty} q_i \\ &\geq c_1 - \sum_{i=N_1}^{\infty} |f(i, x_{\sigma(i)})| - \sum_{i=N_1}^{\infty} |q_i| \\ &\geq c_1 - (c_1 - d_1) = d_1 \end{aligned}$$

and

$$Tx_n \leq c_1 + p_n |d| + \sum_{i=N_1}^{\infty} |f(i, x_{\sigma(i)})| + \sum_{i=N_1}^{\infty} |q_i| \leq c_1 + (1-r)|d| + (r|d| - c_1) = |d|.$$

If (vi) holds, then for $n \geq N_1$ we have

$$\begin{aligned} Tx_n &\geq c_1 + |d|p_n - \sum_{i=N_1}^{\infty} |f(i, d)| - \sum_{i=N_1}^{\infty} |q_i| \\ &\geq c_1 - (1-r)|d| - (c_1 - d_1 - (1-r)|d|) = d_1 \end{aligned}$$

and

$$\begin{aligned}
 Tx_n &\leq c_1 + \sum_{i=N_1}^{\infty} |f(i, d)| + \sum_{i=N_1}^{\infty} |q_i| \\
 &\leq c_1 + c_1 - d_1 - (1-r)|d| \\
 &< d_1 + (2-r)|d| - d_1 - (1-r)|d| \\
 &= |d|.
 \end{aligned}$$

Therefore $T\Omega \subset \Omega$.

Next, we claim that T is a compression mapping on Ω . In fact, for $x, y \in \Omega$, $n \geq N_1$, we have

$$\begin{aligned}
 |Tx_n - Ty_n| &= |p_n(x_{\tau(n)} - y_{\tau(n)}) + \sum_{i=N_1}^{\infty} (f(i, x_{\sigma(i)}) - f(i, y_{\sigma(i)}))| \\
 &\leq |p_n| \sup_{n \geq N} |x_n - y_n| + \sum_{i=N_1}^{\infty} L(i) |x_{\sigma(i)} - y_{\sigma(i)}| \\
 &\leq \left(|p_n| + \sum_{i=N_1}^{\infty} L(i) \right) \sup_{n \geq N} |x_n - y_n| \\
 &\leq \left(1 - r + \frac{r}{2} \right) \|x - y\| \\
 &= \left(1 - \frac{r}{2} \right) \|x - y\|,
 \end{aligned}$$

which implies that

$$\|Tx - Ty\| \leq \left(1 - \frac{r}{2} \right) \|x - y\|.$$

By the Banach fixed point theorem, T has a fixed point $\bar{x} = \{\bar{x}_n\} \in \Omega$. Obviously, \bar{x} is a bounded nonoscillatory solution of (1) with $\liminf_{n \rightarrow \infty} |\bar{x}_n| \geq d_1 > 0$. The proof is complete. \square

The following lemmas show the necessity of condition (2) for the existence of a nonoscillatory solution $\{x_n\}$ with $\liminf_{n \rightarrow \infty} |x_n| > 0$.

Lemma 1. *Assume that (i), (iii), (iv) and (vi) hold. If (1) has a nonoscillatory solution $\{x_n\}$ with $\liminf_{n \rightarrow \infty} |x_n| > 0$, then (2) holds.*

Proof. Without loss of generality, assume that $x_n > d > 0$, $n \geq N$. Let $y_n = x_n - p_n x_{\tau(n)} > 0$. Then

$$\Delta y_n = q_n - f(n, x_{\sigma(n)}).$$

If (2) does not hold, summing the last equation we obtain

$$y_n - y_{N_1} \leq \sum_{i=N_1}^{n-1} q_i - \sum_{i=N_1}^{n-1} f(i, x_{\sigma(i)}) \leq \sum_{i=N_1}^{n-1} |q_i| - \sum_{i=N_1}^{n-1} f(i, d) \rightarrow -\infty, \quad n \rightarrow \infty.$$

Then $\lim_{n \rightarrow \infty} y_n = -\infty$, a contradiction. The proof is complete. \square

Lemma 2. *Assume that (i), (iii), (iv), (v) and $\tau(n) \leq n, n \geq N$ hold. Then the conclusion of Lemma 1 is true.*

Proof. Assume that $x_n \geq d > 0, n \geq N$ is a positive solution of (1). If (2) does not hold, as in the proof of Lemma 1, we have $\lim_{n \rightarrow \infty} y_n = -\infty$. Then x_n is unbounded. Therefore there exists a sequence $\{n_k\}$ with $\lim_{k \rightarrow \infty} n_k = \infty$ such that $x_{n_k} = \max_{n \leq n_k} x_n$. Then

$$y_{n_k} = x_{n_k} - p_{n_k} x_{\tau(n_k)} \geq (1 - p_{n_k}) x_{n_k} > 0,$$

a contradiction. The proof is complete. \square

Combining the above results we obtain

Theorem 2. *Assume that (i), (ii), (iii), (iv) and (vi) hold. Then (2) is a necessary and sufficient condition for (1) to have a nonoscillatory solution $\{x_n\}$ with $\liminf_{n \rightarrow \infty} |x_n| > 0$.*

Theorem 3. *Assume that (i), (ii), (iii), (iv), (v) and $\tau(n) \leq n, n \geq N$ hold. Then (2) is a necessary and sufficient condition for (1) to have a nonoscillatory solution $\{x_n\}$ with $\liminf_{n \rightarrow \infty} |x_n| > 0$.*

Now we consider the case that p_n is oscillatory in (1).

Theorem 4. *Assume that (i), (ii), (iv), (vii) and (2) hold. Then (1) has a bounded nonoscillatory solution $\{x_n\}$ with $\liminf_{n \rightarrow \infty} |x_n| > 0$.*

Proof. Let $\Omega = \{\{x_n\} \in X: d_1 \leq x_n \leq |d|, n \geq N\}$, where $0 < d_1 < (2r-1)|d|$. Define an operator T by (3), where c_1 satisfies $d_1 + (1-r)|d| < c_1 < r|d|$ and N_1 is sufficiently large such that when $n \geq N_1 \geq N, \tau(n) \geq N, \sigma(n) \geq N$ and

$$\sum_{i=N_1}^{\infty} |f(i, d)| + \sum_{i=N_1}^{\infty} |q_i| \leq \min\{c_1 - d_1 - (1-r)|d|, r|d| - c_1\}$$

and

$$\sum_{i=N_1}^{\infty} L(i) \leq \frac{r}{2}.$$

The rest of the proof is similar to that of Theorem 1. \square

Similarly to Lemma 2 we can prove the following assertion.

Lemma 3. *Assume that (i), (iii), (iv), (vii) and $\tau(n) \leq n$, $n \geq N$ hold. Then the conclusion of Lemma 1 is true.*

Combining Theorem 4 and Lemma 3 we obtain

Theorem 5. *Assume that (i), (ii), (iii), (iv), (vii) and $\tau(n) \leq n$, $n \geq N$ hold. Then (2) is a necessary and sufficient condition for (1) to have a nonoscillatory solution $\{x_n\}$ with $\liminf_{n \rightarrow \infty} |x_n| > 0$.*

Remark 1. Theorems 1–5 are discrete analogues of the corresponding results for the neutral differential equation [12].

Finally, we consider the case (viii).

Theorem 6. *Assume that (i) and (viii) hold. Further assume that $\tau(n)$ is increasing, $\tau(n) < n$ for all large n , and*

$$\sum_{n=N}^{\infty} n|f(n, d)| < \infty \text{ for some } d \neq 0$$

and

$$(3) \quad \sum_{n=N}^{\infty} n|q_n| < \infty.$$

Then (1) has a bounded nonoscillatory solution.

Let

$$\begin{aligned} \tau^0(n_0) &= n_0, \quad \tau^{n+1}(n_0) = \tau(\tau^n(n_0)), \quad n = 0, 1, 2, \dots, \\ \tau^{n-1}(n_0) &= \tau^{-1}(\tau^n(n_0)), \quad n = 0, -1, -2, \dots \end{aligned}$$

By a known result [13, Lemma 2.3], (3) is equivalent to

$$\sum_{j=0}^{\infty} \sum_{n=\tau^{-j}(n_0)}^{\infty} |f(n, d)| < \infty$$

and

$$(4) \quad \sum_{j=0}^{\infty} \sum_{n=\tau^{-j}(n_0)}^{\infty} |q_n| < \infty.$$

Proof of Theorem 6. In view of (4), we can choose a sufficiently large n_0 such that

$$\sum_{j=0}^{\infty} \sum_{n=\tau^{-j}(n_0)}^{\infty} |f(n, d)| \leq \frac{1}{2}$$

and

$$(5) \quad \sum_{j=0}^{\infty} \sum_{n=\tau^{-j}(n_0)}^{\infty} |q_n| \leq \frac{1}{2}.$$

Define

$$H_n = \begin{cases} \sum_{i=n}^{\infty} |f(i, d)| + \sum_{i=n}^{\infty} |q_i|, & t \geq n_0, \\ \frac{n - \tau(n_0)}{n_0 - \tau(n_0)} H(n_0), & \tau(n_0) \leq n \leq n_0, \\ 0, & n \leq \tau(n_0). \end{cases}$$

Clearly, $H_n : N \rightarrow R$. Define

$$(6) \quad y_n = \sum_{m=0}^{\infty} H_{\tau^m(n)}, \quad n \geq n_0.$$

It is easy to see that $y_n - y_{\tau(n)} = H_n$, $n \geq \tau^{-1}(n_0)$ and

$$(7) \quad 0 < y_n \leq 1, \quad n \geq n_0.$$

Define a set $\Omega \subset X$ by

$$\Omega = \{ \{x_n\} \subset X : 0 \leq x_n \leq y_n, \quad n \geq n_0 \}$$

and an operator S on Ω by

$$Sx_n = \begin{cases} x_{\tau(n)} + \sum_{i=n}^{\infty} f(i, x_{\sigma(i)}) - \sum_{i=n}^{\infty} q_i, & n \geq n_0, \\ \frac{Sx_{n_0} n y_n}{n_0 y_{n_0}} + y_n \left(1 - \frac{n}{n_0}\right), & \tau(n_0) \leq n \leq n_0. \end{cases}$$

By (5)–(7), $S\Omega \subset \Omega$.

Define a sequence of sequences $\{x_n^k\}_{k=0}^{\infty}$ as follows:

$$x_n^0 = y_n, \quad x_n^k = Sx_n^{k-1}, \quad n \geq n_0, \quad k = 1, 2, \dots$$

By induction, we can prove that

$$y_n = x_n^0 \geq x_n^1 \geq \dots, \quad n \geq n_0.$$

Then there exists a sequence $\{u_n\} \in \Omega$ such that $\lim_{k \rightarrow \infty} x_n^k = u_n$ and $u_n > 0$ for $n \geq n_0, u_n = Su_n$, i.e.,

$$u_n = u_{\tau(n)} + \sum_{i=n}^{\infty} f(i, u_{\sigma(n)}) - \sum_{i=n}^{\infty} q_i.$$

Hence

$$\Delta(u_n - u_{\tau(n)}) + f(n, u_{\sigma(n)}) = q_n.$$

The proof is complete.

Remark 2. We can establish a result similar to Theorem 6 for the neutral differential equation

$$(x(t) - x(\tau(t)))' + f(t, x(\sigma(t))) = q(t).$$

Example 1. Consider the equation

$$(8) \quad \Delta(x_n - \frac{3}{5}x_{\tau(n)}) + n^{-2}x_{\sigma(n)}^3 = e^{-n}, \quad n \geq N$$

where $\lim_{n \rightarrow \infty} \tau(n) = \infty, \lim_{n \rightarrow \infty} \sigma(n) = \infty, q_n = e^{-n}$ and $f(n, x) = n^{-2}x^3$.

For $x, y \in L = [d_1, d_2]$ ($0 < d_1 < d_2$) and $d > d_2$ we have

$$|f(n, x) - f(n, y)| = n^{-2}|x^2 + xy + y^2||x - y| \leq 3d^2n^{-2}|x - y|.$$

Let $L(n) = 3d^2n^{-2}$. Then $\sum_{i=N}^{\infty} L(i) < \infty$ and $\sum_{i=N}^{\infty} |f(i, d)| = \sum_{i=N}^{\infty} |d|^3i^{-2} < \infty$. By Theorem 1, (8) has a nonoscillatory solution $\{x_n\}$ with $\liminf_{n \rightarrow \infty} |x_n| > 0$.

Example 2. Consider the equation

$$(9) \quad \Delta(x_n - (-\frac{1}{3})^n x_{n-1}) + (-\frac{1}{3})^{n-1} \frac{4n^2 - 2n + 1}{3(n-1)^2(n^2+1)} x_n = \frac{4}{3}(-\frac{1}{3})^n - \frac{2n+1}{n^2(n+1)^2},$$

where $p_n = (-\frac{1}{3})^n$ is oscillatory and satisfies (vii), $f(n, x) = (-\frac{1}{3})^{n-1} \frac{4n^2 - 2n + 1}{3(n-1)^2(n^2+1)} x$ and satisfies (i) and (ii), $q_n = \frac{4}{3}(-\frac{1}{3})^n - \frac{2n+1}{n^2(n+1)^2}$ satisfies (iv), $\tau(n) = n - 1 < n$ and

$$\sum_{n=N}^{\infty} |f(n, d)| = \sum_{n=N}^{\infty} (\frac{1}{3})^{n-1} \frac{4n^2 - 2n + 1}{3(n-1)^2(n^2+1)} |d| < \infty.$$

By Theorem 4, (9) has a bounded nonoscillatory solution $\{x_n\}$ with $\liminf_{n \rightarrow \infty} |x_n| > 0$. In fact, $\{x_n\} = \{1 + n^{-2}\}$ is such a solution of (9).

Example 3. Consider the difference equation

$$(10) \quad \Delta(x_n - x_{n-3}) + \frac{1}{n(n+1)(n-3)}x_{n-2} = \frac{6n-5}{(n+1)n(n-2)(n-3)}.$$

It is easy to see that Eq. (10) satisfies all assumptions of Theorem 6. Therefore (10) has a bounded nonoscillatory solution. In fact, $\{x_n\} = \{\frac{1}{n}\}$ is such a solution of (10).

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