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# A posteriori error estimates for parabolic differential systems solved by the finite element method of lines

*Applications of Mathematics*, Vol. 39 (1994), No. 6, 415–443

Persistent URL: http://dml.cz/dmlcz/134269

# A POSTERIORI ERROR ESTIMATES FOR PARABOLIC DIFFERENTIAL SYSTEMS SOLVED BY THE FINITE ELEMENT METHOD OF LINES[1]

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(Received June 3, 1993)

*Summary*. Systems of parabolic differential equations are studied in the paper. Two a posteriori error estimates for the approximate solution obtained by the finite element method of lines are presented. A statement on the rate of convergence of the approximation of error by estimator to the error is proved.

*Keywords*: a posteriori error estimate, system of parabolic equations, finite element method, method of lines

*AMS classification*: 65M15, 65M20

## 1. INTRODUCTION

Recently, a posteriori error estimates are widely used to adjust the grid and to reach the optimal number and optimal distribution of grid points in the finite element as well as finite difference methods.

This approach is very suitable e.g. for solving parabolic partial differential equations by the method of lines. The analysis of the approximate solution at the actual time level based on the calculation of an a posteriori error estimate yields a new grid to be used for the time step leading to the next time level.

A posteriori error estimates have been treated by many authors in various ways (see, e.g., [3], [4]). In the present paper, the quality of error estimates, proposed by Adjerid et al. [1], [2] for a parabolic equation, is studied in the case of a general linear

[1] This research was supported by the Grant Agency of the Academy of Sciences of the Czech Republic under Grant No. 11919.

system of parabolic differential equations with one-dimensional space variable. Note that [2] has parabolic systems in its title but the statements of [2] are concerned only with a scalar equation and can be, of course, readily generalized to parabolic systems with an elliptic operator expressed by a diagonal matrix. Parabolic systems with an elliptic operator expressed by a general symmetric positive definite matrix are studied in the present paper.

We formulate a model problem for a parabolic system in Section 2 and its discretization by the finite element method and the method of lines in Section 3. A parabolic and elliptic error estimate is introduced, and some approximation and a priori estimation results are reviewed in Section 3, too. In Section 4, some auxiliary results are proved. The statements on the convergence rate of the approximation of error by the parabolic and elliptic estimate are presented in Section 5 and 6, respectively.

## 2. Model Problem

The principal ideas as well as algorithmic procedures connected with the use of an adaptive grid for solving linear parabolic partial differential systems can be demonstrated with the help of a simple initial-boundary value model problem. We solve the system of equations

$$(2.1)\quad \frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\Big(A(x)\frac{\partial u}{\partial x}(x,t)\Big) - B(x)u(x,t) + f(x,t), \quad 0 < x < 1,\ 0 < t \leqslant T,$$

with a fixed $T > 0$ for an unknown vector function $u = (u_1, \dots, u_N)^{\mathrm{T}}$, where $A = (A_{ik})$ is a given smooth real $N \times N$ symmetric positive definite matrix, $B = (B_{ik})$ is a given smooth real $N \times N$ symmetric positive semidefinite matrix, and $f = (f_1, \dots, f_N)^{\mathrm{T}}$ is a given $N$-component vector.

We further impose the homogeneous Dirichlet boundary condition

$$(2.2)\qquad u(0,t) = u(1,t) = 0, \quad 0 \leqslant t \leqslant T,$$

and the initial condition

$$(2.3)\qquad u(x,0) = u^0(x), \quad 0 < x < 1,$$

where $u^0 = (u_1^0, \dots, u_N^0)^{\mathrm{T}}$ is a given smooth $N$-component vector. We assume that the boundary and initial conditions are consistent.

We present the variational formulation of the model problem which is the starting point for the finite element discretization. First we introduce some notation.

Denote by

$$(v, w)_0 = \int_0^1 v^{\mathrm{T}}(x) w(x) \, \mathrm{d}x \tag{2.4}$$

the $L_2$ inner product of two vector functions $v = (v_1, \ldots, v_N)^{\mathrm{T}}$ and $w = (w_1, \ldots, w_N)^{\mathrm{T}}$. Let $s$ be a nonnegative integer. Then $H^s = H^s(0,1)$ is the Sobolev space of vector valued functions defined on the interval $(0,1)$ with the inner product given by

$$(v, w)_s = \sum_{r=0}^{s} \left( \frac{\partial^r v}{\partial x^r}, \frac{\partial^r w}{\partial x^r} \right)_0 \tag{2.5}$$

for $v \in H^s$ and $w \in H^s$ and with the norm

$$\|w\|_s^2 = (w, w)_s. \tag{2.6}$$

The case of $s = 1$ is important for the variational formulation. We, moreover, introduce the subspace $H_0^1 = H_0^1(0,1)$ of vector functions $w \in H^1(0,1)$ satisfying the homogeneous Dirichlet boundary conditions, i.e.

$$w(0) = 0, \quad w(1) = 0.$$

Finally we use the energy inner product

$$a(v, w) = \int_0^1 \left( \frac{\partial v^{\mathrm{T}}}{\partial x} A \frac{\partial w}{\partial x} + v^{\mathrm{T}} B w \right) \mathrm{d}x \tag{2.7}$$

for $v \in H^1$, $w \in H^1$ and the energy norm

$$\|w\|_A^2 = a(w, w). \tag{2.8}$$

Notice that under our assumptions on $A$ and $B$ there are positive constants $C_1$, $C_2$ such that, in virtue of the Friedrichs inequality,

$$C_1 \|w\|_1 \leqslant \|w\|_A \leqslant C_2 \|w\|_1 \tag{2.9}$$

for any function $w \in H_0^1$. Further, the second inequality in (2.9), i.e.

$$\|w\|_A \leqslant C_2 \|w\|_1, \tag{2.10}$$

holds for any function $w \in H^1$.

The constants $C$, $C_1$, $C_2$, etc. are generic in the paper, i.e., they may represent different constant quantities in different occurrences.

We say that a vector function $u(x,t)$ is the variational solution of the problem (2.1), (2.2), (2.3) if it maps, as a function of the variable $t$, the interval $[0,T]$ into $H_0^1$, if the identity

$$\left(v, \frac{\partial u}{\partial t}\right)_0 = -a(v,u) + (v,f)_0 \tag{2.11}$$

holds for each $t \in (0,T]$ and all functions $v \in H_0^1$, and if the identity

$$a(v,u) = a(v,u^0) \tag{2.12}$$

holds for $t = 0$ and all functions $v \in H_0^1$.

In this variational formulation as well as in the whole paper, the variable $t$ appears as a parameter. Without explicitly stating, we assume that all the statements and, in particular, constants may depend on $t$.

## 3. Discretization

Finite element solutions of the model problem (2.1), (2.2), (2.3) or, in the variational formulation, (2.11), (2.12) are now constructed in finite dimensional subspaces of $H_0^1$. We first introduce a partition

$$0 = x_0 < x_1 < \ldots < x_{M-1} < x_M = 1$$

of the interval $[0,1]$ into $M$ subintervals $(x_{j-1}, x_j)$, $j = 1,\ldots,M$. We choose a fixed positive integer $p$, put

$$\begin{aligned} S^{M,p} = \{W = (W_1(x),\ldots,W_N(x))^{\mathrm{T}} \mid W \in H^1,\ W_k(x) \in P_p(x) \\ \text{for } x \in [x_{j-1}, x_j],\ j = 1,\ldots,M,\ k = 1,\ldots,N\}, \end{aligned}$$

where $P_p(x)$ is the class of polynomials of degree $p$ in $x$, and take

$$S_0^{M,p} = \{W \mid W \in S^{M,p} \cap H_0^1\}$$

for the approximating finite dimensional subspace to $H_0^1$. We further put

$$h_j = x_j - x_{j-1}, \quad j = 1,\ldots,M,$$

and

$$h = \max_{j=1,\dots,M} h_j.$$

We then say that a vector function $U(x,t) = (U_1, \dots, U_N)^{\mathrm{T}}$ is the finite element approximate solution of the model problem (2.11), (2.12) if it maps, as a function of the variable $t$, the interval $[0,T]$ into $S_0^{M,p}$, if the identity

$$\left(V, \frac{\partial U}{\partial t}\right)_0 = -a(V,U) + (V,f)_0 \tag{3.1}$$

holds for each $t \in (0,T]$ and all functions $V = (V_1, \dots, V_N)^{\mathrm{T}} \in S_0^{M,p}$, and if the identity

$$a(V,U) = a(V,u^0) \tag{3.2}$$

holds for $t = 0$ and all functions $V \in S_0^{M,p}$.

Remark 3.1. Choosing a basis $\{\varrho^{(r)}\}_{r=1}^{R}$, $\varrho^{(r)} = (\varrho_1^{(r)}, \dots, \varrho_N^{(r)})^{\mathrm{T}}$, for the finite dimensional space $S_0^{M,p}$, putting

$$U(x,t) = \sum_{r=1}^{R} c_r(t)\varrho^{(r)}(x)$$

with scalar coefficients $c_r(t)$ depending on $t$, and introducing the test functions

$$V(x) = \varrho^{(r)}(x), \quad r = 1, \dots, R,$$

for (3.1), (3.2), we finally obtain an initial value problem for a system of ordinary differential equations with unknown functions $c_r(t)$. This procedure for constructing the approximate solution $U(x,t)$ is called the *method of lines.* In what follows we assume that this ordinary differential system is integrated exactly. In practice, the system is solved by proper numerical software (e.g. LSODI [6] or DDASSL [8]) that works with a prescribed tolerance for the error of integration in $t$.

To obtain an a posteriori error estimate, we denote the discretization error of the finite element solution $U$ by

$$e(x,t) = u(x,t) - U(x,t), \tag{3.3}$$

$e = (e_1, \dots, e_N)^{\mathrm{T}}$. Then we can substitute

$$u(x,t) = e(x,t) + U(x,t)$$

into (2.11), (2.12) to arrive at the identity

$$(3.4)\qquad \left(v, \frac{\partial e}{\partial t}\right)_0 = -a(v,e) + (v,f)_0 - a(v,U) - \left(v, \frac{\partial U}{\partial t}\right)_0$$

that holds for each $t \in (0,T]$ and all functions $v \in H_0^1$ and the identity

$$(3.5)\qquad a(v,e) = a(v, u^0 - U)$$

holding for $t = 0$ and all functions $v \in H_0^1$.

We construct a finite element approximation $E(x,t) = (E_1, \dots, E_N)^{\mathrm{T}}$ of the discretization error in the finite dimensional subspace $\hat{S}_0^{M,p+1}$ defined as follows. A function $W$ belongs to $\hat{S}_0^{M,p+1}$ if

$$W(x) = \sum_{j=1}^{M} {}^{j}W(x),$$

where the vector functions ${}^{j}W$ belong to $\hat{S}_{0,j}^{p+1}$ and

$$\begin{aligned}\hat{S}_{0,j}^{p+1} = \{Z = (Z_1(x), \dots, Z_N(x))^{\mathrm{T}} \mid Z \in H_0^1,\ Z_k(x) \in P_{p+1}(x) \\ \text{for } x \in [x_{j-1}, x_j],\ Z_k \equiv 0 \text{ elsewhere},\ k = 1, \dots, N\}, \quad j = 1, \dots, M.\end{aligned}$$

Analogically to the formulae (2.4) to (2.8), we introduce local function spaces $H_j^s = H^s(x_{j-1}, x_j)$ and $H_{0,j}^s = H_0^s(x_{j-1}, x_j)$ on the individual intervals $(x_{j-1}, x_j)$, and also the corresponding inner products and norms. We put

$$\begin{aligned}(v,w)_{0,j} &= \int_{x_{j-1}}^{x_j} v^{\mathrm{T}} w \,\mathrm{d}x\,, \\ (v,w)_{s,j} &= \sum_{r=1}^{s} \left(\frac{\partial^r v}{\partial x^r}, \frac{\partial^r w}{\partial x^r}\right)_{0,j}, \\ \|w\|_{s,j}^2 &= (w,w)_{s,j}, \\ a_j(v,w) &= \int_{x_{j-1}}^{x_j} \left(\frac{\partial v^{\mathrm{T}}}{\partial x} A \frac{\partial w}{\partial x} + v^{\mathrm{T}} B w\right) \mathrm{d}x\,, \\ \|w\|_{A,j}^2 &= a_j(w,w)\end{aligned}$$

for functions of the respective spaces. Moreover, there are positive constants $C_1$, $C_2$ such that, due to the Friedrichs inequality,

$$(3.6)\qquad C_1 \|w\|_{1,j} \leqslant \|w\|_{A,j} \leqslant C_2 \|w\|_{1,j}$$

for any function $w \in H^1_{0,j}$. Further, the second inequality in (3.6), i.e.

$$\|w\|_{A,j} \leqslant C_2 \|w\|_{1,j}, \tag{3.7}$$

holds for any function $w \in H^1_j$. The approximation $E \in \hat{S}_0^{M,p+1}$ now satisfies a series of $M$ uncoupled local parabolic problems: Find $E \in \hat{S}_0^{M,p+1}$ such that

$$\left(V, \frac{\partial E}{\partial t}\right)_{0,j} = -a_j(V,E) + (V,f)_{0,j} - a_j(V,U) - \left(V, \frac{\partial U}{\partial t}\right)_{0,j}, \quad 0 < t \leqslant T, \tag{3.8}$$

$$a_j(V,E) = a_j(V, u^0 - U), \quad t = 0, \tag{3.9}$$

hold for all functions $V \in \hat{S}^{p+1}_{0,j}$ and $j = 1, \ldots, M$. Each problem (3.8), (3.9) for the a posteriori error estimate $E$ on $(x_{j-1}, x_j)$ is again solved by the method of lines. In what follows we assume that the corresponding ordinary differential system is integrated exactly. In practice it is solved by proper numerical software which yields the solution with sufficient accuracy.

To save some computation we can neglect the time change of the approximate error and solve, instead of (3.8), (3.9), local elliptic problems: Find $\tilde{E} \in \hat{S}_0^{M,p+1}$ such that

$$0 = -a_j(V,\tilde{E}) + (V,f)_{0,j} - a_j(V,U) - \left(V, \frac{\partial U}{\partial t}\right)_{0,j}, \quad 0 < t \leqslant T, \tag{3.10}$$

$$a_j(V,\tilde{E}) = a_j(V, u^0 - U), \quad t = 0, \tag{3.11}$$

hold for all functions $V \in \hat{S}^{p+1}_{0,j}$ and $j = 1, \ldots, M$.

In Sections 5 and 6 we will show that both the parabolic error estimate $E$ given by (3.8) and (3.9), and the elliptic error estimate $\tilde{E}$ given by (3.10) and (3.11) converge to the exact error $e$ in the $H^1$ norm as $h \to 0$. To this end we will use some known approximation and a priori error estimation results for finite element solutions.

**Lemma 3.1.** *Let $W \in S^{M,p}$ interpolate $w \in H^{p+1}$ at $x_{j-1}, x_j$, and further $p-1$ distinct points on each $(x_{j-1}, x_j)$, $j = 1, \ldots, M$ (i.e., let each component of $W$ interpolate the corresponding component of $w$). Then there exists a positive constant $C$ such that*

$$\|w - W\|_s \leqslant C h^{p+1-s} \|w\|_{p+1}, \quad s = 0, 1, \ldots, p. \tag{3.12}$$

Proof. The scalar version of the statement is proved, e.g., in Oden and Carey [7]. The statement (3.12) is its simple consequence. □

**Lemma 3.2.** *Let $u$ and $U$ be solutions of* (2.11), (2.12) *and* (3.1), (3.2), *respectively. Let $\hat{U} \in S_0^{M,p}$ be the energy projection of $u$ onto $S_0^{M,p}$, i.e., let*

$$a(V, \hat{U}) = a(V, u) \tag{3.13}$$

*hold for all $V \in S_0^{M,p}$ and $0 \leqslant t \leqslant T$. If $u^0 \in H_0^1 \cap H^{p+1}$ and if $\partial u(x,t)/\partial t \in H^{p+1}$ as a function of $x$ for $0 \leqslant t \leqslant T$ then there exists a positive constant $C$ such that*

$$\|\hat{U} - U\|_1^2 \leqslant Ch^{2p+2} \int_0^t \left\| \frac{\partial u}{\partial t}(.\,,\tau) \right\|_{p+1}^2 \mathrm{d}\tau\,, \quad 0 \leqslant t \leqslant T. \tag{3.14}$$

Proof. The proof can be found, e.g., in Wait and Mitchell [11] and easily modified for the nonscalar case. □

**Lemma 3.3.** *Let $u$ and $U$ be solutions of* (2.11), (2.12) *and* (3.1), (3.2), *respectively. If $u^0 \in H_0^1 \cap H^{p+1}$ and if $u$ is smooth enough for all terms in* (3.15), (3.16) *to be bounded then there exist positive constants $\delta$ and $C$ such that*

$$\begin{aligned} \left\| \frac{\partial^r e}{\partial t^r}(.\,,t) \right\|_0 \leqslant Ch^{p+1} \Bigg( \|u^0\|_{p+1} &+ \sum_{l=0}^{r} \left\| \frac{\partial^l u}{\partial t^l}(.\,,t) \right\|_{p+1} \\ &+ \int_{t-\delta}^{t} \left\| \frac{\partial^{r+1} u}{\partial t^{r+1}}(.\,,\tau) \right\|_{p+1} \mathrm{d}\tau \\ &+ \int_0^t \left\| \frac{\partial u}{\partial t}(.\,,\tau) \right\|_2 \mathrm{d}\tau \Bigg), \quad \delta < t \leqslant T, \end{aligned} \tag{3.15}$$

$$\begin{aligned} \left\| \frac{\partial^r e}{\partial t^r}(.\,,t) \right\|_1 \leqslant Ch^{p} \Bigg( \|u^0\|_{p+1} &+ \sum_{l=0}^{r} \sup_{t-\delta<\tau<t} \left\| \frac{\partial^l u}{\partial t^l}(.\,,\tau) \right\|_{p+1} \\ &+ \left( \int_{t-\delta}^{t} \left\| \frac{\partial^{r+1} u}{\partial t^{r+1}}(.\,,\tau) \right\|_{p}^2 \mathrm{d}\tau \right)^{1/2} \\ &+ h\left( \|u^0\|_0 + \int_0^t \|f\|_0 \,\mathrm{d}\tau \right) \Bigg), \quad \delta < t \leqslant T, \end{aligned} \tag{3.16}$$

*where $r$ is a nonnegative integer.*

Proof. The proof is a straightforward modification of the proofs of Theorems 3 and 5 of Thomée [10]. □

## 4. Error Analysis

Let us start the error analysis with the following definition of a scalar interpolation operator possessing particular properties. We keep the assumption that $p$ is an arbitrary but fixed positive integer throughout the paper.

**Definition 4.1.** Consider the interval $[-1,1]$. Denote by $\Pi$ an interpolation operator that maps a scalar function onto $P_p(\xi)$, $\xi \in [-1,1]$. Let the $p+1$ interpolation nodes of $\Pi$ be determined as follows:

Put $p = 2l-1$ if $p$ is odd or $p = 2l$ if $p$ is even. The $2l$ interpolation nodes $\{\pm\xi_i\}_{i=1}^{l}$ are placed symmetrically with respect to the origin. If $p = 2l$ is even then they are augmented with the node $\xi_0 = 0$. In both the cases we put $\xi_l = 1$. We introduce a scalar function

$$\Psi(\xi) = \xi^{p+1} - \Pi\xi^{p+1}, \quad \xi \in [-1,1], \tag{4.1}$$

and determine the remaining $2l-2$ interpolation nodes $\{\pm\xi_i\}_{i=1}^{l-1}$ so that

$$\int_{-1}^{1} \Psi'(\xi)\xi^s \, \mathrm{d}\xi = 0, \quad s = 0,1,\ldots,p-1, \tag{4.2}$$

where the prime denotes the total differentiation since $\Psi$ depends on only one scalar variable.

The dependence of $\Pi$ and $\Psi$ on $p$ is not explicitly expressed, the value of $p$ being assumed fixed. The existence of the operator $\Pi$ and an explicit formula for the function $\Psi$ together with some its properties are presented in the following lemma.

**Lemma 4.1.** *There exists the interpolation operator $\Pi$ of Definition* 4.1. *Moreover, for the scalar function $\Psi$ given by* (4.1) *we have*

$$\Psi(\xi) = \frac{1}{\sqrt{2(2p+1)}}\big(\mathrm{P}_{p+1}(\xi) - \mathrm{P}_{p-1}(\xi)\big), \tag{4.3}$$

*where $\mathrm{P}_r$ is the Legendre polynomial of degree $r$. Further*

$$\Psi'(\xi) = \sqrt{\frac{2p+1}{2}}\,\mathrm{P}_p(\xi) \tag{4.4}$$

*and*

$$\int_{-1}^{1} \Psi^2(\xi)\, \mathrm{d}\xi = \frac{2}{(2p-1)(2p+3)}, \tag{4.5}$$

$$\int_{-1}^{1} \Psi'^2(\xi)\, \mathrm{d}\xi = 1. \tag{4.6}$$

P r o o f. The existence of $\Pi$ is proved in [2]. Briefly, if $p = 2l - 1$ is odd, $\Psi$ can be shown to be an even function, $\Psi'$ is odd, and the conditions (4.2) for $s = 0$ and $s$ even are satisfied identically. The remaining $l - 1$ conditions with $s$ odd determine the $l - 1$ interpolation nodes $\xi_i$, $i = 1, \ldots, l - 1$. Similarly, if $p = 2l$ is even, $\Psi$ is an odd function, $\Psi'$ is even and the conditions (4.2) for $s$ odd are satisfied identically. For $s = 0$, (4.2) is fulfilled, too, with regard to (4.1) and the symmetry of the interpolation nodes. The remaining $l - 1$ conditions with $s$ even determine the $l - 1$ interpolation nodes $\xi_i$, $i = 1, \ldots, l - 1$.

Comparing the conditions (4.1) and (4.2) with the properties of Legendre polynomials $\mathrm{P}_p(\xi)$, we find out (4.4). Integrating (4.4), observing that $\Psi(-1) = \Psi(1) = 0$ (by placement of the nodes), and using the well-known recurrence formulae for Legendre polynomials, we finally obtain (4.3). A straightforward calculation yields (4.5), (4.6). □

R e m a r k 4.1. The functions $\Psi$ defined by (4.1) and expressed by (4.3) form a hierarchical finite element basis for the increasing sequence of $p$, see [9].

**Definition 4.2.** Consider the interval $[x_{j-1}, x_j]$, $j = 1, \ldots, M$. Denote by $\Pi_j$ an interpolation operator that maps a scalar function onto $P_p(x)$, $x \in [x_{j-1}, x_j]$. Let its $p+1$ interpolation nodes be obtained from the interpolation nodes of the operator $\Pi$ by a linear transformation of $[-1, 1]$ onto $[x_{j-1}, x_j]$.

We further introduce a scalar function

$$\begin{aligned} \Psi_j(x) &= x^{p+1} - \Pi_j x^{p+1}, \quad x \in [x_{j-1}, x_j], \\ &= 0 \quad \text{elsewhere.} \end{aligned} \tag{4.7}$$

Let $w = (w_1, \ldots, w_N)^{\mathrm{T}} \in H_j^1$ be a vector function. We introduce an interpolation operator $\pi_j$ which maps each component of a vector function onto $P_p(x)$, $x \in [x_{j+1}, x_j]$, with the help of the operator $\Pi_j$, i.e.

$$\pi_j w = (\Pi_j w_1, \ldots, \Pi_j w_N)^{\mathrm{T}}.$$

Finally we introduce an interpolation operator $\pi$ which maps a vector function $w \in H^1$ onto an interpolate $\pi w$ that agrees with $\pi_j w$ for $x \in [x_{j-1}, x_j]$, $j = 1, \ldots, M$.

R e m a r k 4.2. The interpolation nodes are thus the same for interpolating the individual components of $w$.

We will be interested in the norm of scalar functions, too. For ease, we denote by $\|\Phi\|_{s,j}$ the $H^s$ norm of a scalar function $\Phi$ on $(x_{j-1}, x_j)$, $j = 1, \ldots, M$.

**Lemma 4.2.** *Let $\Psi_j$, $j = 1, \dots, M$, be the functions defined by* (4.7). *Then*

$$\|\Psi_j\|^2_{0,j} = \int_{x_{j-1}}^{x_j} \Psi_j^2(x)\,\mathrm{d}x = \Big(\frac{h_j}{2}\Big)^{2p+3} \frac{2}{(2p-1)(2p+3)}, \tag{4.8}$$

$$\|\Psi'_j\|^2_{0,j} = \int_{x_{j-1}}^{x_j} \Psi'^2_j(x)\,\mathrm{d}x = \Big(\frac{h_j}{2}\Big)^{2p+1}. \tag{4.9}$$

Proof. Due to (4.1), $\Psi$ is a polynomial of degree $p+1$ and its $p+1$ zeros are the interpolation nodes $\xi_i$ of $\Pi$ on the interval $[-1,1]$ (see Definition 4.1). Renumbering the nodes properly, we can write

$$\Psi(\xi) = \prod_{r=1}^{p+1} (\xi - \xi_r). \tag{4.10}$$

The linear transformation

$$x = \frac{1}{2}(h_j\xi + x_{j-1} + x_j) \tag{4.11}$$

maps $[-1,1]$ onto $[x_{j-1}, x_j]$. This transformation maps the interpolation nodes $\xi_r$ of $\Pi$ into the interpolation nodes

$$\eta_r = \frac{1}{2}(h_j\xi_r + x_{j-1} + x_j) \tag{4.12}$$

of $\Pi_j$. The corresponding substitution in $\Psi$ yields

$$\Psi(\xi) = \Big(\frac{2}{h_j}\Big)^{p+1} \Psi_j(x)$$

with regard to (4.10), (4.12).

Similarly,

$$\Psi'(\xi) = \sum_{q=1}^{p+1} \prod_{\substack{r=1\\ r\neq q}}^{p+1} (\xi - \xi_r)$$

and the substitution (4.11) gives

$$\Psi'(\xi) = \Big(\frac{2}{h_j}\Big)^{p} \Psi'_j(x).$$

Performing now the substitution (4.11) with $\mathrm{d}x = \frac{1}{2}h_j\,\mathrm{d}\xi$ in (4.8) and (4.9) and employing (4.5), (4.6) of Lemma 4.1, we arrive at the statement of the lemma by straightforward calculation. □

The difference between $u$ and its interpolate $\pi_j u$ on $[x_{j-1}, x_j]$ can be expressed with the help of the scalar function $\Psi_j$.

**Lemma 4.3.** *Let $u \in H_j^{p+2}$, $j = 1, \ldots, M$. Then*

$$u - \pi_j u = \varphi_j + \gamma_j, \tag{4.13}$$

*where $\varphi_j = (\varphi_{1j}, \ldots, \varphi_{Nj})^{\mathrm{T}}$, $\gamma_j = (\gamma_{1j}, \ldots, \gamma_{Nj})^{\mathrm{T}}$, and*

$$\varphi_{kj}(x,t) = \beta_{kj}(t)\Psi_j(x), \quad k = 1, \ldots, N, \tag{4.14}$$

$$\beta_j = (\beta_{1j}, \ldots, \beta_{Nj})^{\mathrm{T}}. \tag{4.15}$$

*Further, there exists a positive constant $C$ such that, for $0 \leqslant t \leqslant T$,*

$$\left\| \frac{\partial^r \varphi_j}{\partial t^r} \right\|_{s,j} \leqslant C h_j^{p+1-s} \left\| \frac{\partial^r u}{\partial t^r} \right\|_{p+1,j}, \quad \text{where } s = 0, 1 \tag{4.16}$$

*and $r$ is an arbitrary integer for which the expressions exist,*

$$\left\| \frac{\partial \varphi_j}{\partial x} \right\|_{0,j} \leqslant C h_j^{p} \|u\|_{p+1,j}, \tag{4.17}$$

$$\|\gamma_j\|_{s,j} \leqslant C h_j^{p+2-s} \|u\|_{p+2,j}, \quad s = 0, 1, \ldots, p+1, \tag{4.18}$$

$$\left\| \frac{\partial \gamma_j}{\partial x} \right\|_{s,j} \leqslant C h_j^{p+1-s} \|u\|_{p+2,j}, \quad s = 0, 1. \tag{4.19}$$

Proof. Let $V = (V_1, \ldots, V_N)^{\mathrm{T}}$, $V_k(x) \in P_{p+1}(x)$, $x \in [x_{j-1}, x_j]$ and $k = 1, \ldots, N$, be such that $\pi_j V = \pi_j u$ and, moreover, interpolate $u$ at one additional node on $(x_{j-1}, x_j)$. Putting

$$\varphi_j = V - \pi_j V, \tag{4.20}$$

$$\gamma_j = u - V, \tag{4.21}$$

we obtain (4.13). We further have $V_k(x) = \beta_{kj} x^{p+1} + W_k(x)$ where $W_k = \Pi_j W_k \in P_p(x)$, $x \in [x_{j-1}, x_j]$, $k = 1, \ldots, N$. Substituting into (4.20) and taking (4.7) into account, we arrive at (4.14).

In the same way as in the proof of Lemma 3.3 of [2], we find from (4.21) and from Lemma 3.1 that

$$\|\gamma_j\|_{s,j} = \|u - V\|_{s,j} \leqslant C h_j^{p+2-s} \|u\|_{p+2,j}, \quad s = 0, 1, \ldots, p+1,$$

which is (4.18). Moreover, from (4.18) and (4.21) we deduce

$$\|V\|_{p+1,j} = \|V - u\|_{p+1,j} + \|u\|_{p+1,j} \leqslant C h_j \|u\|_{p+2,j} + \|u\|_{p+1,j} \leqslant C \|u\|_{p+1,j}. \tag{4.22}$$

The bound (4.22), (4.20), and Lemma 3.1 then yield the statements (4.16) for $r = 0$, (4.17), and (4.19). Since, due to (4.14), $t$ in $\varphi_j(x,t)$ can be considered as a parameter we easily prove (4.16) for $r > 0$, too. □

The next lemma is concerned with a bound for the bilinear form $a(v,w)$ defined in (2.7).

**Lemma 4.4.** *Let $\pi v \in S^{M,p}$ be an interpolate of $v \in H^{p+2}(0,1)$ that agrees with $\pi_j v$ for $x \in [x_{j-1}, x_j]$, $j = 1, \ldots, M$. Then*

$$|a(W, v - \pi v)| \leqslant Ch^{p+1}\|v\|_{p+2}\|W\|_1$$

*holds for all $W \in S_0^{M,p}$.*

P r o o f. The statement of the lemma follows immediately from the Schwarz inequality, the inequality (2.10), and Lemma 3.1. □

**Lemma 4.5.** *Let $u \in H_0^1 \cap H^{p+2}$ and $U \in S_0^{M,p}$ and $\hat{U} \in S_0^{M,p}$ be solutions of (2.11), (2.12) and (3.1), (3.2) and (3.13), respectively. Further let $\pi u \in S_0^{M,p}$ be an interpolate of $u$ that agrees with $\pi_j u$ for $x \in [x_{j-1}, x_j]$, $j = 1, \ldots, M$, and let the assumptions of Lemma 3.3 be fulfilled. Then*

$$\|\hat{U} - \pi u\|_1 \leqslant Ch^{p+1}\|u\|_{p+2}, \tag{4.23}$$

$$e(x,t) = \varphi(x,t) + \omega(x,t), \tag{4.24}$$

*where $\varphi = (\varphi_1, \ldots, \varphi_N)^{\mathrm{T}}$, $\omega = (\omega_1, \ldots, \omega_N)^{\mathrm{T}}$ and $\omega_j = (\omega_{1j}, \ldots, \omega_{Nj})^{\mathrm{T}}$ with*

$$\omega_j(x,t) = \gamma_j(x,t) + \pi_j u(x,t) - U(x,t), \quad j = 1, \ldots, M, \tag{4.25}$$

*and*

$$\left\|\frac{\partial \varphi}{\partial x}\right\|_0 \leqslant Ch^p\|u\|_{p+1}, \tag{4.26}$$

$$\left\|\frac{\partial \omega}{\partial x}\right\|_0 \leqslant C(u)h^{p+1}, \tag{4.27}$$

$$\left\|\frac{\partial^r \omega}{\partial t^r}\right\|_s \leqslant \left\|\frac{\partial^r e}{\partial t^r}\right\|_s + \left\|\frac{\partial^r \varphi}{\partial t^r}\right\|_s \tag{4.28}$$

*for all nonnegative integers $r$ and $s$ for which the expressions in (4.28) exist.*

*Further, let $\delta > 0$ be the constant from Lemma 3.3. Then*

$$\left\|\frac{\partial^r \omega}{\partial t^r}\right\|_s \leqslant C(u)h^{p+1-s} \quad \text{for } \delta < t \leqslant T,\ s = 0, 1, \tag{4.29}$$

*and any nonnegative integer $r$ for which the terms in (3.15), (3.16) exist, and*

$$\|\omega\|_1 \leqslant C(u)h^{p+1} \quad \text{for } \delta < t \leqslant T. \tag{4.30}$$

Proof. Subtracting $a(V, \pi u)$ from (3.13), using Lemma 4.4, replacing $V$ by $\hat{U} - \pi u$ in the result, and applying (2.9) establishes (4.23).

Keeping the notation (3.3), (4.13) and putting

$$e = u - \pi_j u + \pi_j u - U = \varphi_j + \omega_j, \quad x \in [x_{j-1}, x_j],$$

we arrive at (4.25). If we define $\varphi(x,t)$ and $\omega(x,t)$ as functions on $[0,1]$ whose restrictions to $[x_{j-1}, x_j]$ are $\varphi_j(x,t)$ and $\omega_j(x,t)$, respectively, we obtain (4.24).

Using Lemma 4.3, putting

$$\frac{\partial \omega_j}{\partial x} = \frac{\partial \gamma_j}{\partial x} + \frac{\partial}{\partial x}(\pi_j u - \hat{U}) + \frac{\partial}{\partial x}(\hat{U} - U),$$

and employing again Lemma 4.3, Lemma 3.3, and the bound (4.23), we obtain the statements (4.26), (4.27) in the same way as in the proof of Lemma 3.5 of [2].

Expressing

$$\omega(x,t) = e(x,t) - \varphi(x,t)$$

from (4.24), we obtain (4.28) for all values of $r$ and $s$ for which the expressions exist. We now use the estimates (3.15), (3.16) of Lemma 3.3 for $e$ and (4.16) of Lemma 4.3 for $\varphi$ to show (4.29) for $\delta < t \leqslant T$, $r \geqslant 0$, and $s = 0, 1$.

Further, (4.27) and (4.29) with $r = s = 0$ imply

$$\|\omega\|_1^2 = \left\| \frac{\partial \omega}{\partial x} \right\|_0^2 + \|\omega\|_0^2 \leqslant C(u) h^{2p+2}$$

for $\delta < t \leqslant T$, which is (4.30). □

## 5. Parabolic Error Estimation

To calculate an approximation to error, we use the vector functions

$$\sigma_j^{(k)} = (0, \dots, \Psi_j, \dots, 0)^{\mathrm{T}} \tag{5.1}$$

whose $k$th component is nonzero, $k = 1, \dots, N$, and that belong to $\hat{S}_{0,j}^{p+1}$, $j = 1, \dots, M$. We put

$$E(x,t) = \sum_{j=1}^{M} \sum_{k=1}^{N} b_{kj}(t) \sigma_j^{(k)}(x) = \sum_{j=1}^{M} \left(b_{1j}(t)\Psi_j(x), \dots, b_{Nj}(t)\Psi_j(x)\right)^{\mathrm{T}} \tag{5.2}$$

and show that this function $E(x,t)$ can be taken for an a posteriori approximation of the error $e(x,t)$.

Substituting $\sigma_j^{(l)}$ for $V$ into (3.8), (3.9), we obtain the equations

$$(5.3)\qquad \left(\sigma_j^{(l)}, \frac{\partial E}{\partial t}\right)_{0,j} + a_j(\sigma_j^{(l)}, E) = R_j(\sigma_j^{(l)}, t), \quad 0 < t \leqslant T,$$

$$(5.4)\qquad a_j(\sigma_j^{(l)}, E) = a_j(\sigma_j^{(l)}, u^0 - U), \quad t = 0,\ l = 1, \ldots, N,\ j = 1, \ldots, M,$$

where we denoted by

$$(5.5)\qquad R_j(\sigma_j^{(l)}, t) = (\sigma_j^{(l)}, f)_{0,j} - a_j(\sigma_j^{(l)}, U) - \left(\sigma_j^{(l)}, \frac{\partial U}{\partial t}\right)_{0,j}$$

the residual of the equation (3.1) on $(x_{j-1}, x_j)$ after substituting $\sigma_j^{(l)}$ for $V$. Substituting now (5.2) into (5.3), (5.4), we finally obtain $M$ uncoupled local initial value problems for systems of $N$ ordinary differential equations

$$\sum_{k=1}^{N} b'_{kj}(t)(\sigma_j^{(l)}, \sigma_j^{(k)})_{0,j} + \sum_{k=1}^{N} b_{kj}(t)\, a_j(\sigma_j^{(l)}, \sigma_j^{(k)}) = R_j(\sigma_j^{(l)}, t),$$
$$0 < t \leqslant T,\ l = 1, \ldots, N,\ j = 1, \ldots, M,$$

for the unknown coefficients $b_{kj}(t)$ of (5.2). By (4.8) and (5.1) we have

$$(5.6)\qquad \begin{aligned} &(\sigma_j^{(l)}, \sigma_j^{(k)})_{0,j} = 0 \quad \text{for } l \neq k, \\ &(\sigma_j^{(k)}, \sigma_j^{(k)})_{0,j} = \Sigma_{0j}^2 \neq 0 \text{ independently of } k, \end{aligned}$$

where we used the notation

$$(5.7)\qquad \Sigma_{sj} = \|\Psi_j\|_{s,j}, \quad s = 0, 1.$$

We then arrive at

$$b'_{lj}(t)\Sigma_{0j}^2 + \sum_{k=1}^{N} b_{kj}(t)\, a_j(\sigma_j^{(l)}, \sigma_j^{(k)}) = R_j(\sigma_j^{(l)}, t), \quad 0 < t \leqslant T,\ l = 1, \ldots, N,$$

with the initial conditions

$$\sum_{k=1}^{N} b_{kj}(0)\, a_j(\sigma_j^{(l)}, \sigma_j^{(k)}) = a_j(\sigma_j^{(l)}, u^0 - U), \quad l = 1, \ldots, N,$$

where $j = 1, \ldots, M$.

Introducing the vectors

$$b_j = (b_{1j}, \dots, b_{Nj})^{\mathrm{T}}, \tag{5.8}$$

$$r_j = \left(\Sigma_{0j}^{-2} R_j(\sigma_j^{(1)}, t), \dots, \Sigma_{0j}^{-2} R_j(\sigma_j^{(N)}, t)\right)^{\mathrm{T}} \tag{5.9}$$

and matrices

$$S_j = \left(\Sigma_{0j}^{-2} a_j(\sigma_j^{(l)}, \sigma_j^{(k)})\right)_{l,k=1}^{N} \tag{5.10}$$

for $j = 1, \dots, M$, we arrive at the systems

$$b_j'(t) + S_j b_j(t) = r_j(t), \quad 0 < t \leqslant T, \tag{5.11}$$

with the initial conditions

$$b_j(0) = T_j^{-1} v_j, \quad t = 0,\ j = 1, \dots, M, \tag{5.12}$$

where

$$\begin{aligned} T_j &= \Sigma_{0j}^2 S_j = \left(a_j(\sigma_j^{(l)}, \sigma_j^{(k)})\right)_{l,k=1}^{N}, \\ v_j &= \left(a_j(\sigma_j^{(1)}, u^0 - U), \dots, a_j(\sigma_j^{(N)}, u^0 - U)\right)^{\mathrm{T}}. \end{aligned} \tag{5.13}$$

Using now (4.14), (4.24) and (5.1), we have

$$e(x,t) = \sum_{k=1}^{N} \beta_{kj}(t) \sigma_j^{(k)}(x) + \omega_j(x,t), \quad x \in [x_{j-1}, x_j]. \tag{5.14}$$

Substituting (5.14) for $e$ and (5.1) for $v$ into (3.4), (3.5) and using the notation (5.5), (5.6), and (5.8), we find for $j = 1, \dots, M$ that

$$\beta_{lj}'(t) \Sigma_{0j}^2 + \sum_{k=1}^{N} \beta_{kj}(t)\, a_j(\sigma_j^{(l)}, \sigma_j^{(k)}) = R_j(\sigma_j^{(l)}, t) - G_j(\sigma_j^{(l)}, t),$$
$$0 < t \leqslant T,\ l = 1, \dots, N,$$

where

$$G_j(\sigma_j^{(l)}, t) = \left(\sigma_j^{(l)}, \frac{\partial \omega_j}{\partial t}\right)_{0,j} + a_j(\sigma_j^{(l)}, \omega_j), \tag{5.15}$$

with the initial conditions

$$\sum_{k=1}^{N} \beta_{kj}(0)\, a_j(\sigma_j^{(l)}, \sigma_j^{(k)}) = a_j(\sigma_j^{(l)}, u^0 - \omega_j - U), \quad t = 0,\ l = 1, \dots, N.$$

Introducing now the vector

$$g_j(t)\big(\Sigma_{0j}^{-2}G_j(\sigma_j^{(1)},t),\dots,\Sigma_{0j}^{-2}G_j(\sigma_j^{(N)},t)\big)^{\mathrm{T}}, \tag{5.16}$$

we have finally the systems

$$\beta_j'(t)+S_j\beta_j(t)=r_j(t)-g_j(t),\quad 0<t\leqslant T, \tag{5.17}$$

with the initial conditions

$$\beta_j(0)=T_j^{-1}w_j,\quad t=0,\ j=1,\dots,M, \tag{5.18}$$

where we use the notation (4.15), (5.9), (5.10), (5.13), and

$$w_j=\big(a_j(\sigma_j^{(1)},u^0-U-\omega_j),\dots,a_j(\sigma_j^{(N)},u^0-U-\omega_j)\big)^{\mathrm{T}}.$$

Moreover, putting

$$\alpha_j(t)=b_j(t)-\beta_j(t),$$

and subtracting the system (5.17) from (5.11) and the initial condition (5.18) from (5.12), we obtain the ordinary system

$$\alpha_j'(t)+S_j\alpha_j(t)=g_j(t),\quad 0<t\leqslant T,\ j=1,\dots,M, \tag{5.19}$$

and the initial condition

$$\alpha_j(0)=T_j^{-1}z_j,\quad t=0,\ j=1,\dots,M, \tag{5.20}$$

where

$$z_j=v_j-w_j=\Big(a_j(\sigma_j^{(1)},\omega_j(.\,,0)),\dots,a_j(\sigma_j^{(N)},\omega_j(.\,,0))\Big)^{\mathrm{T}}. \tag{5.21}$$

We will now prove several auxiliary statements.

**Lemma 5.1.** *There exist positive constants $C_1$ and $C_2$ such that*

$$C_1h_j^{2p+3}\leqslant\Sigma_{0j}^2\leqslant C_2h_j^{2p+3},\quad j=1,\dots,M, \tag{5.22}$$

*and*

$$C_1h_j^{2p+1}\leqslant\Sigma_{1j}^2\leqslant C_2h_j^{2p+1},\quad j=1,\dots,M. \tag{5.23}$$

*Further*

$$C_jh_j^{2p+1}\leqslant a_j(\sigma_j^{(k)},\sigma_j^{(k)})=\|\sigma_j^{(k)}\|_{A,j}^2\leqslant C_2h_j^{2p+1},\quad k=1,\dots,N,\ j=1,\dots,M. \tag{5.24}$$

Proof. The bounds (5.22) and (5.23) follow immediately from (5.7), and (4.8) and (4.9) of Lemma 4.2. Finally, (5.24) is a consequence of the equivalence of the norms $\|.\|_{1,j}$ and $\|.\|_{A,j}$ for functions from $H_{0,j}^1$ expressed by (3.6). □

We will use the notation $\|q\|^2 = q^{\mathrm{T}}q$ for the Euclidean norm of a vector $q$ and $\|Q\|$ for the spectral norm of a square matrix $Q$.

**Lemma 5.2.** *Let $T_j$ be given by (5.13) and $S_j$ by (5.10). Then there exist positive constants $C_1$, $C_2$ such that the bounds*

$$C_1 h_j^{2p+1} \leqslant \lambda(T_j) \leqslant C_2 h_j^{2p+1}, \quad j = 1, \dots, M, \tag{5.25}$$

$$C_1 h_j^{-2} \leqslant \lambda(S_j) \leqslant C_2 h_j^{-2}, \quad j = 1, \dots, M, \tag{5.26}$$

*hold for all the eigenvalues $\lambda(T_j)$ and $\lambda(S_j)$ of the matrices $T_j$ and $S_j$, respectively.*

Proof. Introducing the differential equation (2.1), we assumed $A(x)$ to be symmetric positive definite and $B(x)$ symmetric positive semidefinite. We thus deduce that

$$0 < \underline{\gamma}_A \|q\|^2 \leqslant \gamma_A(x)\|q\|^2 \leqslant q^{\mathrm{T}} A(x) q \leqslant \|A(x)\|\,\|q\|^2 \leqslant \bar{\gamma}_A \|q\|^2, \tag{5.27}$$

$$0 \leqslant \underline{\gamma}_B \|q\|^2 \leqslant \gamma_B(x)\|q\|^2 \leqslant q^{\mathrm{T}} B(x) q \leqslant \|B(x)\|\,\|q\|^2 \leqslant \bar{\gamma}_B \|q\|^2 \tag{5.28}$$

for every $N$-component vector $q \neq 0$. Obviously, $\gamma_A(x)$ and $\|A(x)\|$ (or $\gamma_B(x)$ and $\|B(x)\|$) are the smallest and the largest eigenvalue of $A(x)$ (or $B(x)$). Since $x \in [0,1]$, since the entries $a_{ik}(x)$ and $b_{ik}(x)$ are continuous, and since eigenvalues depend continuously on entries, the constants

$$\underline{\gamma}_A = \min_{x\in[0,1]} \gamma_A(x), \qquad \underline{\gamma}_B = \min_{x\in[0,1]} \gamma_B(x),$$
$$\bar{\gamma}_A = \max_{x\in[0,1]} \|A(x)\|, \quad \bar{\gamma}_B = \max_{x\in[0,1]} \|B(x)\|$$

do exist and, moreover, $\underline{\gamma}_A > 0$.

According to (5.1), (5.13), we can calculate the entry $(i,k)$ of the matrix $T_j$,

$$(T_j)_{ik} = a_j(\sigma_j^{(i)}, \sigma_j^{(k)}) = \int_{x_{j-1}}^{x_j} \left(a_{ik}(x)\Psi'^2_j(x) + b_{ik}(x)\Psi_j^2(x)\right) \mathrm{d}x\,, \quad j = 1, \dots, M.$$

Computing $q^{\mathrm{T}}T_j q$, $q \neq 0$, and taking into account (5.27) and (5.28), we arrive at

$$\begin{aligned}
0 &< \|q\|^2 \left( \underline{\gamma}_A \int_{x_{j-1}}^{x_j} \Psi'^2_j(x)\,\mathrm{d}x + \underline{\gamma}_B \int_{x_{j-1}}^{x_j} \Psi_j^2(x)\,\mathrm{d}x \right) \leqslant q^{\mathrm{T}} T_j q \\
&= \int_{x_{j-1}}^{x_j} \left(q^{\mathrm{T}} A(x) q \Psi'^2_j(x) + q^{\mathrm{T}} B(x) q \Psi_j^2(x)\right) \mathrm{d}x \\
&\leqslant \|q\|^2 \left( \bar{\gamma}_A \int_{x_{j-1}}^{x_j} \Psi'^2_j(x)\,\mathrm{d}x + \bar{\gamma}_B \int_{x_{j-1}}^{x_j} \Psi_j^2(x)\,\mathrm{d}x \right).
\end{aligned}$$

Lemma 4.2 now gives

$$0 < \|q\|^2(\underline{\gamma}_A^* h_j^{2p+1} + \underline{\gamma}_B^* h_j^{2p+3}) \leqslant q^{\mathrm{T}} T_j q \leqslant \|q\|^2(\bar{\gamma}_A^* h_j^{2p+1} + \bar{\gamma}_B^* h_j^{2p+3})$$

with some constants $\underline{\gamma}_A^*$, $\bar{\gamma}_A^*$, $\underline{\gamma}_B^*$, and $\bar{\gamma}_B^*$, which implies

$$0 < C_1 h_j^{2p+1} \leqslant \frac{q^{\mathrm{T}} T_j q}{\|q\|^2} \leqslant C_2 h_j^{2p+1}. \tag{5.29}$$

As $T_j$ is Gram matrix defined by the energy inner product (5.13) it is symmetric positive definite. Its smallest and largest eigenvalues are

$$\min_{q\neq 0} \frac{q^{\mathrm{T}} T_j q}{\|q\|^2}, \quad \max_{q\neq 0} \frac{q^{\mathrm{T}} T_j q}{\|q\|^2}$$

and are bounded by (5.29), which implies (5.25).

$S_j$ is a symmetric positive definite matrix as well. The statement (5.26) on $\lambda(S_j)$ thus follows from (5.10), (5.25), and (5.22) of Lemma 5.1. □

**Lemma 5.3.** *The solution $\alpha_j(t)$ of the system* (5.19) *with the initial condition* (5.20) *can be expressed in the form*

$$\alpha_j(t) = \exp(-S_j t)\alpha_j(0) + \int_0^t \exp(-S_j(t-\tau))g_j(\tau)\,\mathrm{d}\tau, \quad 0 \leqslant t \leqslant T,\ j = 1, \dots, M, \tag{5.30}$$

*where $\alpha_j(0)$ is given by* (5.20), (5.21), *and $S_j$ and $T_j$ by* (5.10) *and* (5.13). *Moreover,*

$$\begin{aligned} \|\alpha_j(t)\|^2 \leqslant C\Bigg( &\|\alpha_j(0)\|^2 \exp(-2t\lambda_{\min}(S_j)) \\ &+ \frac{1}{\lambda_{\min}^2(S_j)}\Big\{\|g_j(t)\|^2 + \|g_j(0)\|^2 \exp(-2t\lambda_{\min}(S_j))\Big\} \\ &+ \frac{1}{\lambda_{\min}^3(S_j)} \exp(-2(t-\delta)\lambda_{\min}(S_j)) \int_0^\delta \|g_j'(\tau)\|^2\,\mathrm{d}\tau \\ &+ \frac{1}{\lambda_{\min}^3(S_j)} \int_\delta^t \|g_j'(\tau)\|^2\,\mathrm{d}\tau \Bigg), \quad j = 1, \dots, M, \end{aligned} \tag{5.31}$$

*for any $\delta$, $0 < \delta < t$, where $\lambda_{\min}(S_j)$ is the smallest eigenvalue of $S_j$.*

P r o o f. Solving (5.19), (5.20), we obtain (5.30), where $\alpha_j(0)$ is given by (5.20), see e.g. [5]. Integration by parts yields

$$\begin{aligned}\alpha_j(t) &= \exp(-S_jt)\alpha_j(0) + S_j^{-1}\Big[\exp(-S_j(t-\tau))g_j(\tau)\Big]_{\tau=0}^{t} \\ &\quad - S_j^{-1}\int_0^t \exp(-S_j(t-\tau))g_j'(\tau)\,\mathrm{d}\tau \\ &= \exp(-S_jt)\alpha_j(0) + S_j^{-1}\Big\{g_j(t) - \exp(-S_jt)g_j(0) \\ &\quad - \int_0^\delta \exp(-S_j(t-\tau))g_j'(\tau)\,\mathrm{d}\tau - \int_\delta^t \exp(-S_j(t-\tau))g_j'(\tau)\,\mathrm{d}\tau\Big\},\end{aligned}$$

where $0 < \delta < t$ is arbitrary. Taking the norm and squaring, we have

$$\begin{aligned}(5.32)\ \|\alpha_j(t)\|^2 \leqslant C\Big(&\|\exp(-S_jt)\|^2\|\alpha_j(0)\|^2 + \|S_j^{-1}\|^2\Big\{\|g_j(t)\|^2 \\ &+ \|\exp(-S_jt)\|^2\|g_j(0)\|^2 + \Big\|\int_0^\delta \exp(-S_j(t-\tau))g_j'(\tau)\,\mathrm{d}\tau\Big\|^2 \\ &+ \Big\|\int_\delta^t \exp(-S_j(t-\tau))g_j'(\tau)\,\mathrm{d}\tau\Big\|^2\Big\}\Big).\end{aligned}$$

We can bound the integrals in (5.32) using the Schwarz inequality to get

$$\begin{aligned}(5.33)\quad &\Big\|\int_0^\delta \exp(-S_j(t-\tau))g_j'(\tau)\,\mathrm{d}\tau\Big\|^2 \\ &\qquad \leqslant \frac{C}{\lambda_{\min}(S_j)}\exp(-2(t-\delta)\lambda_{\min}(S_j))\int_0^\delta \|g_j'(\tau)\|^2\,\mathrm{d}\tau\,, \\ (5.34)\quad &\Big\|\int_0^\delta \exp(-S_j(t-\tau))g_j'(\tau)\,\mathrm{d}\tau\Big\|^2 \leqslant \frac{C}{\lambda_{\min}(S_j)}\int_\delta^t \|g_j'(\tau)\|^2\,\mathrm{d}\tau\,,\end{aligned}$$

where we use Lemma 5.2 and the formulae for the spectral matrix norm (see, e.g., [5])

$$(5.35)\qquad \|\exp(-S_j(t-\tau))\| = \exp(-(t-\tau)\lambda_{\min}(S_j)), \quad t \geqslant \tau,$$

$$(5.36)\qquad \|S_j^{-1}\| = \frac{1}{\lambda_{\min}(S_j)}.$$

Finally we obtain the bound (5.31) from (5.32) with the help of (5.33) to (5.36). □

**Lemma 5.4.** *Let the assumptions of Lemma 3.3 be fulfilled with $r = 0, 1, 2$ and let $\delta > 0$ be the constant from the statement of Lemma 3.3. Assume further that*

$\|\partial^r e/\partial t^r\|_1$ *is bounded for* $0 \leqslant t \leqslant \delta$, $r = 0, 1, 2$, *and let* $b_j(t)$ *and* $\beta_j(t)$, $j = 1, \ldots, M$, *be solutions of* (5.11), (5.12) *and* (5.17), (5.18), *respectively. Then there exists such a positive constant* $C(u)$, *where* $u$ *is the solution of* (2.11), (2.12), *that*

$$\|\beta_j(t)\|^2 \Sigma_{1j}^2 \leqslant C(u) h_j^{2p} \quad \text{for } 0 < t \leqslant T,\ j = 1, \ldots, M, \tag{5.37}$$

$$\sum_{j=1}^{M} \|b_j(t) - \beta_j(t)\|^2 \Sigma_{1j}^2 \leqslant C(u) h^{2p+2} \quad \text{for } \delta < t \leqslant T, \tag{5.38}$$

$$\sum_{j=1}^{M} \left(\|b_j(t)\|^2 + \|\beta_j(t)\|^2\right) \Sigma_{1j}^2 \leqslant C(u) h^{2p} \quad \text{for } \delta < t \leqslant T. \tag{5.39}$$

Proof. Employing (4.14), (4.15), and (5.7), we can write

$$\|\varphi_j\|_{0,j}^2 = \sum_{k=1}^{N} \beta_{kj}^2 \|\Psi_j\|_{0,j}^2 = \Sigma_{0j}^2 \|\beta_j\|^2, \quad j = 1, \ldots, M.$$

Further,

$$\|\beta_j(t)\|^2 \Sigma_{1j}^2 = \frac{\Sigma_{1j}^2}{\Sigma_{0j}^2} \|\varphi_j\|_{0,j}^2 \leqslant C h_j^{-2} h_j^{2p+2} \|u\|_{p+1,j}^2 = C(u) h_j^{2p}$$

follows from (5.22), (5.23), and (4.16) of Lemma 4.3 with $s = r = 0$. The bound (5.37) has been established.

We now turn to the proof of (5.38). Using (5.6), (5.15), (5.16), and the Schwarz inequality, we get

$$\begin{aligned} \|g_j(t)\|^2 &= \Sigma_{0j}^{-4} \sum_{k=1}^{N} G_j^2(\sigma_j^{(k)}, t) \\ &\leqslant C \Sigma_{0j}^{-4} \sum_{k=1}^{N} \left( \left(\sigma_j^{(k)}, \frac{\partial \omega_j}{\partial t}\right)_{0,j}^2 + a_j^2(\sigma_j^{(k)}, \omega_j) \right) \\ &\leqslant C \Sigma_{0j}^{-4} \sum_{k=1}^{N} \left( \Sigma_{0j}^2 \left\| \frac{\partial \omega_j}{\partial t} \right\|_{0,j}^2 + \|\sigma_j^{(k)}\|_{A,j}^2 \|\omega_j\|_{A,j}^2 \right) \end{aligned} \tag{5.40}$$

for $j = 1, \ldots, M$. Due to the definition (5.1) of $\sigma_j^{(k)} \in H_{0,j}^1$ and Lemma 5.1, we further have

$$\frac{\Sigma_{0j}}{\|\sigma_j^{(k)}\|_{A,j}} = \frac{\|\sigma_j^{(k)}\|_{0,j}}{\|\sigma_j^{(k)}\|_{A,j}} \leqslant \frac{\|\sigma_j^{(k)}\|_{1,j}}{\|\sigma_j^{(k)}\|_{A,j}} \leqslant C \tag{5.41}$$

and thus

$$(5.42)\qquad \|g_j(t)\|^2 \leqslant C\Sigma_{0j}^{-2}\sum_{k=1}^{N}\frac{\|\sigma_j^{(k)}\|_{A,j}^2}{\Sigma_{0j}^2}\left(\frac{\Sigma_{0j}^2}{\|\sigma_j^{(k)}\|_{A,j}^2}\left\|\frac{\partial\omega_j}{\partial t}\right\|_{0,j}^2+\|\omega_j\|_{A,j}^2\right)$$
$$\leqslant C\Sigma_{0j}^{-2}\sum_{k=1}^{N}\frac{\|\sigma_j^{(k)}\|_{A,j}^2}{\Sigma_{0j}^2}\left(\left\|\frac{\partial\omega_j}{\partial t}\right\|_{0,j}^2+\|\omega_j\|_{A,j}^2\right)$$
$$= C\Sigma_{0j}^{-2}\left(\left\|\frac{\partial\omega_j}{\partial t}\right\|_{0,j}^2+\|\omega_j\|_{1,j}^2\right)\sum_{k=1}^{N}\frac{\|\sigma_j^{(k)}\|_{A,j}^2}{\Sigma_{0j}^2},$$

where we used Lemma 5.1 and (3.7).

Since (5.31) of Lemma 5.3 is the means to prove (5.38) we employ (5.26), (5.42) and Lemma 5.1 to estimate

$$(5.43)\qquad \sum_{j=1}^{M}\frac{1}{\lambda_{\min}^2(S_j)}\|g_j(t)\|^2\Sigma_{1j}^2$$
$$\leqslant C\sum_{j=1}^{M}\frac{1}{\lambda_{\min}^2(S_j)}\frac{\Sigma_{1j}^2}{\Sigma_{0j}^2}\left(\left\|\frac{\partial\omega_j}{\partial t}\right\|_{0,j}^2+\|\omega_j\|_{1,j}^2\right)\sum_{k=1}^{N}\frac{\|\sigma_j^{(k)}\|_{A,j}^2}{\Sigma_{0j}^2}$$
$$\leqslant C\sum_{j=1}^{M}h_j^4h_j^{-2}h_j^{-2}\left(\left\|\frac{\partial\omega_j}{\partial t}\right\|_{0,j}^2+\|\omega_j\|_{1,j}^2\right)\leqslant C\left(\left\|\frac{\partial\omega}{\partial t}\right\|_{0}^2+\|\omega\|_1^2\right).$$

The inequalities (4.29) with $r=1$, $s=0$ and (4.30) of Lemma 4.5 finally give

$$(5.44)\qquad \sum_{j=1}^{M}\frac{1}{\lambda_{\min}^2(S_j)}\|g_j(t)\|^2\Sigma_{1j}^2\leqslant C(u)h^{2p+2},\quad \delta<t\leqslant T,$$

where $\delta>0$ is the constant from Lemma 3.3.

We have assumed that the error $e$ and its first two time derivatives are bounded for $0\leqslant t\leqslant\delta$. We start from (5.43) and use (4.24) and the estimates (4.28) and (4.16) with $r=1$, $s=0$ and $r=0$, $s=1$ to find

$$(5.45)\qquad \sum_{j=1}^{M}\frac{1}{\lambda_{\min}^2(S_j)}\|g_j(t)\|^2\Sigma_{1j}^2\leqslant C(u),\quad 0\leqslant t\leqslant\delta.$$

To proceed further, we differentiate (5.15) with respect to $t$ to get

$$G_j'(\sigma_j^{(k)},t)=\left(\sigma_j^{(k)},\frac{\partial^2\omega_j}{\partial t^2}\right)_{0,j}+a_j\left(\sigma_j^{(k)},\frac{\partial\omega_j}{\partial t}\right).$$

From (5.16) we then, like in (5.40), arrive at

$$
\begin{aligned}
\|g_j'(t)\|^2 &= \Sigma_{0j}^{-4} \sum_{k=1}^{N} G'^2_j(\sigma_j^{(k)}, t) \\
&\leqslant C\Sigma_{0j}^{-4} \sum_{k=1}^{N} \left( \left(\sigma_j^{(k)}, \frac{\partial^2 \omega_j}{\partial t^2}\right)_{0,j}^2 + a_j^2\left(\sigma_j^{(k)}, \frac{\partial \omega_j}{\partial t}\right)\right) \\
&\leqslant C\Sigma_{0j}^{-4} \sum_{k=1}^{N} \left( \Sigma_{0j}^2 \left\| \frac{\partial^2 \omega_j}{\partial t^2} \right\|_{0,j}^2 + \|\sigma_j^{(k)}\|_{A,j}^2 \left\| \frac{\partial \omega_j}{\partial t} \right\|_{A,j}^2 \right).
\end{aligned}
$$

Like in (5.42) we assess

$$
\|g_j'(t)\|^2 \leqslant C\Sigma_{0j}^{-2} \left( \left\| \frac{\partial^2 \omega_j}{\partial t^2} \right\|_{0,j}^2 + \left\| \frac{\partial \omega_j}{\partial t} \right\|_{1,j}^2 \right) \sum_{k=1}^{N} \frac{\|\sigma_j^{(k)}\|_{A,j}^2}{\Sigma_{0j}^2}. \tag{5.46}
$$

Further, like in (5.43),

$$
\sum_{j=1}^{M} \frac{1}{\lambda_{\min}^3(S_j)} \|g_j'(t)\|^2 \Sigma_{1j}^2 \leqslant Ch^2 \left( \left\| \frac{\partial^2 \omega}{\partial t^2} \right\|_0^2 + \left\| \frac{\partial \omega}{\partial t} \right\|_1^2 \right) \tag{5.47}
$$

follows from (5.26), Lemma 5.1, and (5.46). We now use again (4.29) of Lemma 4.5 with $r = 2$, $s = 0$ and $r = 1$, $s = 1$ to show

$$
\sum_{j=1}^{M} \frac{1}{\lambda_{\min}^3(S_j)} \|g_j'(t)\|^2 \Sigma_{1j}^2 \leqslant C(u)h^2(h^{2p+2} + h^{2p}) \leqslant C(u)h^{2p+2}, \quad \delta < t \leqslant T. \tag{5.48}
$$

By assumption, the error and its first two time derivatives are bounded for $0 \leqslant t \leqslant \delta$. Starting from (5.47), and using (4.16) and (4.28) with $r = 2$, $s = 0$ and $r = 1$, $s = 1$, we find

$$
\sum_{j=1}^{M} \frac{1}{\lambda_{\min}^3(S_j)} \|g_j'(t)\|^2 \Sigma_{1j}^2 \leqslant C(u)h^2, \quad 0 \leqslant t \leqslant \delta. \tag{5.49}
$$

Finally we bound $\|\alpha_j(0)\|$. Since

$$
\|T_j^{-1}\| = \frac{1}{\lambda_{\min}(T_j)} \tag{5.50}
$$

we can estimate, by (5.20), (5.21), and the Schwarz inequality,

$$
\begin{aligned}
\|\alpha_j(0)\|^2 \leqslant \|T_j^{-1}\|^2 \|z_j\|^2 &= \frac{1}{\lambda_{\min}^2(T_j)} \sum_{k=1}^{N} a_j^2(\sigma_j^{(k)}, \omega_j(.\,,0)) \\
&\leqslant \frac{1}{\lambda_{\min}^2(T_j)} \|\omega_j(.\,,0)\|_{A,j}^2 \sum_{k=1}^{N} \|\sigma_j^{(k)}\|_{A,j}^2.
\end{aligned}
$$

Further by (5.24), (5.25), and (3.7) we arrive at

$$\|\alpha_j(0)\|^2 \leqslant Ch_j^{-4p-2} h_j^{2p+1} \|\omega_j(.\,,0)\|_{1,j}^2 = Ch_j^{-2p-1} \|\omega_j(.\,,0)\|_{1,j}^2.$$

Substituting (4.25) for $\omega_j$, we come to

$$\|\alpha_j(0)\|^2 \leqslant Ch_j^{-2p-1} (\|\gamma_j(.\,,0)\|_{1,j}^2 + \|\pi_j u(x,0) - \hat{U}(.\,0)\|_{1,j}^2 + \|U(.\,,0) - \hat{U}(.\,,0)\|_{1,j}^2),$$

where $\hat{U}$ is the solution of (3.13) with $t = 0$.

We now multiply by $\Sigma_{1j}^2$, employ (4.18) of Lemma 4.3, sum the inequality over $j$ and apply (4.23) of Lemma 4.5, (3.14) of Lemma 3.2, and (5.23) to obtain

$$\sum_{j=1}^{M} \|\alpha_j(0)\|^2 \Sigma_{1j}^2 \leqslant C(u^0) h^{2p+2}. \tag{5.51}$$

Let us turn back to (5.31) and investigate three of the right-hand part terms. Due to (5.26), (5.51),

$$\sum_{j=1}^{M} \|\alpha_j(0)\|^2 \Sigma_{1j}^2 \exp(-2t\lambda_{\min}(S_j)) \to 0 \text{ as } h \to 0 \text{ and } 0 < t \leqslant T. \tag{5.52}$$

Similarly,

$$\sum_{j=1}^{M} \frac{1}{\lambda_{\min}^2(S_j)} \|g_j(0)\|^2 \Sigma_{1j}^2 \exp(-2t\lambda_{\min}(S_j)) \to 0 \text{ as } h \to 0 \text{ and } 0 < t \leqslant T \tag{5.53}$$

by (5.45) and

$$\sum_{j=1}^{M} \frac{1}{\lambda_{\min}^3(S_j)} \|g_j'(\tau)\|^2 \Sigma_{1j}^2 \exp(-2(t-\delta)\lambda_{\min}(S_j)) \to 0 \tag{5.54}$$
$$\text{as } h \to 0 \text{ and } \delta < t \leqslant T,\ 0 \leqslant \tau \leqslant \delta$$

by (5.49).

Multiplying by $\Sigma_{1j}^2$ and summing over $j$, we finally get from (5.31) that

$$\begin{aligned}\sum_{j=1}^{M} \|\alpha_j(t)\|^2 \Sigma_{1j}^2 &\leqslant C\Big( \sum_{j=1}^{M} \frac{1}{\lambda_{\min}^2(S_j)} \|g_j(t)\|^2 \Sigma_{1j}^2 + \int_\delta^t \sum_{j=1}^{M} \frac{1}{\lambda_{\min}^3(S_j)} \|g_j'(\tau)\|^2 \Sigma_{1j}^2 \,\mathrm{d}\tau \Big) \\ &\leqslant C(u) h^{2p+2}, \quad \delta < t \leqslant T,\end{aligned}$$

since the terms (5.52) to (5.54) tend to zero much faster than $h^{2p+2}$ and since we bounded (5.44) and (5.48). We have proved the statement (5.38).

Since

$$\|b_j(t)\|^2 + \|\beta_j(t)\|^2 \leqslant \|b_j(t) - \beta_j(t)\|^2 + 2\|\beta_j(t)\|^2,$$

we can estimate from (5.37), (5.38)

$$\begin{aligned}\sum_{j=1}^{M} \left(\|b_j(t)\|^2 + \|\beta_j(t)\|^2\right)\Sigma_{1j}^2 &\leqslant \sum_{j=1}^{M} \|b_j(t) - \beta_j(t)\|^2 \Sigma_{1j}^2 + 2\sum_{j=1}^{M} \|\beta_j(t)\|^2 \Sigma_{1j}^2 \\ &\leqslant C(u)(h^{2p+2} + h^{2p}) \leqslant C(u)h^{2p}, \quad \delta < t \leqslant T,\end{aligned}$$

which is (5.39). The lemma has been proved. □

**Theorem 5.1.** *Let $u \in H_0^1 \cap H^{p+2}$ and $U \in S_0^{M,p}$ and $E \in \hat{S}_0^{M,p+1}$ be solutions of* (2.11), (2.12) *and* (3.1), (3.2) *and* (3.14), (3.15), *respectively. Let $u^0 \in H_0^1 \cap H^{p+1}$ and let $u$ be smooth enough for all terms in* (3.15), (3.16) *to be bounded with $r = 0, 1, 2$. Further let $\delta > 0$ be the constant from the statement of Lemma* 3.3 *and let $\|\partial^r e/\partial t^r\|_1$ be bounded for $0 \leqslant t \leqslant \delta$, $r = 0, 1, 2$. Then*

$$\|e(.\,,t)\|_1^2 = \|u(.\,,t) - U(.\,,t)\|_1^2 = \|E(.\,,t)\|_1^2 + \varepsilon, \quad \delta < t \leqslant T,$$

*and there exists a positive constant $C$ such that*

$$|\varepsilon| \leqslant C(u)h^{2p+1}.$$

Proof. Using (4.24), we have

$$\|e\|_{1,j}^2 = \|\varphi_j + \omega_j\|_{1,j}^2 = \|\varphi_j\|_{1,j}^2 + 2(\varphi_j, \omega_j)_{1,j} + \|\omega_j\|_{1,j}^2.$$

Summing over $j$ and using (4.14), (5.2), (5.7), and the Schwarz inequality, we find

$$\varepsilon = \|e\|_1^2 - \|E\|_1^2 = \sum_{j=1}^{M} \Sigma_{1j}^2 \sum_{k=1}^{N} \left(\beta_{kj}^2(t) - b_{kj}^2(t)\right) + 2\|\varphi\|_1\|\omega\|_1 + \|\omega\|_1^2.$$

Employing again (4.14), (4.30), and (5.37) of Lemma 5.4, we arrive at

$$|\varepsilon| \leqslant \sum_{j=1}^{M} \Sigma_{1j}^2 \sum_{k=1}^{N} |\beta_{kj}^2(t) - b_{kj}^2(t)| + C(u)h^p h^{p+1} + C(u)h^{2p+2}, \quad \delta < t \leqslant T,$$

where $\delta > 0$ is the constant from Lemma 3.3. By the Schwarz inequality, we finally obtain

$$
\begin{aligned}
|\varepsilon| &\leqslant \sum_{j=1}^{M} \Sigma_{1j}^2 \sum_{k=1}^{N} \left| (\beta_{kj}(t) - b_{kj}(t)) (\beta_{kj}(t) + b_{kj}(t)) \right| + C(u) h^{2p+1} \\
&\leqslant \sum_{j=1}^{M} \Sigma_{1j} \|\beta_j(t) - b_j(t)\| \Sigma_{1j} \|\beta_j(t) + b_j(t)\| + C(u) h^{2p+1} \\
&\leqslant C \Big( \sum_{j=1}^{M} \Sigma_{1j}^2 \|\beta_j(t) - b_j(t)\|^2 \Big)^{1/2} \Big( \sum_{j=1}^{M} \Sigma_{1j}^2 (\|\beta_j(t)\|^2 + \|b_j(t)\|^2) \Big)^{1/2} + C(u) h^{2p+1} \\
&\leqslant C(u)(h^{p+1} h^p + h^{2p+1}) = C(u) h^{2p+1}
\end{aligned}
$$

by virtue of Lemma 5.4. The theorem has been proved. □

Remark 5.1. The statement of Theorem 5.1 shows that the function $E(x,t)$ can be used to obtain an a posteriori error estimate in practical computation. The error on each interval $(x_{j-1}, x_j)$ of the partition is characterized by a single number called the local *error indicator*, i.e., in our case, by the quantity $\|E(.\,,t)\|_{1,j}$. The error on the whole interval $[0,1]$ can also be characterized by a single number, e.g. by $\|E(.\,,t)\|_1$. This number is called the *error estimator* (cf., e.g., [4]).

Remark 5.2. Introducing positive weights for the individual components in the definition (2.4) of inner product, we may emphasize or suppress the influence of the error of some components of the solution on the error indicator and estimator.

## 6. Elliptic Error Estimation

The elliptic error estimation (3.10), (3.11) can be analyzed in a similar way. Replacing $V$ by $\sigma_j^{(k)}$ in (3.10) we come to

$$a_j(\sigma_j^{(l)}, \tilde{E}) = R_j(\sigma_j^{(l)}, t), \quad 0 < t \leqslant T,\ l = 1, \dots, N,\ j = 1, \dots, M,$$

where $R_j(\sigma_j^{(l)}, t)$ is given by (5.5). Expressing $\tilde{E}(x,t)$ in the form

$$\tilde{E}(x,t) = \sum_{j=1}^{M} \sum_{k=1}^{N} \tilde{b}_{kj}(t) \sigma_j^{(k)}(x),$$

we have $M$ uncoupled linear algebraic systems

$$\sum_{k=1}^{N} \tilde{b}_{kj}(t)\, a_j(\sigma_j^{(l)}, \sigma_j^{(k)}) = R_j(\sigma_j^{(l)}, t), \quad l = 1, \dots, N, \tag{6.1}$$

for the unknown coefficients $\tilde{b}_{kj}$, $k = 1, \dots, N$, where $j = 1, \dots, M$. Putting

$$y_j(t) = \left(R_j(\sigma_j^{(1)}, t), \dots, R_j(\sigma_j^{(N)}, t)\right)^{\mathrm{T}} \tag{6.2}$$

and using the notation (5.13), we can write the solution of (6.1) in the form

$$\tilde{b}_j(t) = T_j^{-1} y_j(t), \quad j = 1, \dots, M. \tag{6.3}$$

Substituting $\sigma_j^{(l)}$ for $v$ into (3.4), we get

$$\left(\sigma_j^{(l)}, \frac{\partial e}{\partial t}\right)_{0,j} = -a_j(\sigma_j^{(l)}, e) + (\sigma_j^{(l)}, f)_{0,j} - a_j(\sigma_j^{(l)}, U) - \left(\sigma_j^{(l)}, \frac{\partial U}{\partial t}\right)_{0,j}, \quad j = 1, \dots, M,$$

and, comparing it with (5.5), we conclude

$$R_j(\sigma_j^{(l)}, t) = \left(\sigma_j^{(l)}, \frac{\partial e}{\partial t}\right)_{0,j} + a_j(\sigma_j^{(l)}, e). \tag{6.4}$$

Further substituting (5.14) for $e$ into $a_j$ in (6.4), we arrive at

$$\sum_{k=1}^{N} \beta_{kj}(t)\, a_j(\sigma_j^{(l)}, \sigma_j^{(k)}) = R_j(\sigma_j^{(l)}, t) - \left(\sigma_j^{(l)}, \frac{\partial e}{\partial t}\right)_{0,j} - a_j(\sigma_j^{(l)}, \omega_j), \quad l = 1, \dots, N, \ j = 1, \dots, M. \tag{6.5}$$

The system (6.5) can now be rewritten as

$$\beta_j(t) = T_j^{-1} y_j(t) + \tilde{g}_j(t), \quad j = 1, \dots, M, \tag{6.6}$$

where $T_j$ is given by (5.13), $y_j$ by (6.2) and

$$\tilde{g}_j(t) = T_j^{-1} s_j(t), \tag{6.7}$$

$$s_j(t) = \left(\left(\sigma_j^{(1)}, \frac{\partial e}{\partial t}\right)_{0,j} - a_j(\sigma_j^{(1)}, \omega_j), \dots, \left(\sigma_j^{(N)}, \frac{\partial e}{\partial t}\right)_{0,j} - a_j(\sigma_j^{(N)}, \omega_j)\right)^{\mathrm{T}}. \tag{6.8}$$

We establish an analog of Lemma 5.4.

**Lemma 6.1.** *Let $\tilde{b}_j(t)$ and $\beta_j(t)$, $j = 1, \dots, M$, be solutions of (6.3) and (5.17), (5.18), respectively. Let the assumptions of Lemma 3.3 be fulfilled. Then there exist positive constants $C(u)$, where $u$ is the solution of (2.11), (2.12), and $\delta$ such that*

$$\sum_{j=1}^{M} \|\tilde{b}_j(t) - \beta_j(t)\|^2 \Sigma_{1j}^2 \leqslant C(u) h^{2p+2} \quad \text{for } \delta < t \leqslant T, \tag{6.9}$$

$$\sum_{j=1}^{M} \left(\|\tilde{b}_j(t)\|^2 + \|\beta_j(t)\|^2\right) \Sigma_{1j}^2 \leqslant C(u) h^{2p} \quad \text{for } \delta < t \leqslant T. \tag{6.10}$$

Proof. We proceed like in the proof of Lemma 5.4. Since (6.6) is equivalent to (5.17) we calculate from (6.3), (6.6), and (6.7) that

$$\tilde{b}_j(t) - \beta_j(t) = \tilde{g}_j(t) = T_j^{-1} s_j(t), \quad j = 1, \ldots, M.$$

From (6.8), (5.50), and the Schwarz inequality we have

$$\begin{aligned}
(6.11) \qquad \|\tilde{b}_j(t) - \beta_j(t)\|^2 &\leqslant \|T_j^{-1}\|^2 \|s_j(t)\|^2 \\
&\leqslant \frac{C}{\lambda_{\min}^2(T_j)} \sum_{k=1}^{N} \left( \left( \sigma_j^{(k)}, \frac{\partial e}{\partial t} \right)_{0,j} + a_j^2(\sigma_j^{(k)}, \omega_j) \right) \\
&= \frac{C}{\lambda_{\min}^2(T_j)} \sum_{k=1}^{N} \|\sigma_j^{(k)}\|_{A,j}^2 \left( \frac{\Sigma_{0j}^2}{\|\sigma_j^{(k)}\|_{A,j}^2} \left\| \frac{\partial e}{\partial t} \right\|_{0,j}^2 + \|\omega_j\|_{A,j}^2 \right).
\end{aligned}$$

We further use (5.41) and (3.7) to obtain

$$\|\tilde{b}_j(t) - \beta_j(t)\|^2 \leqslant C \frac{1}{\lambda_{\min}^2(T_j)} \sum_{k=1}^{N} \|\sigma_j^{(k)}\|_{A,j}^2 \left( \left\| \frac{\partial e}{\partial t} \right\|_{0,j}^2 + \|\omega_j\|_{1,j}^2 \right).$$

Multiplying by $\Sigma_{1j}^2$ and summing up over $j$, we obtain with respect to (3.15), (4.30), (5.23), (5.24), and (5.25) that

$$\sum_{j=1}^{M} \|\tilde{b}_j(t) - \beta_j(t)\|^2 \Sigma_{1j}^2 \leqslant C(u) h^{-4p-2} h^{2p+1} h^{2p+2} h^{2p+1} \leqslant C(u) h^{2p+2} \qquad \text{for } \delta < t \leqslant T,$$

which is the inequality (6.9). The value of $\delta$ is given by Lemma 3.3.

Since now again, like in the proof of Lemma 5.4,

$$\|\tilde{b}_j(t)\|^2 + \|\beta_j(t)\|^2 \leqslant \|\tilde{b}_j(t) - \beta_j(t)\|^2 + 2\|\beta_j(t)\|^2$$

the statement (6.11) is a simple consequence of (6.9) and (5.37). The proof has been completed. □

**Theorem 6.1.** *Let $u \in H_0^1 \cap H^{p+2}$ and $U \in S_0^{M,p}$ and $\tilde{E} \in \hat{S}_0^{M,p+1}$ be solutions of (2.11), (2.12) and (3.1), (3.2) and (3.10), (3.11), respectively. Let $u^0 \in H_0^1 \cap H^{p+1}$ and let $u$ be smooth enough for all terms in (3.15) to be bounded with $r = 0, 1$. Then there exist positive constants $C$ and $\delta$ such that*

$$\|e(.,t)\|_1^2 = \|u(.,t) - U(.,t)\|_1^2 = \|\tilde{E}(.,t)\|_1^2 + \tilde{\varepsilon}, \quad \delta < t \leqslant T,$$

*and*

$$|\tilde{\varepsilon}| \leqslant C(u)h^{2p+1}.$$

P r o o f. The proof is carried out in the same way as the proof of Theorem 5.1. Lemma 6.1 is employed instead of Lemma 5.4. □

R e m a r k 6.1. According to Theorem 6.1, an error indicator and error estimator from Remark 5.1 can also be introduced with the help of the function $\tilde{E}(x,t)$.

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