Jiří Jarušek

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REGULARITY AND OPTIMAL CONTROL OF QUASICOUPLED AND COUPLED HEATING PROCESSES

JIŘÍ JARUŠEK, Praha[1]

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Summary. Sufficient conditions for the stresses in the threedimensional linearized coupled thermoelastic system including viscoelasticity to be continuous and bounded are derived and optimization of heating processes described by quasicoupled or partially linearized coupled thermoelastic systems with constraints on stresses is treated. Due to the consideration of heating regimes being "as nonregular as possible" and because of the well-known lack of results concerning the classical regularity of solutions of such systems, the technique of spaces of Běsov-Sobolev type is essentially employed and the possibility of its use when solving optimization problems is studied.

Keywords: heat equation, Lamé system, coupled system, viscoelasticity, optimal control, state space constraints, bounded stresses

AMS classification: 73U05 (35B65, 35M05, 35R05, 49J20, 49K20)

0. INTRODUCTION AND NOTATION

In the series of papers [5] regularity of the solution of a thermoelastic system in the sense of boundedness and continuity of stresses was proved for the quasicoupled system consisting of the quasilinear heat equation in the form

$$\beta_0\dot{u} = \Delta u \;\text{ on } Q = I \times \Omega,\; I := (0, \mathcal{T}), \tag{1}$$
$$\frac{\partial u}{\partial \nu} = g(T) - g(u) \text{ on } S = I \times \partial\Omega, \quad u(0,\cdot) = 0 \;\text{ on } \Omega$$

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and of the Lamé system e.g. in its homogeneous and isotropic version

$$(2) \qquad (1-2\sigma)\Delta v + \nabla \operatorname{div} v = (2+2\sigma)\nabla\gamma(u) \text{ on } \Omega, \quad t \in \bar{I} := \langle 0, \mathcal{T} \rangle,$$

$$(1-2\sigma)\Big(\frac{\partial v}{\partial \nu} + ((\nu, \nabla_i v)_i)\Big) + 2\sigma\nu \operatorname{div} v = (2+2\sigma)\gamma(u)\nu \text{ on } \partial\Omega, \quad t \in \bar{I},$$

where Ω is a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with a sufficiently smooth boundary which can have isolated nonsmoothnesses of the convex type (convex edges or vertices), u is the temperature, v the displacement, g a given sufficiently smooth nondecreasing function with $g(0) = 0$, γ a given sufficiently smooth function, σ a given constant (the Poisson ratio) and the dot denotes the time derivative. Nonregular heating regimes T (e.g. having jumps) were admitted (in harmony with the requirements of technical practice). Certain sufficient conditions for such a regularity were established not only for the character of nonregularity of T, but also for the case of a supported body, where the boundary value condition in (2) is replaced on some part of the boundary by a Neumann or some mixed condition. In the third part of [5] the two-dimensional case of the linearized coupled system including viscoelasticity (see (3)) was studied from the same point of view.

The main purpose of this paper is to finish the investigation of regularity of solutions of such systems in the coupled three-dimensional case (Sec. 1) and then to treat the optimization of such systems considering a (pointwise) constraint on the magnitude of stresses. This is done in Sec. 2. For the formulation of such a problem and its treatment the employed kind of regularity of the solutions seems to be the main base as it ensures the reasonability of the constraint.

The practical importance of the state-space constraint on stresses is obvious—the body must not be damaged by the heating. For real processes we must preserve the nonlinear boundary value condition for the heat equation.

In the paper we often use anisotropic Sobolev spaces $H^\alpha(\mathbb{R} \times M)$ for a domain $M \subset \mathbb{R}^N$, $N = 2$ or 3, where H denotes the square integrability of the generalized (possibly fractional) derivatives (for the integrability index $p \neq 2$ we denote them $W^\alpha_p(M)$). If $\alpha \in \mathbb{R}^2$, then its first coordinate denotes the time regularity, the second the space-coordinate regularity of its elements. h denotes the local tangential shift and for a function $f\colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^m$, $m \in \mathbb{N}$, we put $f_{-h}\colon [t,x] \mapsto f(t, x+h)$. In the Bochner spaces $L_p(J; H^\alpha(M))$, for an interval $J \subset \mathbb{R}$ and a subdomain $M \subset \Omega$ the first coordinate of α denotes the (local) normal regularity, the second the (local) tangential regularity of its elements, both with respect to $\partial\Omega$. The same convention will be held for $B(J; H^\alpha(M))$, where for a Banach space X, $B(J; X)$ is the space of bounded maps from J into a Banach space X equipped with the sup norm. Under regularity we understand here the number (possibly noninteger) of generalized derivatives in the respective direction. $\mathbb{R}_+$ will denote the interval

$(0, +\infty)$ and for $a \in \mathbb{R}_+^k$, $D^a \equiv \frac{\partial^{|a|}}{\partial x^{a_1} \ldots \partial x^{a_k}}$. For a real number or a real function z we put $z^+ \equiv \max(z, 0)$.

1. Regularity of stresses in three-dimensional linearized coupled thermoelastic systems including viscoelasticity

In this section we shall proceed with the investigation of the regularity of stresses for the coupled model described below assuming again the noncontinuity of the heating regime and a nonlinear boundary value condition in the heat equation admitting e.g. Stefan-Boltzmann radiation law. We consider a bounded domain Ω in $\mathbb{R}^3$ with a C_3-smooth boundary. Our system is a partial linearization of the original physically motivated system and is assumed to have the form

(3)
$$\begin{cases} \left.\begin{aligned} &\beta_0 \dot{u} = \Delta u + \delta_0 \operatorname{div} \dot{v} \\ &\ddot{v} = (1 - 2\sigma_1)\Delta \dot{v} + (1 - 2\sigma)\Delta v + \nabla \operatorname{div} v - (2 + 2\sigma)\nabla(\gamma u) \end{aligned}\right\} \text{ on } Q, \\ \left.\begin{aligned} &\frac{\partial u}{\partial \nu} = g(T) - g(u) \\ &(1 - 2\sigma_1)\frac{\partial \dot{v}}{\partial \nu} + (1 - 2\sigma)\Big(\frac{\partial v}{\partial \nu} + ((\nu, \nabla_i v)_i)\Big) + 2\sigma\nu \operatorname{div} v = (2 + 2\sigma)\gamma u \nu \end{aligned}\right\} \text{ on } S, \\ \qquad u(0, \cdot) = 0, \quad v(0, \cdot) = 0, \quad \dot{v}(0, \cdot) = 0 \text{ on } \Omega. \end{cases}$$

with g being nondecreasing, bounded below and satisfying a growth condition in the infinity which will be specified in the sequel. Here β_0, γ, δ_0, σ, σ_1 are positive constants with σ, $\sigma_1 < \frac{1}{2}$. As in [5], Part III we assume that the heating regime is bounded on Q and satisfies the relation $g(T) \in \bigcap\limits_{\varepsilon > 0} H^{\frac{1}{2}-\varepsilon, 1+\eta}(S)$ for some $\eta > 0$.

We recall the variational formulation of the problem

(4)
$$\begin{aligned} &\int\limits_Q \beta_0 \dot{u} w_0 + (\nabla u, \nabla w_0) + (1 - 2\sigma_1)(\nabla \dot{v}, \nabla w_1) \\ &\quad + (1 - 2\sigma)(\nabla v, \nabla w_1) + \operatorname{div} v \operatorname{div} w_1 + \ddot{v} w_1 \, \mathrm{d}x \, \mathrm{d}t + \int\limits_S g(u) w_0 \, \mathrm{d}x \, \mathrm{d}t \\ &= \int\limits_Q (2 + 2\sigma)\gamma u \operatorname{div} w_1 - \delta_0 \operatorname{div} \dot{v} w_0 \, \mathrm{d}x \, \mathrm{d}t + \int\limits_S g(T) w_0 \, \mathrm{d}x \, \mathrm{d}t \end{aligned}$$

for a test function $w = [w_0, w_1] = [w_0, [w_1^1, w_1^2, w_1^3]] \in L_2(I; H^1(\Omega;\, \mathbb{R}^4))$. Putting $w_0 = u$, $w_1 = \dot{v}$ in (3) we immediately obtain the usual energy estimate and via the Galerkin method we prove existence of a solution. Our aim is to prove

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a $C_{\frac{5}{2}+\eta}$-smooth boundary for some $\eta > 0$, let g from (3) be C_2-smooth, bounded below and non-decreasing on $\mathbb{R}$, let $g(0) = 0$, g, g' and g'' be polynomially bounded at infinity, let g' be bounded on $(-\infty, 0)$ and let it satisfy the additonal growth condition*

$$\underset{\delta_0>0}{\exists}\ \underset{z>0}{\exists}\ \underset{c>0}{\exists}\ \underset{y\geqslant z}{\forall}\ g(y) \geqslant c|y|^{\delta_0}. \tag{5}$$

Let β_0, σ, γ from (3) be positive constants with $\sigma < \frac{1}{2}$. Let the heating regime be bounded on Q and non-negative with $T(0) = 0$ and let the relation $g(T) \in \bigcap_{\varepsilon>0} H^{\frac{1}{2}-\varepsilon,1+\theta}(S)$ be satisfied for some $\theta > 0$. Then the corresponding stress tensor belongs to $C_0(\overline{Q};\, \mathbb{R}^9)$.

The proof will be divided in several steps.

Step 1. First some preliminary regularity results for the solution of (3) will be mentioned. In the same way as in [5], Part III, Sec. 3, we start with the proof of some partial regularity of ∇u and $\nabla \dot{v}$ in time (up to the time regularity of T) via the shift method. From it, the fact $\dot{u} \in L_2(Q)$, $\ddot{v} \in L_2(Q;\, \mathbb{R}^3)$ will be derived by virtue of partial regularity of traces, if g is bounded below on $\mathbb{R}$. This will be made by putting $w = [\dot{u}, \ddot{v}]$ in (4). For details of the estimation employed we refer to [5], Part III, pp. 284–285. (We remark that the described procedure works for $T \in H^{\alpha}(I;\, L_2(\Omega))$ with $\alpha > \frac{1}{3}$.) Then we prove the tangential regularity up to the existence of the generalized second tangential derivatives. To prove it, we use the local straightening of the boundary, the shift method with second order differences and the renormation technique—see Appendix. Taking Γ_ϑ as a ball in $\mathbb{R}^2$ (a part of the straightened $\partial\Omega$) and $S_\vartheta :\ = I \times \Gamma_\vartheta$, we obtain (after the use of the renormation lemma—see Appendix) the nonlinear boundary term in the form $\int_{S_\vartheta} g'(u)|\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t$. Due to the monotonicity of g this term can be estimated in this step on the left hand side of the appropriate "energy-type" inequality which is here important, because the boundedness of u has not yet been proved. Then, calculating the normal regularity directly from the system (3) and the above proved results, we obtain $u \in H^{1,2}(Q)$. Analogously, differentiating the Lamé system successively in time, in the tangential and normal variables and using the positivity of coefficients at the normal derivatives of the highest order (cf. (23) in [5], Part III), we prove

$$\begin{aligned} &\nabla\ddot{v} \in L_2(Q;\, \mathbb{R}^9) \quad \text{and} \quad v,\ \dot{v} \in H^{1,3}(Q;\, \mathbb{R}^3) \\ &\Longrightarrow\ v, \dot{v} \in \bigcap_{\varepsilon>0} H^{\frac{5}{6}-\varepsilon,\, \frac{5}{2}-\varepsilon}(S;\, \mathbb{R}^3) \\ &\Longrightarrow\ v, \dot{v} \in L_p(Q;\, \mathbb{R}^3) \cap L_p(S;\, \mathbb{R}^3) \quad \text{for every } p \in \langle 1, +\infty) \end{aligned} \tag{6}$$

Furthermore, like in [5], Sec. 3 (cf. [5], Part III, Secs.2 and 3, too), the use of time shifts, of the results mentioned in (6) and of the trace theorem yield that $\nabla u \in \bigcap_{\varepsilon>0} H^{\frac{5}{8}-\varepsilon}(I;\, L_2(\Omega))$. Such results, however, do not suffice to prove the continuity of the stress tensor.

Step 2. The present aim is to improve essentially the tangential-regularity result. As in the two-dimensional case we must be careful about the nonlinear term. We use the shift method with second order differences. The nonlinear term has not the suitable sign and can not be estimated on the left hand side any more. After the straightening of $\partial\Omega$ (with S_ϑ as above) and renormation, it has (locally) the estimate[2]

$$
\begin{aligned}
(7)\qquad & \int\limits_{S_\vartheta} \int\limits_{(-\eta,\eta)^2} |h|^{-2-2\alpha} (g'(u_{-h})\nabla u_{-h} - g'(u)\nabla u, \nabla u_{-h} - \nabla u)\, \mathrm{d}h\, \mathrm{d}x\, \mathrm{d}t \\
& \leqslant \tilde{c}\Big(\int\limits_{S_\vartheta} (g'(u)|\nabla u|)^2 \,\mathrm{d}x\, \mathrm{d}t \Big)^{\frac{1}{2}} \Big(\int\limits_{S_\vartheta} \int\limits_{(-\eta,\eta)^2} |h|^{-2-4\alpha} |\nabla u_{-h} - \nabla u|^2 \,\mathrm{d}h\, \mathrm{d}x\, \mathrm{d}t \Big)^{\frac{1}{2}} \\
& \leqslant \varepsilon \|\nabla u\|^2_{L_{2p}(S_\vartheta)} \|g'(u)\|^2_{L_{\frac{2p}{p-1}}(S_\vartheta)} + \frac{\tilde{c}^2}{2\varepsilon} \|u\|^2_{L_2(I;\, H^{1+2\alpha}(\partial\Omega_\vartheta))}, \\
& p > 1,\ \alpha \in (0, \tfrac{1}{2}),\ \varepsilon \ \text{small},
\end{aligned}
$$

where $\tilde{c}$ is a suitable positive constant, cf. Lemma in Appendix. (Using the partition of unity we can assume that $\operatorname{supp} u \subset S_\vartheta$ and we take $\eta > 0$ sufficiently small.) From the above mentioned facts we know that $u/_S$ is bounded in $L_2(I;\, H^{\frac{3}{2}}(\partial\Omega))$ and in $\bigcap_{\varepsilon>0} H^{\frac{5}{8}-\varepsilon}(I;\, H^{\frac{1}{2}}(\partial\Omega))$. The norms of these spaces will be called the old norms. Let $\tilde{r} > 0$ be so small that $\Omega_\vartheta \equiv \Gamma_\vartheta \times (0, \tilde{r}) \subset \Omega$ after the local straightening of the boundary and let $Q_\vartheta \equiv I \times \Omega_\vartheta$. Now, our effort is to prove

$$
\begin{aligned}
(8)\qquad & \sup_{t\in I} \beta_0 \int\limits_{\Omega_\vartheta} \int\limits_{(-\eta,\eta)^2} |h|^{-4-2\alpha} (u_{-h} + u_h - 2u)^2(t,\cdot)\, \mathrm{d}h\, \mathrm{d}x \\
& + \int\limits_{Q_\vartheta} \int\limits_{(-\eta,\eta)^2} |h|^{-4-2\alpha} |\nabla u_{-h} + \nabla u_h - 2\nabla u|^2 \,\mathrm{d}h\, \mathrm{d}x\, \mathrm{d}t < +\infty, \\
& \text{i.e. } u \in L_2(I;\, H^{2,2+\alpha}(\Omega)) \quad \text{and} \quad u \in L_2(I;\, H^{\frac{3}{2}+\alpha}(\partial\Omega)),\ \alpha \in (0, \tfrac{1}{2}).
\end{aligned}
$$

[2] Here and in the sequel, in boundary integrals ∇ denotes the gradient with respect to the tangential space variables.

These norms will be denoted as new. The second term on the right hand side of (7) will be estimated by

$$
\begin{aligned}
(9)\qquad & \frac{\tilde{c}^2}{2\varepsilon}\|u\|^2_{L_2(I;\ H^{1+2\alpha}(\partial\Omega_\vartheta))} \leqslant \frac{\tilde{c}^2}{2\varepsilon}\|u\|^{1-\theta}_{L_2(I;\ H^{\frac{3}{2}}(\partial\Omega_\vartheta))}\|u\|^{\theta}_{L_2(I;\ H^{\frac{3}{2}+\frac{4\alpha-1}{2\theta}}(\partial\Omega_\vartheta))} \\
& \leqslant \zeta\|u\|_{L_2(I;\ H^{\frac{3}{2}+\frac{4\alpha-1}{2\theta}}(\partial\Omega_\vartheta))} + \frac{(\tilde{c}^2\theta^\theta)^{\frac{1}{1-\theta}}(1-\theta)}{(2\varepsilon\zeta^\theta)^{\frac{1}{1-\theta}}}\|u\|_{L_2(I;\ H^{\frac{3}{2}}(\partial\Omega_\vartheta))}, \\
& \theta \in (0,1) \ \text{arbitrary}, \ \zeta > 0 \ \text{arbitrarily small.}
\end{aligned}
$$

To be able to estimate the first term on the right hand side of (9) by an appropriate new norm, we need to have $4\alpha - 1 < 2\alpha\theta$ which enables us to have $\alpha \in (0, \frac{1}{2})$.

For the estimates of the first term on the right hand side of (7) we need

Proposition 1. (Cf. [1], Sec. IV.) *Let J be an interval in $\mathbb{R}$, let real numbers $\beta > 0$, $p > 1$ be given. If $f \in H^\beta(J)$ has a bounded support, then $f \in L_p(J)$ for each p satisfying the inequality*

$$
(10)\qquad \frac{1}{\beta}\left(\frac{1}{2} - \frac{1}{p}\right) < 1 \quad \Longleftrightarrow \quad p < \frac{2}{1-2\beta}.
$$

If $\beta > \frac{1}{2}$, then $f \in C_0(J)$. For J bounded the corresponding imbedding $H^\beta(J) \hookrightarrow L_p(J)$ or $C_0(J)$ is compact.

Introducing the vector-valued spaces $H^\alpha(J\,;\mathcal{H})$ for a Hilbert space $\mathcal{H}$ as usual and using Proposition 1 to $\|f\|_{\mathcal{H}}$ for arbitrary f, we can easily generalize the proposition to the following case:

Corollary. *Let $J_1, \ldots, J_k$ be bounded intervals in $\mathbb{R}$ for an integer k. Then the compact imbedding $H^{\alpha_1}(J_1\,; H^{\alpha_2}(J_2\,; \ldots H^{\alpha_k}(J_k)\ldots)) \hookrightarrow L_{p_1}(J_1\,; L_{p_2}(J_2\,; \ldots L_{p_k}(J_k)\ldots))$ holds provided for each $i \in \{1, \ldots, k\}$, (10) is valid for p_i, α_i. For $\alpha_i > \frac{1}{2}$, $i = 1, \ldots, k$, the imbedding $H^\alpha(J_1\,; H^\alpha(J_2\,; \ldots H^\alpha(J_k)\ldots)) \hookrightarrow C_0(\prod_{i=1}^{k} J_k)$ holds.*

We remark that trace theorems for spaces employed can be easily based on Corollary.

More general investigation of vector-valued Sobolev (or Běsov-Sobolev) spaces is done in [11] on the base of a generalized approach via the Fourier transformation. We recall the renormation technique of Chapter 2 of [11], particularly Definition 2 of the appropriate class of the Lizorkin spaces $S^{\bar{r}}_{\bar{p},\bar{q}}B(\mathbb{R}^2)$ in 2.2.1 there which are, due to Section 2.3.4, Theorem 2 and Remark 4 there, equivalent to our class of the

Běsov-Sobolev spaces. Using these facts and the appropriate Nikol'skii lemma—see Theorem in 1.6.2 there (whose validity for our difference norm is obvious, cf. also [1]), we can prove another generalization of Proposition 1.

Proposition 2. *Let Ω be a bounded domain in $\mathbb{R}^k$ with a sufficiently smooth boundary, let α, $a \in \mathbb{R}_+^k$, let $p \in (1, +\infty)$, $q \in (1, +\infty)^k$ satisfy the inequality*

$$\sum_{i=1}^{k} \frac{1}{\alpha_i}\left(\frac{1}{p} - \frac{1}{q_i} + a_i\right) < 1. \tag{11}$$

Then the operator $D^a\colon f \mapsto D^a f$ is compact from $W_p^\alpha(\Omega)$ into $L_q(\Omega)$. If (11) *holds for $\frac{1}{q_i} = 0$, $i = 1, \ldots, k$, then D^a is compact from $W_p^\alpha(\Omega)$ into $C_0(\overline{\Omega})$.*

We remark that for for a noninteger the assertion of Proposition 2 signifies the complete continuity of the imbedding of $W_p^\alpha(\Omega)$ into the space whose norm is a sum of the $L_q(\Omega)$ norm and the appropriate fractional-derivative seminorm. For $p = 2$ Proposition 2 can be proved directly on the base of Proposition 1 and its Corollary (cf. (13) below).

In the first term on the right hand side of (7)

$$\varepsilon \|\nabla u\|^2_{L_{2p}(S_\vartheta)} \|g'(u)\|^2_{L_{\frac{2p}{p-1}}(S_\vartheta)} \tag{12}$$

we estimate its first factor by the old norm and the second by the new norm for α near $\frac{1}{2}$. Then for sufficiently small ε and ζ we are able to complete the proof of (8) for such α. The estimation of both factors in (12) will be based on the use of the extension technique (cf. [9]), of the Fourier transformation, Lemma from Appendix and of the Hölder inequality for the appropriate transforms. We denote the extended u by u again and use the notation τ, ξ_n, ξ_t for the dual variable to the time variable, to the normal space variable and the tangential space variable, respectively. Of course, τ and ξ_n are one-dimensional and ξ_t is two-dimensional. The old norms make possible to estimate $\nabla u \in \bigcap_{\varepsilon>0} H^{\frac{5}{16}-\varepsilon, \frac{1}{2}}(S_\vartheta) \hookrightarrow \bigcap_{\varepsilon>0} L_{\frac{36}{13}-\varepsilon}(S_\vartheta)$, $\varepsilon > 0$ arbitrary (cf. Proposition 2), because $|\tau|^{\frac{5}{8}-\varepsilon_1}|\xi|^2|\xi_n|^{1+\eta} \leqslant |\tau|^{\frac{5}{4}-\varepsilon}|\xi|^2 + |\xi|^4$ with $\varepsilon_1 \searrow 0$ for $\eta, \varepsilon \searrow 0$ and Corollary of Proposition 1 can be used. Therefore $p \in (1, \frac{18}{13})$, $\frac{2p}{p-1} > \frac{36}{5}$ in (12). Using the new norms (8) we first estimate for $\upsilon_0 \equiv |\xi_{t,1}\xi_{t,2}|$, $\upsilon \equiv \upsilon_0|\xi_n|$

$$\begin{aligned} &\upsilon_0^{\frac{10}{11}-\varepsilon_1}|\tau|^{\frac{10}{11}-\varepsilon_1} \leqslant \mathrm{const}(|\tau|^{\frac{5}{4}-\varepsilon}\upsilon_0^{\frac{1}{2}} + \upsilon_0^{2-\varepsilon}) \quad \text{with } \varepsilon_1(\varepsilon) \searrow 0 \ \text{ for } \varepsilon \searrow 0 \\ &\Longrightarrow u \in \bigcap_{\varepsilon>0} L_{22-\varepsilon}(S_\vartheta), \\ &\upsilon^{\frac{25}{27}-\varepsilon_1}|\tau|^{\frac{25}{27}-\varepsilon_1} \leqslant \mathrm{const}(\upsilon^{\frac{5}{3}-\varepsilon} + |\tau|^{\frac{5}{4}-\varepsilon}\upsilon^{\frac{2}{3}}) \quad \text{with } \varepsilon_1(\varepsilon) \searrow 0 \ \text{ for } \varepsilon \searrow 0 \\ &\Longrightarrow u \in \bigcap_{\varepsilon>0} L_{27-\varepsilon}(Q_\vartheta). \end{aligned} \tag{13}$$

From these estimates we can immediately see: if there is $\varepsilon > 0$ (arbitrarily small) such that

$$g'(z) = O(z^{3\frac{1}{18}-\varepsilon}), \ z \to +\infty, \tag{14}$$

then the second factor in (12) can be estimated by the new norms and the proof is complete.

The growth condition (14) for g' need not be always satisfactory. Hence we shall approach the problem more finely. The (only) other term in the heat part of (4) which must be checked in our estimation is its "right hand side" $\operatorname{div} \dot{v}$. Applying integration by parts to it, we obtain the volume and the boundary integral containing $\dot{v}$ only and both of them have their L_p-norms bounded just by the old norms of u for each $p \in (1, +\infty)$, cf. (6). We shall denote the maximum of those norms by m_p. Putting $w = [|u|^{2\beta}u, 0]$ for a suitable $\beta > 0$ in (4), we obtain

$$\begin{aligned}
(15)\quad & \int_S g(u)\, u|u|^{2\beta} \,\mathrm{d}x\,\mathrm{d}t + \beta_0 \frac{1+2\beta}{(1+\beta)^2} \int_Q |\nabla(|u|^\beta u)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
& + (1+2\beta) \sup_{t\in I} \int_\Omega |u|^{2\beta+2}(t,\cdot)\,\mathrm{d}x \leqslant \sup_Q |g(T)| \int_S |u|^{2\beta+1} \,\mathrm{d}x\,\mathrm{d}t \\
& + m_{\frac{p}{p-1}} \Bigg(\Big(\int_S |u|^{p(2\beta+1)} \,\mathrm{d}x\,\mathrm{d}t \Big)^{\frac{1}{p}} + \Big(\int_Q |\nabla(|u|^{2\beta}u)|^p \,\mathrm{d}x\,\mathrm{d}t \Big)^{\frac{1}{p}} \Bigg) \\
\leqslant\ & \sup_Q |g(T)| \int_S |u|^{2\beta+1} \,\mathrm{d}x\,\mathrm{d}t + m_{\frac{p}{p-1}}^{\frac{p}{p-1}} \frac{p-1}{p^{\frac{p}{p-1}}} \\
& + \Big(\int_S |u|^{p(2\beta+1)} \,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon p}{2} \int_Q |\nabla(|u|^\beta u)|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{2-p}{2\varepsilon^{\frac{2}{p(2-p)}}} \int_Q |u|^{\beta\frac{2p}{2-p}} \,\mathrm{d}x\,\mathrm{d}t \Big)
\end{aligned}$$

with $1 < p$ arbitrarily close to 1 and $0 < \varepsilon$ arbitrarily small. We modify the above mentioned procedure putting $w = \big[\max(z, |u|^{2\beta}u), 0\big]$ for a constant $z > 0$. Denoting $M_z := \{[t,x] \in \overline{Q}; u(t,x) \geqslant z\}$, $Q_z \equiv M_z \cap Q$, analogously S_z and Ω_z and calculating like in (15), we finally obtain

$$\begin{aligned}
(16)\quad & \int_{S_z} g(u)\, u|u|^{2\beta} \,\mathrm{d}x\,\mathrm{d}t + (1+2\beta) \sup_{t\in I} \int_{\Omega_z} |u|^{2\beta+2}(t,\cdot)\,\mathrm{d}x \\
& \leqslant k_0(p, \sup_Q |g(T)|, \beta_0) \\
& \quad \times \Big(\int_{S_z} |u|^{2\beta+1} \,\mathrm{d}x\,\mathrm{d}t + \int_{S_z} |u|^{p(2\beta+1)} \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_z} |u|^{\beta\frac{2p}{2-p}} \,\mathrm{d}x\,\mathrm{d}t \Big) + k_1
\end{aligned}$$

with k_1 dependent on z, p, $m_{\frac{p}{p-1}}$ and the old norms of u only. Now we employ the assumed additional growth condition (5) for g. As $p-1$ can be arbitrarily small, we derive from (5) and (16) that $u^+ \in L_q(Q) \cap L_q(S)$ for every $q \in (1, +\infty)$. Hence we have proved (8) under the assumption (5) for g with polynomially bounded growth at infinity of itself and its first derivative. For such g we have also proved that $u^+ \in B(I;\, L_p(\Omega))$, $u^+ \in L_p(S)$ and therefore $g(u^+) \in L_p(\Omega)$ and $g(u^+) \in L_p(S)$ for each $p \in (1, +\infty)$.

Step 3. Now, we prove a better time regularity of u. We still assume (5) and, moreover, boundedness of g' on $(-\infty, 0)$. We use the procedure described in [5], Part III., Secs.1 and 2, but with necessary modifications respecting the up-to-now non-proved boundedness of u. We put $w_1 = \dot{u}_\ell - \dot{u}$, where ℓ is the time shift, and $w_2 = 0$ in (4). On the left hand side of the resulting variational inequality we estimate in fact the terms

$$
\begin{aligned}
(17)\qquad & \beta_0 \int_{-\delta}^{\delta} \int_Q (\dot{u}_{-\ell} - \dot{u})^2 |\ell|^{-1-2\alpha} \, \mathrm{d}\ell \, \mathrm{d}x \, \mathrm{d}t \\
& + \frac{1}{2} \int_{-\delta}^{\delta} \int_\Omega (\nabla(u_{-\ell} - u))^2 (x, t_0) |\ell|^{-1-2\alpha} \, \mathrm{d}\ell \, \mathrm{d}x, \\
& t_0 \in \langle 0, T \rangle,
\end{aligned}
$$

on the right hand side we have the volume term whose estimation is very easy due to the above mentioned regularity of $\operatorname{div} \dot{v}$ (cf. (6)), and two boundary terms. The most important estimate is that of the term

$$
\begin{aligned}
(18)\qquad & \int_{-\delta}^{\delta} \int_S (g(T_{-\ell}) - g(T))(\dot{u}_{-\ell} - \dot{u}) |\ell|^{-1-2\alpha} \, \mathrm{d}\ell \, \mathrm{d}x \, \mathrm{d}t \\
& \leqslant c_0 \int_{-\delta}^{\delta} |\ell|^{-1+\varepsilon} \|\dot{u}_{-\ell} - \dot{u}\|^2_{H^{-\frac{3}{16}-\varepsilon_0}(I;\, L_2(S))} \, \mathrm{d}\ell \\
& \int_{-\delta}^{\delta} |\ell|^{-1-4\alpha-\varepsilon} \|g(T_{-\ell}) - g(T)\|^2_{H^{\frac{3}{16}-\varepsilon_0}(I;\, L_2(S))} \, \mathrm{d}\ell \\
& \leqslant c_1 + c_2 \|g(T)\|^2_{H^{\frac{3}{16}+2\alpha-\frac{1}{\varepsilon_0}}(I;\, L_2(S))}
\end{aligned}
$$

with c_0, c_1, c_2 suitable positive constants and ε_0, $\varepsilon > 0$ arbitrarily small. In the last estimate we have used also the renormation lemma. The last norm in (18) will be estimated by $\sup\limits_{R(T)} |g| \|T\|^2_{H^{\frac{3}{16}+2\alpha-\frac{1}{\varepsilon_0}}(I;\, L_2(S))}$, where $R(T)$ is the range of T, and this can be done for $\alpha < \frac{5}{32}$.

A little more difficult is the estimation of the last term which differs from that in (18) by $g(T)$ being consistently replaced by $g(u)$. First we proceed as in (18), but the estimate of the admissible α needs the knowledge of boundary regularity of $g(u)$, because we have not yet proved the essential boundedness of u. If (5) holds and g'' is again polynomially bounded, then we are able to prove $g'(u)\,\dot u \in L_{2-\varepsilon}(Q)$ and $\nabla^2 g(u) \equiv g''(u)\nabla u \otimes \nabla u + g'(u)\nabla^2 u \in L_{\frac{54}{29}-\varepsilon}(Q)$ for $\varepsilon > 0$ sufficiently small. Using Proposition 2 we prove that $g(u) \in \bigcap\limits_{\varepsilon>0} H^{\frac{49}{54}-\varepsilon,\, \frac{49}{27}-\varepsilon}(Q)$, hence $g(u) \in H^{\omega}(I;\, \partial\Omega)$ for each $\omega \in (0, \frac{71}{108})$. Thus the above mentioned procedure can be also executed for this term with $\alpha < \frac{5}{32}$. Iterating the procedure like in [5], Part III, Sec. 1, we improve the admissible interval for α to $(0, \frac{5}{24})$. This result gives us a certain better time regularity of the trace of u. Now we use the iterative method of [5], Part III, Sec. 2 consisting in the estimation of the better time regularity of ∇u and then of the trace of u with the help of time shifts in the arguments of (4) (by putting $w_1 = (w_1)_{-\ell} \equiv u_\ell - u$ and $w_2 = 0$ in (4)) and in combination with the above described procedure. Carrying out this iterative procedure, we prove that $u \in \bigcap\limits_{\varepsilon>0} H^{\frac{5}{4}-\varepsilon}(I;\, L_2(\Omega))$.

Step 4. Here we shall complete the proof of Theorem 1. The proof of the normal regularity of u under the assumption (5) is again analogous to that in [5], Part III. We are able to prove that $g(u) \in L_2(I;\, H^1(S))$ which enables us to use the interpolation method like there (treating $\dot u$ as a part of the right hand side of a parametric elliptic equation) and prove that $u \in \bigcap\limits_{\varepsilon>0} H^{\frac{5}{4}-\varepsilon,\, \frac{5}{2}-\varepsilon}(Q) \cap \bigcap\limits_{\varepsilon>0} H^{1-\varepsilon,2-\varepsilon}(S)$.

It remains to prove a little better tangential regularity of u. We return to the estimate (7), but for $\alpha = \frac{1}{2} + \tilde\theta$, $\tilde\theta \in (0, \frac{1}{2})$. Unlike (7) we estimate it with the help of the renormation lemma (see Appendix) as follows:

$$
\begin{aligned}
(19)\qquad & \int\limits_{S_\vartheta} \int\limits_{(-\eta,\eta)^2} |h|^{-2-2\alpha}(g'(u_{-h})\nabla u_{-h} - g'(u)\nabla u, \nabla u_{-h} - \nabla u)\,\mathrm{d}h\,\mathrm{d}x\,\mathrm{d}t \\
& \leqslant \varepsilon_0 \|u\|^2_{L_2(I;\, H^{2+\alpha}(\partial\Omega))} + \frac{1}{\varepsilon_0} \int\limits_{S_\vartheta} \int\limits_{(-\eta,\eta)^2} |h|^{-2-2\tilde\theta}\Big((g'(u))^2 |\nabla(u_{-h} - u)|^2 \\
& \quad + |\nabla u_{-h}|^2 (g'(u_{-h}) - g'(u))^2\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}h,
\end{aligned}
$$

where $\varepsilon > 0$ is arbitrarily small. The first term in (19) will be estimated by the "new" norm whose boundedness is being proved. The second term (without the coefficient) can be estimated by

$$
\begin{aligned}
(20)\qquad & \|g'(u)\|^2_{L_p(S)} \|u\|^2_{L_{\frac{2p}{p-1}}(I;\, W^{1+\tilde\theta+\varepsilon_1}_{\frac{2p}{p-1}}(\partial\Omega))} \\
& + \|\nabla u\|^2_{L_{4-\varepsilon_1}(S))} \|g'(u)\|^2_{L_{4+\frac{2\varepsilon_1}{2-\varepsilon_1}}(I;\, W^{\tilde\theta+\varepsilon_2}_{4+\frac{2\varepsilon_1}{2-\varepsilon_1}}(\partial\Omega))}
\end{aligned}
$$

with $p \geqslant 1$ arbitrarily large and $\varepsilon_1, \varepsilon_2$ arbitrarily small. As we can prove the boundedness of all terms in (20) for a suitable choice of $p, \varepsilon_1, \varepsilon_2$ at least for $\tilde{\theta} \in (0, \frac{1}{2})$ using the up-to-now proved regularity of u and Proposition 2, and the estimation of the other terms on the right hand side of the appropriate energy-type inequality is standard, we have proved that for such $\tilde{\theta} > 0$

$$(21) \qquad T \in \bigcap_{\varepsilon>0} H^{\frac{1}{2}-\varepsilon, 1+\tilde{\theta}}(S) \quad \Longrightarrow \quad u \in \bigcap_{\varepsilon>0} H^{\frac{5}{4}-\varepsilon, \frac{5}{2}-\varepsilon, \frac{5}{2}+\tilde{\theta}, \frac{5}{2}+\tilde{\theta}}(Q).$$

From it we can prove $u \in C(\overline{Q})$ and a possible further regularization procedure is quite standard. Particularly, (21) holds for every $\tilde{\theta} > 0$.

The proof of the regularity of v is then quite standard, too. Some of its steps were mentioned above. Via the shift method we prove better time and tangential regularity corresponding to the result for u in (21). In fact, it is sufficient to use an estimate like (26) of [5], Part III and via Corollary of Proposition 1 we prove the boundedness and continuity of $\nabla \dot{v}$ and therefore of the stresses. Theorem 1 is proved. □

Remark 1. Without the assumption (5) we could proceed with the proof under the only growth condition (14). As (5) seems to be satisfactory and the estimation would be more complicated, we avoid this process.

2. If in (3) a function $\gamma(u)$ with globally bounded γ', γ'', γ''' occurs instead of γu (γ a constant), the proof of $v \in H^{1,3}(Q; R^3)$ requires to prove $\nabla u \in L_4(Q)$. To prove it, however, we need the proof of (8) which can be proved e.g. under assumption (14) without further knowledge about $\dot{v}$. If e.g. (5) and (14) hold simultaneously, we are able to do all the other estimation, to arrive at (21) and, finally, to prove the assertion of Theorem 1.

3. For $\dim \Omega = 2$ we can prove $\nabla u \in L_4(Q)$ if $\nabla u \in H^{\frac{12+8\alpha}{7+6\alpha}, \frac{1}{2}+\alpha}(Q)$ and $\alpha > \alpha_0 \equiv \frac{1}{\sqrt{2}} - \frac{3}{8}$, which holds for $T \in \bigcap_{\varepsilon>0} H^{\frac{1}{2}-\varepsilon, \alpha}(S)$ for $\alpha > \alpha_0$.

Replacing the condition of Theorem 1 for T by some spatially asymmetric condition admitting jumps also in one of the spatial direction that is, e.g.

$$(22) \qquad g(T) \in \bigcap_{\varepsilon>0} H^{\frac{1}{2}-\varepsilon, \frac{1}{2}-\varepsilon, \frac{11}{6}+\theta}(S) \quad \text{for some } \theta > 0,$$

we can prove the basic regularity $\dot{u} \in L_2(Q)$, $u \in \bigcap_{\varepsilon>0} L_2(I; H^{1,2-\varepsilon}(\Omega_\vartheta))$ and analogous results for v for every choice of the region Ω_ϑ. Then $\Delta u \in L_2(Q)$ and, in fact, the normal regularity can be calculated directly (from the tangential shift method we in fact obtain $\frac{\partial u}{\partial x_t} \in \bigcap_{\varepsilon>0} L_2(I; H^{1-\varepsilon}(\Omega_\vartheta))$ and then we can prove $\frac{\partial^2 u}{\partial x_n^2} \in$

$\bigcap\limits_{\varepsilon>0} L_2\left(I\,;\,(H^{\varepsilon}(\Omega_{\vartheta}))^*\right)$, where x_t is an arbitrary unit tangential vector and x_n is the unit normal vector). This result finally gives $u \in \bigcap\limits_{\varepsilon>0} H^{1,2-\varepsilon}(Q)$. Using the same arguments we can prove (6) with $3-\varepsilon$ instead of 3 for $\varepsilon > 0$ arbitrarily small. Then we can execute the first two steps of the further regularization procedure as above under the assumption (5) and all other growth conditions of Theorem 1. We prove

$$
\begin{aligned}
(23) \qquad & u \in \bigcap_{\varepsilon>0} H^{\frac{5}{4}-\varepsilon,2-\varepsilon,2-\varepsilon,\frac{5}{2}-\varepsilon}(Q_{\vartheta}) \cap \bigcap_{\varepsilon>0} H^{1-\varepsilon,\frac{3}{2}-\varepsilon,2-\varepsilon}(S_{\vartheta}) \quad \text{and} \\
& u^+ \in \bigcap_{p\in(1,+\infty)} L_p(Q) \cap \bigcap_{p\in(1,+\infty)} L_p(S)
\end{aligned}
$$

for every choice of the region Ω_{ϑ}. There is no possibility to prove any better normal regularity of u. Thus we shall proceed in the proof of better tangential regularity in the "regular" tangential direction (we denote it by x_r) similarly to (19) and (20) with the necessary changes: the indices for derivatives will hold for x_r only and we have proved up to now only that $\frac{\partial u}{\partial x_r} \in \bigcap\limits_{\varepsilon>0} H^{\frac{1}{2}-\varepsilon,\frac{5}{6}-\varepsilon,1-\varepsilon}(S_{\vartheta}) \hookrightarrow \bigcap\limits_{\varepsilon>0} L_{\frac{42}{11}-\varepsilon}(S_{\vartheta})$ (its regularity in the remaining tangent direction further denoted by x_e follows from the shift and renormation technique, but not directly from (23)). We can assume in fact that $\tilde{\theta} \in (0,1)$ because of the renormation lemma (for $\tilde{\theta} \geqslant \frac{1}{2}$ we need to start in (19) from the second differences, its right hand side, however, remains unchanged). As in (20) $p \in (1,+\infty)$ can be arbitrary, the first term of (20) can be estimated via Proposition 2 for every $\tilde{\theta} \in (0,1)$ thanks to (23) and the assumption to g'. Using the fact that the differences of the positive part of a function can be estimated by the differences of the function and the assumption $|g'(z)| \leqslant c'(1+(z^+)^{2n})$ for some $n \in \mathbb{N}$, we can estimate the second factor of the second term of (20) by

$$
\begin{aligned}
(24) \qquad & \int_{-\delta}^{\delta} \int_{S_{\vartheta}} |h|^{-1-\alpha q} |g(u_{-h}) - g(u)|^q \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}h \\
&= \int_{-\delta}^{\delta} \int_{S_{\vartheta}} |h|^{-1-\alpha q} \left| \int_{u(t,x_e,x_r)}^{u(t,x_e,x_r+h)} g'(\zeta)\,\mathrm{d}\zeta \right|^q \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}h \\
&\leqslant c' \|u\|^q_{W_q^{0,0,\alpha}(S_{\vartheta})} \\
&\quad + \frac{c'}{2n+1} \int_{-\delta}^{\delta} \int_{S_{\vartheta}} |h|^{-1-\alpha q} |u_{-h} - u|^q |P_0(u^+, u^+_{-h})|^q \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}h,
\end{aligned}
$$

where P_0 is a polynomial of degree $2n$. Due to (23) and the above proved integrability of $\frac{\partial u}{\partial x_r}$ the application of the Hölder inequality to the last integral in (24) does not change the estimate (with a possible exception of an arbitrarily small $\varepsilon > 0$) and, in fact, we work with q greater than but arbitrarily close to $\frac{21}{5}$ and arbitrary $\alpha > \tilde{\theta}$. Using Proposition 2 we can see that the estimate can be made for $\tilde{\theta} < \frac{109}{126}$. Like in [5], Part III, Sec. 2 it suffices to restrict ourselves to $\tilde{\theta} \leqslant \frac{5}{6} + \theta$, $\theta > 0$ arbitrarily small. Thus we have proved for some $\theta > 0$ that $u \in \bigcap_{\varepsilon>0} H^{\frac{5}{4}-\varepsilon,2-\varepsilon,2-\varepsilon,\frac{10}{3}+\theta}(Q) \cap C_0(I; H^{\frac{4}{3}-\varepsilon,\frac{4}{3}-\varepsilon,2+\frac{3}{5}\theta}(\Omega)) \hookrightarrow C_0(\bar{Q})$. (For some possible better regularity of u we can then use standard procedures.) Then we prove the better regularity of v in the x_r-direction. We left the details to the reader. Thus the following theorem is proved:

Theorem 1′. *The assertion of Theorem* 1 *also holds, provided its requirement of regularity of* T *is replaced by* (22) *and the other assumptions of Theorem* 1 *remain valid.*

Remark. The condition (22) is valid e.g. if T is monotonic in time for almost every $x \in \partial\Omega$, has uniformly bounded variation in the "non-regular" space variable (independently of time and of the other space variable) and is sufficiently smooth in the regular space variable (e.g. with the Hölder continuous derivative with the exponent corresponding to (22)).

2. Optimal heating of the thermoelastic processes

In this part we will treat the optimal control of processes described by the above defined systems. We shall keep the general scheme of the optimal control problem

$$(25) \qquad \Lambda(\mathcal{U}) \equiv J(\mathcal{U}, \Psi\mathcal{U}) \to \inf \qquad \text{subject to } \mathcal{U} \in U_{\mathrm{ad}} \subset U,\ \Psi\mathcal{U} \in Y_{\mathrm{ad}} \subset Y.$$

To avoid any confusion in the sequel, we preserve the notation of Sec. 1 and denote the control variable $\mathcal{U}$ by T, provided the heating regime is the only control variable. The state operator $\Psi\colon T \mapsto [u, v]$ (in the notation of the preceding section) will be denoted by Ψ_1 for the system $\{(1), (2)\}$ and by Ψ_2 for the system (3). The set of admissible controls U_{ad} will be a closed subset of the set of functions defined on S with the range in the interval $\langle 0, D\rangle$ being nondecreasing in time and belonging to the control space U. Suitable possibilities of the choice of U, U_{ad} having practical reasons are introduced below. The state-space constraint is defined in the following way: We assume that for a displacement v, a reference stress $r(v) \equiv r(v(t,x))$, $[t,x] \in \bar{Q}$, dependent only on the stress tensor to v at $[t,x]$ is defined. It is well known (cf. e.g. [10]) that such a definition of r ensures that the mapping $T \mapsto r(v)$ is well defined even when the semicoercive problem (2) does not ensure unicity of its solution formulated in displacements. Moreover, we assume that for any temperature u some bound for the maximal admissible stress $s(u)$ is prescribed. Then

$$Y_{\mathrm{ad}} := \{y \equiv [u, v] \in Y\,;\ (r(v))^2 \leqslant s(u) \ \text{ on } \bar{Q}\}. \tag{26}$$

Of course, we assume that r is at least Lipschitz and s is C_1-smooth and positive. The choice of the state space, the particular choice of the reference stress and the form and the qualities of the cost function will be treated below again.

To be able to do it, we need to recall the following facts from [5] and the first part of this paper: As the Hook law for homogeneous isotropic bodies is included in (2) and (3), we have the stress tensor to the state operator Ψ_i, $i = 1, 2$ in the particular form

$$\begin{aligned}
(27)\qquad \tau_{ij}(v) &\equiv \frac{E}{(2+2\sigma)(1-2\sigma)}\Bigl((1-2\sigma)e_{ij}(v) + \sigma\delta_{ij}\sum_{k=1}^{N} e_{kk}(v)\Bigr),\\
&\quad i, j = 1, \ldots, N,\\
\tau_{ij}(v) &\equiv \frac{E}{(1+\sigma)(1-2\sigma)}\Bigl((1-2\sigma)e_{ij}(v) + (1-2\sigma_1)e_{ij}(\dot{v})\\
&\quad + 2\sigma\delta_{ij}\sum_{k=1}^{N} e_{kk}(v) + (2\sigma_1 - 1)\delta_{ij}\sum_{k=1}^{N} e_{kk}(\dot{v})\Bigr),\\
&\quad i, j = 1, \ldots, N,
\end{aligned}$$

respectively, with the Young modulus of elasticity E, the small strain tensor $e_{ij}(v) \equiv \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$, $i, j = 1, \ldots, N$, and the Kronecker symbol δ_{ij}. However, the majority of the results recalled further is valid for the general case of the linear elasticity, too. If Ω has a sufficiently smooth boundary, i.e. $\partial\Omega \in C_{\frac{3}{2}+\eta}$, $C_{\frac{5}{2}+\eta_0}$ for $\dim\Omega \equiv N = 2, 3$, respectively, with $\eta_0 > 0$ arbitrarily small and if $g(T) \in H^{\frac{1}{2}-\varepsilon}(I\,;\ L_2(\partial\Omega))$, $H^{\frac{1}{2}-\varepsilon, 1+\eta}(S)$ for $N = 2, 3$ respectively again, where ε is a fixed very small positive number and $\eta \equiv \eta(\varepsilon) > 0$ can be arbitrarily small for arbitrarily small ε, then both operators Ψ_1, Ψ_2 give bounded and continuous stresses. For $\dim\Omega = 3$ the same assertion is true for $g(T)$ from $H^{\frac{1}{2}-\varepsilon, \frac{1}{2}-\varepsilon, \frac{11}{6}+\eta}(S)$, if analogous requirements for ε and η are satisfied. For the state operator Ψ_1 and the isolated convex boundary nonsmoothnesses of $\Omega \subset \mathbb{R}^2$, the same is true for $g(T)$ satisfying the above mentioned condition. In the three-dimensional case we have proved it for the nonsmoothnesses having the form of edges, and the possibility to combine these nonsmoothnesses with the nonregularity of the control in one space variable is limited (see [5], Part III, Sec. 1). For $\dim\Omega = 2$ also certain combinations of the boundary value condition in (2) on a part $I \times (\partial\Omega)_1$ of S with the condition describing the influence of a support on another part $I \times (\partial\Omega)_2$, namely

$$\mathcal{S}_t = 0,\ v_n = 0 \tag{28}$$

$$v = 0, \tag{29}$$

will be possible in order to obtain bounded stresses, if $\partial\Omega_i$ consists of finite number of components, $i = 1, 2$ and, moreover, for (28) the angle between $\partial\Omega_1$ and $\partial\Omega_2$ is less than or equal to $\frac{\pi}{2}$ and for (29) less than $\arcsin\sqrt{1-\sigma}$ at their every contact point. The condition has a clear generalization to $\Omega \subset \mathbb{R}^3$, the common boundary Γ of $\partial\Omega_1$ and $\partial\Omega_2$ must consist of sufficiently smooth simple closed curves. Of course, for such cases the requirements for the admissible control spaces remain unchanged.

The above defined control constraints yield directly that $T \in U_{\mathrm{ad}}$ implies $T \in \bigcap_{\varepsilon>0} H^{\frac{1}{2}-\varepsilon}(I;\ L_2(\partial\Omega))$ (cf. [5], Lemma 2) and the same holds for $g(T)$ if g is non-decreasing.

Generalizing the procedure made in Sec. 1 as well as in [5] and denoting the anisotropic Sobolev spaces in such a way that the first index corresponds to the time, the second one to the normal and the third one to the tangential space variable, we prove that for $g(T) \in H^{\alpha,\beta}(S)$, for the original system [(1),(2)] or (3) and for Ω without boundary nonsmoothnesses, the corresponding state variable belongs to

$Y \equiv Y_1(\alpha,\beta) \times Y_2(\alpha,\beta)$, where for $N = \dim\Omega$ and $\tilde{n} = \min(2\alpha,\beta)$

$$Y_1(\alpha,\beta) := \left\{ w \in H^{\alpha+\frac{3}{4},\tilde{n},\beta+\frac{3}{2}}(Q);\ \frac{\partial^{\frac{3}{2}} w}{\partial x_n^{\frac{3}{2}}} \in H^{\alpha,0,\beta}(Q) \right\}, \tag{30}$$

$$Y_2(\alpha,\beta) \equiv \tilde{Y}_2(\alpha,\beta) := \left\{ w;\ \nabla w \in H^{\alpha+\frac{3}{4},\tilde{n},\beta+\frac{3}{2}}(Q;\ \mathbb{R}^{N^2}),\ \frac{\partial^{\frac{5}{2}} w}{\partial x_n^{\frac{5}{2}}} \in H^{\alpha,0,\beta}(Q) \right\}$$

for the system$[(1),(2)]$,

$$Y_2(\alpha,\beta) \equiv \tilde{\tilde{Y}}_2(\alpha,\beta) := \left\{ w;\ \nabla\dot{w} \in H^{\alpha+\frac{3}{4},\tilde{n},\beta+\frac{3}{2}}(Q;\ \mathbb{R}^{N^2}),\ \frac{\partial^{\frac{5}{2}} \dot{w}}{\partial x_n^{\frac{5}{2}}} \in H^{\alpha,0,\beta}(Q) \right\}$$

for the system (3).

These facts are written formally. In fact, the local behaviour of the "localizations" of the solution made with the help of the partition of unity is displayed here, provided these "localizations" are considered after their prolongation to the whole space and after use of the Fourier transformation to those prolongation as usual. The proof of (30) is based on the well-known behaviour of the solution for the right-hand side (or for the right-hand side of the appropriate boundary value condition) belonging to a suitable anisotropic space, in which the space-regularity index equals twice the time-regularity index, and on the posibility to consider the nonlinear solution-dependent terms as additional parts of the right-hand sides, provided their regularity is better than that of the appropriate original right-hand side. For more general spaces, the above mentioned results (for the the right-hand sides in L_2 and in $H^{\tilde{n},2\tilde{n}}$) and the appropriate shift technique must be used (here possibly for mixed time-space derivatives). To avoid further technicalities with these methods used in Sec. 1, we do not present the detailed proofs here. The nonlinearities in systems, however, restrict the possibility in the use of these methods as it was seen e.g. in Sec. 1 or in different parts of [5]. However, the relations (30) hold for the system [(1),(2)] or (3) at least for $\alpha \in (\alpha_0, 1)$ for a suitable $\alpha_0 \in (0, \frac{1}{2})$ and $\beta \geqslant 0$ for $N = 2$, $\beta \geqslant \frac{1}{2}$ for $N = 3$ in the isotropic case. For $N = 3$ and $\beta \equiv [\beta_1, \beta_2]$ with $\beta_2 \geqslant \frac{1}{2}$, $\beta_1 \geqslant \frac{1}{2} - \tilde{\varepsilon}$ for $0 < \tilde{\varepsilon}$ small our method proves (30) with $\tilde{n} = \min(2\alpha, \beta_1, \beta_2)$ for the system [(1),(2)] without any additional restrictions; for the system (3) for g bounded below we have proved it under some additional restriction on the growth of g and its first derivative at $+\infty$ (see the preceding section). The magnitude of this restriction depends on $\tilde{\varepsilon}$. The methods of proofs of these facts yield also continuity of the operators Ψ_i, $i = 1, 2$, considered to act between the mentioned spaces. Any occuring nonsmoothnesses of the boundary or any combination of the original boundary value condition and (28) or (29) limit the possibility to prove (30) in the space variable which is normal to the nonsmoothness or to Γ in the neighbourhood of such singularity—there is a strict limit for the regularity in this variable which is dependent on the magnitude of the angle. As the implication $T \in H^{\alpha,\beta}(Q) \to g(T) \in H^{\alpha,\beta}(Q)$ is not true in general, there are three important problems:

1^0 the recommended choice of U_{ad} to preserve the validity of $g(T) \in U$ for a given choice of U and each $T \in U_{\mathrm{ad}}$. This is clearly true for every U_{ad} if $U = H^{\alpha,\beta}(S)$ with $\alpha, \beta \leqslant 1$, which can be satisfied for two-dimensional problems. For three-dimensional problems higher-order derivatives of composed functions must be used which increase the request for the "weak differentiability" of the functions belonging to U_{ad}. E.g. in the situation of Theorem 1 T must belong to $\bigcap_{\varepsilon>0} H^{\frac{1}{2}-\varepsilon,\frac{5}{2}+\eta_0}(S)$ for some $\eta_0 > 0$ to obtain that $g(T)$ satisfies the assumptions of that theorem (with the use of Proposition 2). For practical problems in the context of the above mentioned Lemma 2 from [5] it seems to be reasonable e.g. to assume additional conditions in the definition of U_{ad} in such a way that in the spatial direction, where (locally) some discontinuities of the heating regimes are allowed, there is a constant K such that for all $T \in U_{\mathrm{ad}}$ and every choice of $t \in I$ and of the possible other spatial variable, the variation of T in the variable considered is bounded by K. A suitable boundedness in the space of functions with the Hölder continuous derivatives with the appropriate Hölder index represents one of the easiest possibilities how to cope with the "regular" spatial variable. Moreover, a bound for the maximal admissible C_2-magnitude of the local coordinate transformation can be obtained. Then for $u \in U_{\text{ad}}$ the above mentioned implication is valid (cf. Remark at the end of Sec. 1);

2^0 the compactness of U_{ad} which is important for the existence of an optimal control. It depends on the introduction of some additional assumption to U_{ad}—the compactness can be then ensured by Proposition 2. Adopting e.g. the above recommended assumptions, we can find suitable spaces U in which U_{ad} is compact and the state-space constraint has still sense in the corresponding Y (given by (30));

3^0 the differentiability of the state operator, particularly of the superposition operators (in the sequel, the superposition operator is the operator $u \mapsto F \circ u$ for a given function F).

We remark that such a choice of U_{ad} does not seem to exceed requests of the technical practice.

We choose the control space U and the state space Y to obtain continuity of the cost function, some kind of differentiability of the state operator and compactness of U_{ad} ensuring that there is an optimal solution of (25). We define spaces

$$
\begin{aligned}
&\mathcal{Y}_1^1(\alpha,\beta) := \left\{u \in Y_1(\alpha,\beta)\,;\, u(0,\cdot) = 0 \text{ on } \Omega,\ \beta_0\frac{\partial u}{\partial t} = \Delta u \text{ a.e. in } Q\right\},\\
(31)\quad &\mathcal{Y}_1^2(\alpha,\beta) := \left\{v \in \tilde{Y}_2(\alpha,\beta)\,;\, v(0,\cdot) = 0,\ \text{on } \Omega\right\},\\
&\mathcal{Y}_1(\alpha,\beta) \equiv \mathcal{Y}_1^1(\alpha,\beta) \times \mathcal{Y}_1^2(\alpha,\beta),\\
&\mathcal{Y}_2 := \{[u,v] \in Y_1(\alpha,\beta) \times \tilde{\tilde{Y}}_2(\alpha,\beta);\\
&\qquad \text{all equations, all initial conditions}\\
&\qquad \text{and the boundary condition to the Lamé part of (3) hold}\}.
\end{aligned}
$$

It is not difficult to see that the relation between the space from which T are taken and the resulting regularity of $[u,v]$ expressed in (30) can be inverted in the sense that if $[u,v]$ is in the space defined in (31), then $g(T) \in H^{\alpha,\beta}(S)$. The operator Ψ_1 has two components: $\Psi_{1,1}$ assigning the temperature part of the state to a given control and $\Psi_{1,2}$ assigning the displacement part of the state to it. We can write $\Psi_{1,1} \equiv \Xi_2 \circ \Xi_1$ and $\Psi_{1,2} \equiv \Xi_4 \circ \Xi_3 \circ \Xi_2 \circ \Xi_1$, where $\Xi_1 : T \mapsto g(T)$ and $\Xi_3 : u \mapsto \gamma(u)$ are the superposition operators, $\Xi_2 : \tilde{T} \mapsto u$, the solution of (1) with $g(T) \equiv \tilde{T}$ is onto $\mathcal{Y}_1^1$ and the linear operator $\Xi_4 : \tilde{u} \mapsto v$, where v is the solution of (2) with $\gamma(u) = \tilde{u}$, is onto $\mathcal{Y}_1^2$.

The appropriate differentiability of the operators employed depends then on the differentiability of the superposition operators on Běsov-Sobolev spaces. As the problem does not seem to be frequently studied (cf. [12]) and the author has not found sufficiently satisfactory literature, we shall slightly touch this problem on $\mathbb{R}^m$— its extension to the bounded domains with a sufficiently smooth boundary via the local coordinates method is obvious from the above mentioned technique. We start with the difference seminorm

$$\|\cdot\|'_{m,k,\alpha} : u \mapsto \left(\int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^k} \frac{(\Delta_h u)^2}{|h|^{k+2\alpha}} \,\mathrm{d}x \,\mathrm{d}h \right)^{\frac{1}{2}}, \tag{32}$$

where $\Delta_h u(x) = u(x+h) - u(x)$, $\alpha \in (0,1)$ and $h \in \mathbb{R}^k \times \{0\}$. Let $F \in C_3(\mathbb{R})$ and $u, \ell \in H^{\alpha,0}(\mathbb{R}^m) \cap B(\mathbb{R}^m)$ (the space of bounded functions) be such that $\|u\|_{m,k,\alpha}, \|\ell\|_{m,k,\alpha} < +\infty$. Then, easily, the same assertion holds for $F \circ (u + \mu\ell)$, $\ell\, F' \circ (u + \mu\ell)$, $\ell^2 F'' \circ (u + \mu\ell)$ for any $\mu \in \langle 0, +\infty)$. Thus we have

$$\begin{aligned}
&\|F \circ (u+\mu\ell) - F \circ u - \ell\, F' \circ u\|'_{m,k,\alpha} \\
&= \left\| \ell \int\limits_0^\mu F' \circ (u + \kappa\ell) - F' \circ u) \,\mathrm{d}\kappa \right\|'_{m,k,\alpha} \\
&= \left\| \ell^2 \int\limits_0^\mu (\mu - \kappa) F'' \circ (u + \kappa\ell) \,\mathrm{d}\kappa \right\|'_{m,k,\alpha} \\
&\leqslant \left(\int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^k} \left(\int\limits_0^\mu (\mu - \kappa) |\Delta_h (u + \kappa\ell)| \,\mathrm{d}\kappa \right)^2 |h|^{-k-2\alpha} \,\mathrm{d}h \,\mathrm{d}x \right)^{\frac{1}{2}} \\
&\quad \times \|F\|_{C_3(\mathbb{R})} \, \|\ell^2\|_{B(\mathbb{R}^m)} \\
&\quad + \|\ell^2\|'_{m,k,\alpha} \, \|F\|_{C_3(\mathbb{R})} \left\| \int\limits_0^\mu (\mu - \kappa) |u + \kappa\ell| \,\mathrm{d}\kappa \right\|_{B(\mathbb{R}^m)} .
\end{aligned} \tag{33}$$

As $\int\limits_0^\mu (\mu - \kappa)|\Delta_h(u + \kappa\ell)(x)| \,\mathrm{d}\kappa \leqslant \frac{\mu^2}{2} |\Delta_h u(x))| + \frac{\mu^3}{6} |\Delta_h \ell(x)|$, it is easy to see that the expression in (33) is $o(\mu)$, $\mu \searrow 0$. From this we can see that $\ell\, F' \circ u$ is the appropriate directional derivative of the superposition operator in the space $H^\alpha(\mathbb{R}^m)$ for every vector $\alpha \in \mathbb{R}^m$ with all coordinates in $\langle 0, 1)$ if both u and ℓ are bounded functions. A suitable generalization to our case of operators is a little cumbersome, though (with the help of the renormation and other technique employed in the paper) nearly obvious.

The operator Ξ_1 is unbounded on the chosen type of spaces, but it is clearly bounded on $U = B(\Omega)$. Let U_0 be the linear span of U_{ad} (we take its modified definition from 1^0) in U. Then, using the above mentioned result which can be extended to our space U due to the Hölder continuity of the appropriate derivatives of the elements of U_0, we prove the Gâteaux differentiability of Ξ_1 on U_0 (with respect to the norm of U) provided g is sufficiently smooth (e.g. in $C_5(\mathbb{R})$).

For differentiability of Ξ_3 we must proceed in the above started investigation. In general, there are three questions concerning differentiability of the superposition operators on Běsov-Sobolev spaces, namely

1. whether the operator is defined on the whole space,
2. whether the element $F' \circ u$ belongs again to the space,
3. whether it is the directional derivative of F in u with respect to the direction ℓ.

The operator can be Gâteaux differentiable on the whole space provided the space is an algebra (with respect to the pointwise multiplication of its elements). Then the answer to all the questions is positive, if e.g. $g \in C_{[|\alpha|+2]}(\mathbb{R})$, where $|\cdot|$ denotes the max norm and $[a]$ the integer part of the number a. It is cumbersome but not too difficult to prove it with help of the procedure in (33) used for higher order derivatives. In fact, the validity of the assertion is based on the possibility to imbed the employed spaces into $C_0(\bar{Q})$ using the imbedding theorem (Proposition 2). The Fréchet differentiability is then a consequence of the continuity of the differential as a mapping.

As the operator $u \mapsto [Au, Bu]$, where $Au \equiv \beta_0 \dot{u} - \Delta u$ on Q and $Bu \equiv \frac{\partial u}{\partial \nu} + g(u)$ on S, is Fréchet differentiable and onto the product of the range of A with U, we use the implicit function theorem and prove the Fréchet differentiability of Ξ_2 on U. Its Fréchet differential at a point $\tilde{T}$ in the direction ℓ is the solution of the equation (1) modified in such a way that ℓ stands for $g(T)$, and $g(u)$ is replaced by $g'(\tilde{u})u$ with $\tilde{u}$ the solution of (1) for $\tilde{T}$ replacing $g(T)$. As Ξ_4 is continuous linear, its differentiability is obvious.

Thus we ensure at least the Gâteaux differentiability of the state operator Ψ_1 on U_{ad}, where the representative of the differential at a point T acting along a direction $\tilde{T}$ is the solution $[\tilde{u}, \tilde{v}]$ of the modified system [(1),(2)] with $g'(T)\tilde{T}$ instead of $g(T)$, $\gamma'(u)(\tilde{u})$ instead of $\gamma(u)$ and $g'(u)\tilde{u}$ instead of $g(u)$ with u the "temperature part" of the solution of the respective original system. The case of the state operator Ψ_2 is quite analogous, hence we avoid details.

The nonuniqueness of the solution of the Lamé system (2) can occur and the residual space $\mathcal{R}$ of all displacements solving (2) with $\gamma(u) = 0$ can be nonempty. Therefore we shall assume that, besides its possible dependence on the control T, on the temperature u and on the independent time and space variables, the cost function J depends on the projection of v to the complement of $\mathcal{R}$ only (which is satisfied e.g. when it depends on strains and/or stresses related to v). We shall denote this condition as ♠. It ensures that the cost function is single-valued for any control from U_{ad}.

We sum up the investigations into the following proposition:

Proposition 3. *Let $U = H^{\alpha,\beta}(S)$ with α, β satisfying the assumption of Theorem 1 or 1′ for $N = 3$ or $\alpha = \frac{1}{2} - \varepsilon$ with a suitably small $\varepsilon > 0$ and $\beta \geqslant 0$ for $N = 2$. Let Y be as in* (30) *or* (31), *let $Y_{\rm ad}$ be given as in* (26) *and let the assumptions on r, s mentioned at the beginning of the section hold. Let J be continuous on $U \times Y$ and, in the case of the operator Ψ_1, let the condition $\spadesuit$ be satisfied. Let $U_{\rm ad}$ be compact in U and let the superposition operator $T \mapsto g \circ T$ be continuous from $U_{\rm ad}$ (endowed with the topology of U) into U. Then there exists at least one solution of the problem* (25). *If, moreover, J is Fréchet differentiable on $U \times Y$ then Λ is Gâteaux differentiable on $U_{\rm ad}$.*

Remark 1. There are several reasonable choices of the cost function for problems arising in technical practice. One of them—the time minimization—was just treated in [6] in the particular case of spatially constant control variables. In this case the "time length" $\mathcal{T}$ of the heating process is one of the coordinates of the control variable and norming this length like in [6] we obtain the control variable in the explicit form $[T, \mathcal{T}]$. A reasonable choice of the cost function seems to be $\int_\Omega (u(\mathcal{T}, \cdot))^2 \,\mathrm{d}x$ which is to be maximized. Another possibility is the minimum of energy taken as the $L_2(S)$-integral of the control variable T or minimization of $\int_Q s(u) - (r(v))^2 \,\mathrm{d}x\,\mathrm{d}t$ or another value measuring the distance of the reference stress to the maximal admissible one. In these cases, to obtain a reasonable optimal control problem, the definition of $U_{\rm ad}$ must be completed by a terminal condition $u(\mathcal{T}, \cdot) = u_{\rm req}$, where $u_{\rm req}$ is the required final temperature of the body or a penalized distance of $u(\mathcal{T}, \cdot)$ and $u_{\rm req}$ must be added into the cost function. All the above choices are taken in order to select a heating regime "as quick as possible".

2. A particular choice of the reference stress can be done in several ways. For a given tensor in the form of a symmetric matrix of the degree $N = \dim \Omega$, denoting by $\omega_1, \ldots, \omega_N$ its eigenvalues we can take e.g.

$$(34) \qquad \max_{i \in \{1,\ldots,N\}} |\omega_i| \quad \text{or} \quad \Big(\sum_{1 \leqslant i < j \leqslant N} (\omega_i - \omega_j)^2 \Big)^{\frac{1}{2}}.$$

If it is the stress tensor corresponding to the displacement v, we shall take the value in (34) as the reference stress $r(v)$. We remark that another choice of the reference stress is the well known Hencky-Huber-Mises stress.

In the sequel we make some remarks concerning the optimality conditions and the adjoint system. We restrict ourselves to the case when the bound s is a constant and e.g. $r(v) = \|\nabla v\|_{C_0(Q;\, \mathbb{R}^{N^2})}$. We shall derive an optimality condition for our problem, where the main task is to remove the state-space constraint. If we linearize it, i.e. supposing g and γ in (1) and (2) are positive constants and having a convex cost function to be minimized (or a concave function to be maximized, as e.g. $\int_\Omega u(\mathcal{T},\cdot)\,dx$) we can use the vertical perturbation, introducing the problem

$$(25)_p \qquad \Lambda(T) \to \inf \quad \text{subject to} \quad T \in U_{\mathrm{ad}},\ r(u) \leqslant s + p.$$

As the Slater condition can be easily satisfied, we can use the perturbation theory of duality (cf. [3]) and prove the existence of a Karush-Kuhn-Tucker vector.

Under the above introduced restriction to the form of the state-space constraint our general optimization problem has the the form

$$(35) \qquad \begin{aligned} &\Lambda(T) \to \inf \\ &\text{subj. to} \\ &\min_S T \geqslant 0,\ \max_S T \leqslant D,\ T \text{ nondecreasing in time on } S, \\ &\|T\|_{X_0} \leqslant q, \quad h(T) \equiv r(\Psi_i(T)) \leqslant s \text{ for suitable } i \in \{1,2\}, \end{aligned}$$

where X_0 is a "better" space occuring in the definition of U_{ad} (cf. $1^0, 2^0$ above). The first three control constraints can be described with help of Lipschitz functions. This is clearly impossible, however, for the X_0-norm which must be unbounded on U. Anyway, such a description defines U_{ad} as a convex compact subset of U. To be able to use the well-developed Clarke calculus, we are forced to work with the problem, where the Ξ_1-images of T are controls (though a suitable choice of X_0 can ensure its Lipschitz continuity on U_{ad}). After this change, due to the monotonicity and smoothness of g all the control constraints can preserve their character (possibly with a different space X_0 and different bounds) and therefore we shall denote it by (35′). Of course, if we were to preserve the equivalence to the original problem, we should not be able to assume the set in the new X_0 to be a ball. We respect here, however, mainly the consequences for practice and the auxiliary character of this constraint which, in fact, is added here to ensure the compactness of U_{ad} only. Then the state-space constraint and the cost function are Lipschitz and we can use the Clarke generalization of the Lagrange multiplier theory ([2], §6.1) which yields the existence of numbers $\lambda_0 \geqslant 0$, $\lambda_1 \geqslant 0$, $\mathcal{K} \geqslant 0$ not all equal to 0 such that for every solution $T_\star$ of the problem (35′)

$$(36) \qquad 0 \in \lambda_0 \partial \Lambda(T_\star) + \lambda_1 \partial h(T_\star) + \mathcal{K}(\lambda_0, \lambda_1) \partial d_{U_{\mathrm{ad}}}(T_\star)$$

provided the problem has the form of minimization of Λ (its transcription to the maximization problem is obvious). Here ∂ denotes the Clarke generalized gradient of the functions employed and $d_{U_{\rm ad}}$ is $\operatorname{dist}(\cdot, U_{\rm ad})$ which is convex due to our new formulation of the problem. Moreover, the proposed cost functions are continuously differentiable, therefore $\partial\Lambda \equiv \Lambda'$. For solutions $T_\star$ of (35′), where the Mangasarian-Fromowitz constraint qualification

$$\text{(37)} \quad \underset{\tilde{T}\in U}{\exists} \ \underset{\lambda_0>0}{\exists} \ \|\tilde{T}\|_U = 1,\ T_\star + \lambda\tilde{T} \in U_{\rm ad},\ \lambda \in \langle 0, \lambda_0\rangle, \text{ and } (r\circ\Psi_{1,2})^0(T_\star;\tilde{T}) = r_0 < 0$$

(where $\ell^0(z_0; z_1)$ is the Clarke directional derivative of a function ℓ in z_0 and a direction z_1) is satisfied, we can assume in (36) that $\lambda_0 = 1$ (see [2]). Due to the nonempty interior of $Y_{\rm ad}$ such an assumption is reasonable.

The calculation of the employed Clarke gradient as well as of Λ' needs the calculation of the adjoint operators to the gradients of Ψ_i. Let $Y \equiv Y_1 \times Y_2$ be as in (30). For $i = 1$ we have the adjoint operator to $\Xi_2'(T)$ mapping a given $z \in Y_1^*$ to $y \equiv w/_S$, where w is a solution of the equation

$$\text{(38)} \qquad \beta_0\frac{\partial w}{\partial t} = -\Delta w - z \ \text{ on } Q, \quad \frac{\partial w}{\partial \nu} = -g'(u)w \ \text{ on } S, \quad w(\mathcal{T},\cdot) = 0 \ \text{ on } \Omega$$

with $u \equiv \Xi_2(T)$. The adjoint operator to $(\Xi_4 \circ \Xi_3)'(u)$ has the form $\tilde{z} \mapsto \tilde{y} = \gamma'(u)\operatorname{div}\tilde{w}$, where $\tilde{w}$ is the solution of the Lamé system

$$\text{(39)} \qquad \begin{gathered} (1-2\sigma)\Delta\tilde{w} + \nabla\operatorname{div}\tilde{w} = -(2+2\sigma)\tilde{z} \text{ on } \Omega,\ t\in\bar{I}, \\ (1-2\sigma)\left(\frac{\partial\tilde{w}}{\partial\nu} + ((\nu,\nabla_i\tilde{w})_i)\right) + 2\sigma\nu\operatorname{div}\tilde{w} = 0 \text{ on } \partial\Omega,\ t\in\bar{I}. \end{gathered}$$

For the operator $\Psi_2'(T)$ we obtain the adjoint operator in the form $\tilde{z} \mapsto \tilde{u}/_S$, where $\tilde{u}$ is a component of the solution of the system

$$\text{(40)} \quad \begin{cases} \left.\begin{aligned} &\beta_0\dot{\tilde{u}} + \Delta\tilde{u} = \gamma'(u)\operatorname{div}\dot{\tilde{v}}, \\ &\ddot{\tilde{v}} + (1-2\sigma_1)\Delta\dot{\tilde{v}} - (1-2\sigma)\Delta\tilde{v} - \nabla\operatorname{div}\tilde{v} = (2+2\sigma)(-\tilde{Z} - \delta_0\nabla\tilde{u}) \end{aligned}\right\} \text{ on } Q, \\ \left.\begin{aligned} &g'(u)\tilde{u} = -\frac{\partial\tilde{u}}{\partial\nu}, \\ &-(1-2\sigma_1)\frac{\partial\dot{\tilde{v}}}{\partial\nu} + (1-2\sigma)\Big(\frac{\partial\tilde{v}}{\partial\nu} + ((\nu,\nabla_i\tilde{v})_i)\Big) + 2\sigma\nu\operatorname{div}\tilde{v} \\ &\quad = (2+2\sigma)\delta_0\tilde{u}\nu \end{aligned}\right\} \text{ on } S, \\ \qquad \tilde{u}(\mathcal{T},\cdot) = 0, \quad \tilde{v}(\mathcal{T},\cdot) = 0, \quad \dot{\tilde{v}}(\mathcal{T},\cdot) = 0 \text{ on } \Omega \end{cases}$$

with $\tilde{Z}(t,\cdot) \equiv \int_t^{\mathcal{T}} \tilde{z}(\tau,\cdot)\,\mathrm{d}\tau$. This fact is a consequence of the Green theorem. We remark that the well-known general properties of adjoint operators ensure the existence and unicity of a solution for both linear adjoint systems [(38),(39)] and (40), if

u is the temperature component of a solution of the original system for $T \in U_{\text{ad}}$. Of course, for the input data of (39) the condition that $\mathcal{R}$ lies in their kernel must be fulfilled. Note the terminal-time condition in both adjoint systems corresponding to the opposite sign at $\dot{\tilde{u}}$ and $\ddot{\tilde{v}}$! Due to the bad input data from the dual spaces, the solutions are in certain classes of "very weak" solutions in general. Further calculation of the Clarke gradient represents an appropriate use of the corresponding chain rule and the theorem concerning the differentiation of the max-function (cf. [2]).

Finally, we consider the problem [(1),(2)] and the cost function in the form

$$J(u, \Psi_1 u) \equiv \int_\Omega (u(\mathcal{T}, \cdot))^2 \, \mathrm{d}x \tag{41}$$

which is to be maximized. We ensure on the base of the comparison theorem for the heat equation (cf. e.g. [13]) that for any solution $T_\star$ of the problem (25), $r(v_\star) = s$, where $v_\star = \Psi_{1,2} T_\star$. Indeed, if it is not true, it is not difficult to find in a small neighbourhood of $T_\star$ a heating regime $T \in U_{\text{ad}}$ which heats faster than $T_\star$ and the state-space constraint remains valid. With the same argument we are even able to prove that $\|\nabla v(T)\|_{C_0(\Omega)} = s$ for $v = \Psi_{1,2}(T)$ for any solution v of (25). Let us assume that $T_\star$ is an isolated optimal solution of (25) such that the condition (37) holds with the usual directional derivative. (We remark that, in a particular case $\gamma(u) \equiv \gamma_0 u$ with γ_0 a constant, such a situation occurs e.g. if $g'(cz) \geqslant cg'(z)$, $z \in \langle 0, D \rangle$, $c \in (1-\varepsilon, 1)$, $\varepsilon > 0$ arbitrarily small and $g(T_\star) - g(\Psi_{1,1} T_\star) + g'(\Psi_{1,1} T_\star) > 0$ on Q which holds e.g. for strictly convex g.) Let B be such a ball in the space U that in $T_\star + B$ there is no solution of (25) but $T_\star$. Denote by $(25)_{Bp}$ the problem of the form $(25)_p$, where U_{ad} is replaced by $U_{\text{ad}} \cap (T_\star + B)$. Denoting by T_p the solutions of $(25)_{Bp}$, we prove that $T_p \overset{U}{\to} T_\star$ for $p \to 0$. Indeed, if a sequence $p_n \searrow 0$, then there is some its subsequence $\{p_{n_k}\}$ and some $T_0 \in U_{\text{ad}}$ such that $T_{p_{n_k}} \to T_0$. As Λ is continuous, $\Lambda(T_{p_n}) \geqslant \Lambda(T_\star)$ and $\Psi_1(T_0) \in Y_{\text{ad}}$, clearly $T_0 = T_\star$, because on $T_\star + B$, $T_\star$ is the only solution of (25). On the other hand, if $p_n \nearrow 0$, T_0 is a limit of a subsequence of T_{p_n} and $\Lambda(T_0) < \Lambda(T_\star)$, then we easily obtain a contradiction with the assumption concerning the existence of $\tilde{T}$. Therefore $T_0 = T_\star$ in this case, too. In general, for T_{p_n} being a solution of $(25)_{p_n}$, the above described procedure proves that every cluster point of $\{T_{p_n}\}$ satisfying (37) is a solution of (25).

3. Appendix—a short survey of regularization technique employed

As the employed detailed regularization technique is essential for the results of the preceding sections and not all readers are necessarily familiar with it, we offer this survey. For the definition of Běsov-Sobolev spaces we refer to [1], [11]. For those being square integrable together with all their existing derivatives, the following renormation lemma (Lemma 1 in [5] proved on p. 428) is very useful:

Lemma. *For an arbitrary function $f\colon \mathbb{R}^N \to \mathbb{R}$, any $k \in \mathbb{N}$ and any $\alpha \in (0,k)$ we have*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\frac{\Delta_k^h f}{|h|^{\frac{N}{2}+\alpha}}\right)^2 \mathrm{d}x\,\mathrm{d}h = \widetilde{c}_{N,k}(\alpha) \int_{\mathbb{R}^N} |\hat{f}|^2 |\xi|^{2\alpha}\,\mathrm{d}\xi, \tag{42}$$

where by hat the Fourier transform is denoted, $\Delta_1^h\colon f(x) \mapsto (f(x+h)-f(x))$, $x,h \in \mathbb{R}^N$, $\Delta_k^h \equiv \Delta_1^h \circ \Delta_{k-1}^h$, $k \in \mathbb{N}$ and

(43)

$$\widetilde{c}_{N,k}(\alpha) = \widetilde{c}_N^0(\alpha) 2^{2k-2\alpha} \int_{\mathbb{R}} \frac{\sin^{2k} t}{|t|^{1+2\alpha}}\,\mathrm{d}t,\ \widetilde{c}_N^0(\alpha) = \begin{cases} 1, & N=1,\\ \displaystyle\int_{\mathbb{R}^{N-1}} (1+|s|^2)^{-\frac{N}{2}-\alpha}\,\mathrm{d}s, & N \geqslant 2. \end{cases}$$

We remark that Lemma together with the well-known results for "entire" derivatives (cf. e.g. [9], Chapter 1, Sec. 1.2) makes different renormations for fractional-derivative norms and seminorms possible. Particularly, Lemma together with the localization technique enables us to rewrite the nonlinear boundary terms occuring in the proof of Theorem 1 or Theorem 1′. Another fact proved in that paper (Lemma 2 there) and used in the preceding sections is the bounded $H^{\frac{1}{2}-\varepsilon}$-norm of functions having bounded variation on a bounded interval for any $\varepsilon > 0$.

Assume that for any $x \in \partial\Omega$ there is a function $\psi\colon \mathbb{R}^{N-1} \to \mathbb{R}$, $\psi(0) = 0$ such that, after a suitable rotation and shift of the coordinate system, $x = 0$ and there is a neighbourhood U of 0 and a number $\eta > 0$ such that $\{[y,z] \in \mathbb{R}^{N-1}\times\mathbb{R};\ y \in U \ \&\ z \in (\psi(y), \psi(y)+\eta)\} \subset \Omega$ and $\{[y,z] \in \mathbb{R}^{N-1}\times\mathbb{R};\ y \in U \ \&\ z \in (\psi(y)-\eta, \psi(y))\} \subset \mathbb{R}^N \setminus \Omega$. The local straightening of the boundary is than the mapping $[y,z] \mapsto [y, z-\psi(y)]$. If the boundary is smooth at x, we can assume that $\nabla\psi(0) = 0$. The localization technique requires the existence of a sufficiently smooth partition of unity on Ω. Let $x \in \operatorname{supp} \varrho \cap \partial\Omega$, where ϱ belongs to the partition of unity. If the test function in a variational equation has support in a small neighbourhood of x, then (after the straightening of the boundary) the appropriate volume and boundary integrals in the variational formulation can be replaced by certain integrals over a half-space and subspace, respectively. If $\operatorname{supp} \varrho$ is sufficiently small, u is a solution of the problem and h is the tangential direction to the straightened boundary, then $\varrho^2 \Delta_k^h u$ can be put as a test function. Adding together suitable shifts of the variational problem with the same test function, we can usually obtain, after some calculation, a certain estimate for the seminorm of the type (42) at ϱu. (If $x \in \Omega$, the straightening of the boundary and the restriction to tangential directions is redundant.) For the sake of simplicity, the localization technique is used "implicitly" in proofs in Sec. 1, particularly u stands for ϱu there and the estimates of terms containing differences of ϱ or of its derivatives, easily based on the assumed smoothness of that function, are avoided. The terms created by the straightening of the boundary (small perturbations of the elliptic operators) are also avoided; the assumed smoothness of $\partial\Omega$, i.e. of the function ψ, yields easily their estimates. For further details we refer again to [5] and also to [4], where the above mentioned technique is explained in all details on a slightly different problem with the Lamé system.

An analogous technique is used when the extension of functions from Ω to whole $\mathbb{R}^N$ is necessary such that the (Běsov-)Sobolev norm on $\mathbb{R}^N$ of the extension is bounded by a certain multiple of the norm on Ω. In addition, an extension procedure from the half-space to the whole space described in [9], Chapter 1, Sec. 2.2 is employed here.

The trace theorem is based on the technique of the straightening of the boundary, on the Fubini theorem and on Corollary of Proposition 1 (its version in which the imbedding into C_0 in the normal direction holds).

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Author's address: *Jiří Jarušek*, Institute of Information Theory and Automation, Czech Acad. Sci., Pod vodárenskou věží 4, 182 08 Praha 8, Czech Republic.