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PROX-REGULARIZATION AND SOLUTION OF ILL-POSED ELLIPTIC VARIATIONAL INEQUALITIES[1]

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Abstract. In this paper new methods for solving elliptic variational inequalities with weakly coercive operators are considered. The use of the iterative prox-regularization coupled with a successive discretization of the variational inequality by means of a finite element method ensures well-posedness of the auxiliary problems and strong convergence of their approximate solutions to a solution of the original problem.

In particular, regularization on the kernel of the differential operator and regularization with respect to a weak norm of the space are studied. These approaches are illustrated by two nonlinear problems in elasticity theory.

Keywords: prox-regularization, ill-posed elliptic variational inequalities, finite element methods, two-body contact problem, stable numerical methods

MSC 2000: 35J85, 49A29, 49D45, 65K10, 73C30

1. Introduction

The idea of iterative regularization for solving ill-posed variational problems was formulated first by Bakushinski/Polyak in [9]. However, in implicit form it has appeared already in a number of previous publications, including the fundamental paper [33] of Mosco on the stable approximation of variational inequalities.

A theoretical foundation of iterative regularization by means of the prox-mapping was given by Rockafellar in [40]. The first concrete algorithms of this type, destined for solving finite dimensional convex programming problems, are based on augmented Lagrangian methods (cf. Rockafellar [41], Antipin [4]) and penalty methods (cf. Kaplan [25]). In the recent past the development in this field has been extremely intensive. We refer to several papers only: The results by Gueler [18],

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HA [20] and LUQUE [31] are concerned with the investigation of the properties of the prox-mapping and the proximal point algorithm, introduced by MARTINET in [32]; IBARAKI/FUKUSHIMA/IBARAKI have used in [24] prox-regularization with respect to the primal and dual variables with decomposition of the arising auxiliary problems; AUSLENDER [7] and FUKUSHIMA [16] have considered convex programming algorithms which couple prox-regularization with special cutting plane procedures; SPINGARN in [43] and [44] has described a prox-technique of the partial inverses of maximal monotone operators to construct decomposition algorithms for convex programming problems. In a number of papers (cf. ALART [1], ALART/LEMAIRE [2], AUSLENDER/CROUZEIX/FEDIT [8], MOUALLIF/TOSSINGS [34], TOSSINGS [45]) iterative prox-regularization has been performed in different penalty methods.

In the papers referred above there is no discretization of the problems considered, moreover, the main results are concentrated on finite dimensional problems.

General approaches for constructing stable methods of the discretization of convex variational problems in Hilbert spaces, based on the principle of iterative prox-regularization, have been developed by KAPLAN/TICHATSCHKE in [26], [28] and LEMAIRE in [30]. In [30] the conditions for the approximation of the data are formulated in terms of the variational convergence (cf. ATTOUCH [5], ATTOUCH/WETS [6]). The papers [26] and [28] are mainly oriented towards elliptic variational inequalities, discretized by means of finite element methods, as well as towards semi-infinite programming problems, approximated via standard approaches. A special penalty method for solving convex, semi-infinite problems, using an adaptive discretization coordinated with prox-regularization, was suggested by the authors in [27].

The present paper deals with a modified principle of iterative prox-regularization described in KAPLAN/TICHATSCHKE [28] for solving variational problems

$$\min\{J(u):\ u \in K \subset V\}$$

with K a convex, closed subset of a Hilbert space V and J a convex, lower semicontinuous (lsc) functional on V. This principle can be sketched briefly as follows: In the i-th step of a chosen standard discretization method a convex problem

$$\min\{J_i(u):\ u \in K_i \subset V_i\},$$

is constructed on a finite dimensional subspace V_i of the Hilbert space V. Starting with a point $u^{i,0}$, approximate prox-iterations

$$u^{i,s} \approx \arg\min\{J_i(u) + \|u - u^{i,s-1}\|^2:\ u \in K_i\} \tag{1.1}$$

are performed with the data J_i and K_i while

$$\|u^{i,s} - u^{i,s-1}\| > \delta_i,$$

where δ_i is a given value. The last iterate on the i-th discretization level is used as starting point $u^{i+1,0}$ in order to continue the procedure on the next level (step $(i+1)$).

It should be remarked that in the usual scheme of iterative regularization the discretization has to be improved after each prox-iteration, i.e. formally $u^{i,1}$ is calculated by means of (1.1) and after that one has to put $u^{i+1,0}=u^{i,1}$, $i := i+1$.

The aim of this modification is quite clear, especially, if J_i is close to J and $K_i \subset K$: We do not increase the exactness of the approximation if the prox-iterations guarantee a sufficiently fast decrease of the objective function. This is particularly important, if there is no possibility to choose a priori a starting point which is close to the sought solution. Indeed, using the standard way of iterative prox-regularization, with high probability we may obtain auxiliary problems of large dimension in situations where the iterates are still far away from the sought solution of the original problem. It is obvious that in this case an approximation with high accuracy does not have any reason and only enlarges the numerical expense.

In order to distinguish in the sequel the standard and modified methods of iterative prox-regularization, we will call them the one-step regularization (OSR-method) and the multi-step regularization (MSR-method), respectively.

Here the investigation of MSR-methods is mainly concerned with elliptic variational inequalities with weakly coercive operators. Due to additional assumptions on the structure of the problems and the behaviour of the discretization methods, reflecting the peculiarities of elliptic variational inequalities and finite element methods, we get essentially weaker conditions on the choice of the controlling parameters of the MSR-method than in Kaplan/Tichatschke [28] (compare Theorem 1 in [28] with Theorems 3.5 and 4.4 here). These conditions ensure strong convergence of the iterates to a solution of the original problem. Moreover, in this description it is not essential what kind of algorithm is used for solving the regularized auxiliary problems: it is only important how precisely they are solved.

This paper deals also with two new variants of MSR-methods: *regularization on the kernel* of the objective functional and the so-called *weak regularization* (with respect to a weaker norm of the space V). These modifications take account of the structure of the optimal set, which is known a priori for some variational inequalities.

Two examples illustrate our consideration. The first is a contact problem of two elastic bodies without friction and the second is a static problem of the linear theory of elasticity with given friction, investigated by Duvaut/Lions in [13] (see also Panagiotopoulos [38]). Both models, considered in the framework of the plane theory of elasticity, are quite popular as concerns the investigation of the peculiarities of elliptic variational inequalities with weakly coercive operators.

2. Three variants of MSR-methods

In a Hilbert space Y we consider the following variational problem:

$$\min\{J(u):\ u \in K\}, \tag{2.1}$$

with K a convex, closed subset of Y and

$$J(u) = \frac{1}{2}a(u,u) + j(u) - \langle f, u\rangle\,. \tag{2.2}$$

Assumption 2.1.
(i) $a(\cdot,\cdot)$ is a continuous, symmetric bilinear form on $Y \times Y$, $a(u,u) \geqslant 0$ on Y;
(ii) $j(\cdot)$ is a convex, lower semi-continuous (lsc) functional on Y;
(iii) $f \in Y'$, where Y' is the dual space to Y with $\langle\cdot,\cdot\rangle$ the duality pairing.

We assume that Problem (2.1) is solvable and denote by $U^\star$ its optimal set.

The investigation of different MSR-methods for Problem (2.1) will be performed in the framework of a general scheme.

Let H be a Hilbert space such that Y can be continuously embedded into H; let Y_1 be a closed (concerning $\|\cdot\|_Y$ or $\|\cdot\|_H$) subspace of Y and $\mathcal{P}$ an orthoprojector (in the same norm) onto the subspace Y_1; $b(\cdot,\cdot)$ a symmetric bilinear form on $Y \times Y$.

Assumption 2.2.
(i) $a(u,u) \geqslant b(u,u) \geqslant 0$ on Y;
(ii) for some $\beta > 0$ the inequality

$$\frac{1}{2}b(u,u) + \|\mathcal{P}u\|_H^2 \geqslant \beta\|u\|_Y^2 \tag{2.3}$$

is fulfilled for all $u \in Y$.

Under Assumptions 2.1(i) and 2.2 the relation

$$|u|^2 = \frac{1}{2}b(u,u) + \|\mathcal{P}u\|_H^2 \tag{2.4}$$

defines on Y a new norm $|\cdot|$. The space Y with this norm is denoted by $\mathcal{Y}$, and its conjugate by $\mathcal{Y}'$.

The norms $\|\cdot\|_Y$ and $|\cdot|$ are equivalent. Indeed, there exist two constants c and M such that

$$\|u\|_H \leqslant c\|u\|_Y \tag{2.5}$$

and

$$|b(u,v)| \leqslant M\|u\|_Y\|v\|_Y, \quad |a(u,v)| \leqslant M\|u\|_Y\|v\|_Y \tag{2.6}$$

and for the orthoprojector $\mathcal{P}$ we have

$$\|\mathcal{P}u\|_H \leqslant c\|u\|_Y. \tag{2.7}$$

Using the inequalities (2.3) and (2.5)–(2.7), we immediately obtain

$$\beta\|u\|_Y^2 \leqslant |u|^2 \leqslant \Big(\frac{1}{2}M + c^2\Big)\|u\|_Y^2 \text{ for } u \in Y. \tag{2.8}$$

2.1. The general scheme of MSR-methods

In order to solve Problem (2.1), a family of auxiliary problems

$$\Psi_{i,s}(u) = J_i(u) + \|\mathcal{P}u - \mathcal{P}u^{i,s-1}\|_H^2 \to \min_{u \in K_i} \qquad s = 1, \ldots, s(i); i = 1, 2, \ldots \tag{2.9}$$

is constructed with K_i convex, closed subsets of Y and

$$J_i(u) = \frac{1}{2}a(u,u) + j_i(u) - \langle f, u\rangle\,,$$

where $j_i\colon\, Y \to \mathbb{R}$ are convex, Gâteaux-differentiable functionals. The values $s(i)$ are specified in the course of the following iteration procedure, starting with $s(i) = 1$ for each i.

Let a point $u^{1,0}$ and sequences $\{\delta_i\}$ and $\{\varepsilon_i\}$ be given with $\delta_i > 0$, $\varepsilon_i \geqslant 0$, $\lim \varepsilon_i = 0$. For a fixed pair (i,s) the point $u^{i,s}$ is defined such that

$$\|\nabla\Psi_{i,s}(u^{i,s}) - \nabla\Psi_{i,s}(\bar{u}^{i,s})\|_{Y'} \leqslant \varepsilon_i' \tag{2.10}$$

with

$$\varepsilon_i' \leqslant \beta\Big(\frac{1}{2}M + c^2\Big)^{-1/2}\varepsilon_i \tag{2.11}$$

and

$$\bar{u}^{i,s} = \arg\min\{\Psi_{i,s}(u)\colon\; u \in K_i\}. \tag{2.12}$$

If $\|\mathcal{P}u^{i,s} - \mathcal{P}u^{i,s-1}\|_H > \delta_i$, we agree that $s(i) = s+1$, continue with the pair $(i, s+1)$ and compute $u^{i,s+1}$, otherwise put $s(i) = s$ (this is the final value of $s(i)$), $u^{i+1,0} := u^{i,s(i)}$ and compute $u^{i+1,1}$.

In the sequel we refer to this scheme as Method (2.9)–(2.12).

In (2.9) the functionals J_i are used in order to include in this scheme a smoothing procedure for the objective functional J, if J is not differentiable.

For the choice $Y_1 = Y = H$ we obtain the *basic variant of the MSR-method* studied by KAPLAN/TICHATSCHKE [28].

If Y_1 belongs to the kernel of the bilinear form $a(\cdot,\cdot)$ in (2.2), we deal with a *regularization on the kernel.*

Finally, if the norm of the space H is weaker than the norm of Y, then a *weak regularization* is introduced.

2.2. Convergence result for the general scheme

In order to investigate convergence of Method (2.9)–(2.12), we need the following auxiliary statement.

Lemma 2.3. *Suppose that Assumptions* 2.1 *and* 2.2 *are fulfilled, that* G *is a convex, closed subset of the space* Y *and that* $\bar{j}$ *is a convex, lsc functional on* G *such that*

$$\sup_{u\in G} |\bar{j}(u) - j(u)| \leqslant \sigma. \tag{2.13}$$

Moreover, let $z^0 \in Y$ *be an arbitrarily chosen point and*

$$z^1 = \arg\min\{\bar{J}(u) + \|\mathcal{P}u - \mathcal{P}z^0\|_H^2 :\ u \in G\}$$

with $\bar{J}(u) = \frac{1}{2}a(u,u) + \bar{j}(u) - \langle f, u\rangle$. *Then, for each* $u \in G$ *the estimates*

$$|z^1 - u|^2 - |z^0 - u|^2 \leqslant -\|\mathcal{P}z^1 - \mathcal{P}z^0\|_H^2 + J(u) - J(z^1) + 2\sigma \tag{2.14}$$

and

$$|z^1 - u| \leqslant |z^0 - u| + \eta(u) \tag{2.15}$$

hold, where the norm $|\cdot|$ *is defined by* (2.4) *and*

$$\eta(u) = \begin{cases} 0 & J(u) \leqslant J(z^1) - 2\sigma \\ (J(u) - J(z^1) + 2\sigma)^{1/2} & \end{cases}.$$

If, moreover, $\|\mathcal{P}z^1 - \mathcal{P}z^0\|_H \geqslant \delta$ *and* $\delta \geqslant \eta(u)$, *then*

$$|z^1 - u| \leqslant |z^0 - u| + \frac{\eta^2(u) - \delta^2}{2|z^0 - u|}. \tag{2.16}$$

P r o o f. With regard to (2.7) the bilinear form $(\mathcal{P}u, \mathcal{P}v)_H$ is bounded on the space $Y \times Y$. Thus, in view of the choice of z^1, the inequality

$$\bar{j}(u) - \bar{j}(z^1) + a(z^1, u - z^1) - \langle f, u - z^1 \rangle + 2(\mathcal{P}u - \mathcal{P}z^1, \mathcal{P}z^1 - \mathcal{P}z^0)_H \geqslant 0$$

is fulfilled for all $u \in G$ and, due to (2.13),

$$\begin{aligned} &j(u) - j(z^1) + a(z^1, u - z^1) - \langle f, u - z^1 \rangle \\ &\quad +2(\mathcal{P}u - \mathcal{P}z^1, \mathcal{P}z^1 - \mathcal{P}z^0)_H + 2\sigma \geqslant 0. \end{aligned} \tag{2.17}$$

Now, using the definition of the norm $|\cdot|$, we obtain

$$\begin{aligned} |z^1 - u|^2 - |z^0 - u|^2 = {}& \frac{1}{2} b(z^1 - u, z^1 - u) - \frac{1}{2} b(z^0 - u, z^0 - u) \\ & - \|\mathcal{P}z^1 - \mathcal{P}z^0\|_H^2 + 2(\mathcal{P}z^1 - \mathcal{P}z^0, \mathcal{P}z^1 - \mathcal{P}u)_H, \end{aligned}$$

and with regard to (2.17) and Assumption 2.2(i), a straightforward calculation leads to

$$\begin{aligned} |z^1 - u|^2 - |z^0 - u|^2 \leqslant {}& -\frac{1}{2} a(z^1, z^1) + \frac{1}{2} a(u, u) + j(u) - j(z^1) \\ & - \langle f, u - z^1 \rangle - \|\mathcal{P}z^1 - \mathcal{P}z^0\|_H^2 + 2\sigma \\ = {}& J(u) - J(z^1) - \|\mathcal{P}z^1 - \mathcal{P}z^0\|_H^2 + 2\sigma. \end{aligned}$$

Thus, inequality (2.14) is proved and (2.15), (2.16) follow immediately. □

Let $S_r = \{u \in \mathcal{Y} \colon |u| \leqslant r\}$. We suppose that the values $r^\star$ and $r \geqslant r^\star$ are fixed such that

$$U^\star \cap S_{r^\star/8} \neq \emptyset, \ u^{0,1} \in S_{r^\star/4} \tag{2.18}$$

and introduce the sets

$$Q = K \cap S_r, \ Q' = U^\star \cap S_{r^\star}, \ Q_i = K_i \cap S_r.$$

Assumption 2.4.

(i) For each $i = 1, 2, \ldots$ the estimates

$$\sup_{u \in S_r} |j(u) - j_i(u)| \leqslant \sigma_i \tag{2.19}$$

and

$$\varrho(Q_i, Q) \leqslant \varphi_i, \ \varrho(Q', Q_i) \leqslant \varphi_i \tag{2.20}$$

hold, with $\{\sigma_i\}$ and $\{\varphi_i\}$ given non-negative sequences converging to 0 and $\varrho(A,B) = \sup_{v\in A}\inf_{u\in B} |v-u|$ for $A \subset \mathcal{Y}$, $B \subset \mathcal{Y}$;

(ii) a value $\bar{\nu}(r) < \infty$ is known such that for each $u^1, u^2 \in S_r$

$$|j(u^1) - j(u^2)| \leqslant \bar{\nu}(r)|u^1 - u^2|. \tag{2.21}$$

Theorem 2.5. *Let r and $r^\star$ be chosen as above and Assumptions 2.1, 2.2 and 2.4 be valid. Moreover, let the controlling sequences $\{\varphi_i\}$, $\{\sigma_i\}$, $\{\varepsilon_i\}$ and $\{\delta_i\}$ satisfy the conditions*

$$\frac{1}{4r}\Big\{2\nu(r)\varphi_i + 2\sigma_i - \Big(\delta_i - \frac{1}{2}\varepsilon_i\Big)^2\Big\} + \frac{1}{2}\varepsilon_i < 0 \tag{2.22}$$

and

$$\sum_{i=1}^{\infty}\Big\{\Big(2\nu(r)\varphi_i + 2\sigma_i\Big)^{1/2} + \frac{1}{2}\varepsilon_i + 2\varphi_i\Big\} < \frac{1}{2}r^\star, \tag{2.23}$$

with $\nu(r) = M\beta^{-\frac{1}{2}}r + \|f\|_{\mathcal{Y}'} + \bar{\nu}(r)$. Then

(i) *Method (2.9)–(2.12) is well-defined, i.e., $s(i) < \infty$ for all i;*

(ii) *$|u^{i,s}| < r^\star$, $|\bar{u}^{i,s}| < r^\star$ for all pairs (i,s);*

(iii) *$\{u^{i,s}\}$ converges weakly to an element $u^\star \in U^\star$.*

If, moreover, the subspace Y_1 is finite dimensional, then

(iv) *$\{u^{i,s}\}$ converges to $u^\star$ in the norm of the space Y.*

Proof. Applying Lemma 2.3, the statements (i) and (ii) can be established following the proof of Lemma 2 in Kaplan/Tichatschke [28] (see also the proof of Lemma 3.2 below, where the same ideas are used). Because of (ii) the equality

$$\bar{u}^{i,s} = \arg\min\{J_i(u) + \|\mathcal{P}u - \mathcal{P}u^{i,s-1}\|_H^2 :\ u \in Q_i\}$$

holds true. Now, we prove the statements (iii) and (iv).

Due to Assumption 2.2 and (2.8) the functions $\Psi_{i,s}$ are strongly convex (with constant β) on Y. Hence, (2.10) and (2.12) lead to

$$\|\bar{u}^{i,s} - u^{i,s}\|_Y \leqslant \frac{1}{2}\beta^{-1}\varepsilon_i'$$

and because of (2.8) and (2.11),

$$|\bar{u}^{i,s} - u^{i,s}| \leqslant \frac{1}{2}\varepsilon_i. \tag{2.24}$$

From (2.7) and (2.11) we obtain that

$$\|\mathcal{P}\bar{u}^{i,s} - \mathcal{P}u^{i,s}\|_H \leqslant \frac{1}{2}c\beta^{-1}\varepsilon_i' = \frac{1}{2}c\Big(\frac{1}{2}M + c^2\Big)^{-1/2}\varepsilon_i \leqslant \frac{1}{2}\varepsilon_i. \tag{2.25}$$

But (2.22) and (2.23) yield

$$\delta_i > \frac{1}{2}\varepsilon_i.$$

The last inequality together with (2.25) ensures that for each i and for $1 \leqslant s < s(i)$

$$\|\mathcal{P}\bar{u}^{i,s} - \mathcal{P}u^{i,s-1}\|_H > \|\mathcal{P}u^{i,s} - \mathcal{P}u^{i,s-1}\|_H - \frac{1}{2}\varepsilon_i > \delta_i - \frac{1}{2}\varepsilon_i > 0. \tag{2.26}$$

Now, let $w \in U^\star \cap S_{r^\star}$ be chosen arbitrarily. According to (2.20) one can take points $v^i \in Q_i$ and $\bar{v}^{i,s} \in Q$ such that for each i

$$|v^i - w| \leqslant \varphi_i \text{ and } |\bar{u}^{i,s} - \bar{v}^{i,s}| \leqslant \varphi_i,\ \ s = 1, \dots, s(i).$$

Since the functional J is Lipschitz-continuous on S_r with the constant $\nu(r)$, this leads to

$$J(v^i) - J(w) \leqslant \nu(r)\varphi_i,\ \ J(\bar{v}^{i,s}) - J(\bar{u}^{i,s}) \leqslant \nu(r)\varphi_i$$

and, due to the choice of w, $J(w) \leqslant J(\bar{v}^{i,s})$ holds. Hence,

$$J(v^i) - J(\bar{u}^{i,s}) \leqslant 2\nu(r)\varphi_i. \tag{2.27}$$

Now, using Lemma 2.3 with $\bar{j} = j_i$, $\sigma = \sigma_i$, $G = Q_i$, $u = v^i$, $z^0 = u^{i,s-1}$, we get

$$|\bar{u}^{i,s} - v^i|^2 - |u^{i,s-1} - v^i|^2 \leqslant -\|\mathcal{P}\bar{u}^{i,s} - \mathcal{P}u^{i,s-1}\|_H^2 + J(v^i) - J(\bar{u}^{i,s}) + 2\sigma_i, \tag{2.28}$$

and, in view of $|v^i| \leqslant r$, $|u^{i,s-1}| < r$ and the relations (2.16), (2.22), (2.26) and (2.27), one can conclude that

$$|\bar{u}^{i,s} - v^i| < |u^{i,s-1} - v^i| + \frac{1}{4r}\Big\{2\nu(r)\varphi_i + 2\sigma_i - \Big(\delta_i - \frac{1}{2}\varepsilon_i\Big)^2\Big\}$$

for each i and $1 \leqslant s < s(i)$. Due to (2.15) and (2.27),

$$|\bar{u}^{i,s(i)} - v^i| \leqslant |u^{i,s(i)-1} - v^i| + (2\nu(r)\varphi_i + 2\sigma_i)^{1/2}.$$

With regard to (2.22) and (2.24) the last two inequalities yield for each i and $1 \leqslant s < s(i)$

$$|u^{i,s} - v^i| < |u^{i,s-1} - v^i| + \frac{1}{4r}\Big\{2\nu(r)\varphi_i + 2\sigma_i - \Big(\delta_i - \frac{1}{2}\varepsilon_i\Big)^2\Big\} + \frac{1}{2}\varepsilon_i,$$

hence

$$|u^{i,s} - v^i| < |u^{i,s-1} - v^i| \text{ for } 1 \leqslant s < s(i). \tag{2.29}$$

Also

$$\begin{aligned} |u^{i,s(i)} - v^i| &\leqslant |u^{i,s(i)-1} - v^i| + (2\nu(r)\varphi_i + 2\sigma_i)^{1/2} + \frac{1}{2}\varepsilon_i \\ &= |u^{i,s(i)-1} - v^i| + \tau_i \end{aligned} \tag{2.30}$$

with $\tau_i = (2\nu(r)\varphi_i + 2\sigma_i)^{1/2} + \frac{1}{2}\varepsilon_i$. Thus, for each i,

$$|u^{i+1,0} - v^i| \leqslant |u^{i,0} - v^i| + \tau_i$$

holds and, taking into account the choice of v^i,

$$|u^{i+1,0} - w| \leqslant |u^{i,0} - w| + \tau_i + 2\varphi_i.$$

However, from (2.23) it follows that $\sum_{i=1}^{\infty} \tau_i < \infty$ and $\sum_{i=1}^{\infty} \varphi_i < \infty$. Therefore, Lemma 2.2.2 in Polyak [39] ensures convergence of $\{|u^{i,0} - w|\}$ for each $w \in U^\star \cap S_{r^\star}$.

Now, the inequalities (2.29) and (2.30) lead to

$$-\tau_i + |u^{i+1,0} - v^i| = -\tau_i + |u^{i,s(i)} - v^i| \leqslant |u^{i,s} - v^i| \leqslant |u^{i,0} - v^i|,$$

hence

$$-\tau_i - 2\varphi_i + |u^{i+1,0} - w| < |u^{i,s} - w| \leqslant |u^{i,0} - w| + 2\varphi_i.$$

Thus, $\{|u^{i,s} - w|\}$ converges for each $w \in U^\star \cap S_{r^\star}$ and it is obvious that $\{|\bar{u}^{i,s} - w|\}$ converges to the same limit. By using inequality (2.28) together with

$$\begin{aligned} |u^{i,s-1} - v^i| &\leqslant |u^{i,s-1} - w| + \varphi_i, \\ |\bar{u}^{i,s} - w| &\leqslant |\bar{u}^{i,s} - v^i| + \varphi_i, \end{aligned}$$

we obtain that

$$\begin{aligned} &|u^{i,s-1} - w|^2 - |\bar{u}^{i,s} - w|^2 \\ &\quad \geqslant J(\bar{v}^{i,s}) - (J(\bar{v}^{i,s}) - J(\bar{u}^{i,s})) - J(w) \\ &\qquad + (J(w) - J(v^i)) - 2\sigma_i - 8r\varphi_i - 2\varphi_i^2 \\ &\quad \geqslant J(\bar{v}^{i,s}) - J(w) - 2\nu(r)\varphi_i - 2\sigma_i - 8r\varphi_i - 2\varphi_i^2. \end{aligned} \tag{2.31}$$

Since the limits of $\{|u^{i,s}-w|\}$ and $\{|\bar{u}^{i,s}-w|\}$ coincide and $J(\bar{v}^{i,s}) \geqslant J(w)$, inequality (2.31) guarantees that

$$\lim_{i\to\infty} \sup_{1\leqslant s\leqslant s(i)} [J(\bar{v}^{i,s}) - J(w)] = 0,$$

hence

$$\lim_{i\to\infty} \sup_{1\leqslant s\leqslant s(i)} [J(\bar{u}^{i,s}) - J(w)] = 0 \tag{2.32}$$

and

$$\lim_{i\to\infty} \sup_{1\leqslant s\leqslant s(i)} [J(u^{i,s}) - J(w)] = 0. \tag{2.33}$$

Because Q is a closed convex set, any weak cluster point of $\{\bar{v}^{i,s}\}$ belongs to Q. Thus, any weak cluster point of $\{\bar{u}^{i,s}\}$ or $\{u^{i,s}\}$ is contained in Q, too and belongs to $U^\star$ in view of (2.32) and (2.33). Now, Opial's lemma [36] yields that $\{u^{i,s}\}$ and $\{\bar{u}^{i,s}\}$ converge weakly to some $u^\star \in U^\star$, i.e., statement (iii) is true.

But if the subspace Y_1 is finite dimensional, then weak convergence of $\{\bar{v}^{i,s}\}$ to $u^\star$ leads to

$$\lim_{i\to\infty} \sup_{1\leqslant s\leqslant s(i)} \|\mathcal{P}\bar{v}^{i,s} - \mathcal{P}u^\star\|_H = 0. \tag{2.34}$$

Using

$$\begin{aligned} J(\bar{v}^{i,s}) + \|\mathcal{P}\bar{v}^{i,s} - \mathcal{P}u^\star\|_H^2 - J(u^\star) &= \frac{1}{2}a(\bar{v}^{i,s} - u^\star, \bar{v}^{i,s} - u^\star) + a(u^\star, \bar{v}^{i,s} - u^\star) \\ - \langle f, \bar{v}^{i,s} - u^\star\rangle + j(\bar{v}^{i,s}) - j(u^\star) + \|\mathcal{P}\bar{v}^{i,s} - \mathcal{P}u^\star\|_H^2 \end{aligned}$$

together with (2.4), one can conclude that

$$\begin{aligned} &J(\bar{v}^{i,s}) + \|\mathcal{P}\bar{v}^{i,s} - \mathcal{P}u^\star\|_H^2 - J(u^\star) - a(u^\star, \bar{v}^{i,s} - u^\star) + \langle f, \bar{v}^{i,s} - u^\star\rangle \\ &- (j(\bar{v}^{i,s}) - j(u^\star)) \geqslant |\bar{v}^{i,s} - u^\star|^2, \end{aligned}$$

and statement (iv) immediately follows. □

Remark 2.6. In fact, instead of (2.23) we have used in the proof of the statements (iii) and (iv) essentially weaker conditions:

$$\sum_{i=1}^{\infty} \varphi_i^{1/2} < \infty, \ \sum_{i=1}^{\infty} \sigma_i^{1/2} < \infty, \ \sum_{i=1}^{\infty} \varepsilon_i < \infty \text{ and } \delta_i > \frac{1}{2}\varepsilon_i. \tag{2.35}$$

Hence, if the data $r^\star$, $\{K_i\}$ and $\{\varepsilon_i\}$ are chosen such that

$$K_i \subset \text{ int } S_{r_i} \text{ with } r_i = r^\star - \frac{1}{2}\varepsilon_i, \ i = 1, 2, \ldots,$$

then Theorem 2.5 remains true with (2.35) instead of condition (2.23).

R e m a r k 2.7.

1. We do not assume that the non-regularized auxiliary problems

$$\min\{J_i(u):\ u \in K_i\}$$

are solvable.

2. Theorem 2.5 does not require that $\lim \delta_i = 0$. Putting $\delta_i > 2r^\star$ we obtain, due to $|u^{i,s}| < r^\star$, that $s(i) = 1$ for each i, i.e., a usual iterative prox-regularization method (OSR-method) arises (cf. Kaplan/Tichatschke [26]). In this case condition (2.22) is superfluous.

3. Convergence of MSR-methods in case of $K_i \subset K$

Throughout this section we assume that H, Y_1, $\mathcal{P}$ and $b(\cdot,\cdot)$ are chosen as at the beginning of Section 2 and that Assumptions 2.1 and 2.2 are valid.

In the sequel, Method (2.9)–(2.12) will be studied for Problem (2.1) provided the feasible sets K_i of the auxiliary problems (2.9) possess the additional property $K_i \subset K$.

Let $u^{\star\star}$ be a fixed element of the optimal set $U^\star$ of Problem (2.1).

Assumption 3.1.

(i) For each $i = 1, 2, \ldots$ the inclusion $K_i \subset K$ and the inequality

$$\varrho(u^{\star\star}, K_i) \leqslant \varphi_i \tag{3.1}$$

hold, with $\{\varphi_i\}$ a given sequence tending to 0;

(ii) for each $r > 0$ the functional j satisfies the Lipschitz condition with a constant $\bar{\nu}(r)$ on the sphere $S_r(u^{\star\star}) = \{u \in \mathcal{Y}:\ |u - u^{\star\star}| \leqslant r\}$;

(iii) r_0 and $r^\star$ are chosen such that $r_0 \geqslant \sup \varphi_i$ and $r^\star \geqslant 8r_0$.

If Assumption 3.1(ii) is fulfilled, then for all u^1, $u^2 \in S_r(u^{\star\star})$ the inequality

$$|J(u^1) - J(u^2)| \leqslant \nu(r)|u^1 - u^2| \tag{3.2}$$

is obvious with

$$\nu(r) \geqslant M\beta^{-\frac{1}{2}}(r + |u^{\star\star}|) + \|f\|_{\mathcal{Y}'} + \bar{\nu}(r). \tag{3.3}$$

From now we suppose that $\nu(r)$ is a given non-decreasing function.

Lemma 3.2. *Let Assumption* 3.1 *be fulfilled and let* $r^\star$ *be chosen such that in addition*

$$\sum_{i=1}^{\infty}\Big((\nu_0\varphi_i+2\sigma_i)^{1/2}+\frac{1}{2}\varepsilon_i+2\varphi_i\Big)<\frac{r^\star}{2}. \tag{3.4}$$

Moreover, assume that

$$\frac{1}{r^\star}\Big(\nu_0\varphi_i+2\sigma_i-\Big(\delta_i-\frac{1}{2}\varepsilon_i\Big)^2\Big)+\varepsilon_i<0,\ i=1,2,\ldots, \tag{3.5}$$

where

δ_i,ε_i *are the controlling parameters in Method* (2.9)–(2.12),

$\nu_0=\nu(r_0)$ *and*

$$\sigma_i\geqslant\sup_{u\in Y}|j(u)-j_i(u)|. \tag{3.6}$$

Then, starting with $u^{1,0}\in S_{r^\star/4}(u^{\star\star})$, *in Method* (2.9)–(2.12) *the internal iteration cycle is finite, i.e.,* $s(i)<\infty$ *for each* i, *and the inclusions*

$$\bar{u}^{i,s}\in\operatorname{int}S_{r^\star}(u^{\star\star}),\quad u^{i,s}\in\operatorname{int}S_{r^\star}(u^{\star\star})$$

are valid for all pairs (i,s).

Proof. Consider a fixed pair (i,s) with $s\geqslant 1$. In view of $K_i\subset K$ and (2.12) the relation $J(u^{\star\star})\leqslant J(\bar{u}^{i,s})$ holds. Choosing $v^i\in K_i$ such that $|v^i-u^{\star\star}|\leqslant\varphi_i$, Assumption 3.1(ii) ensures

$$J(v^i)\leqslant J(u^{\star\star})+\nu_0\varphi_i.$$

Hence

$$J(v^i)\leqslant J(\bar{u}^{i,s})+\nu_0\varphi_i$$

and, in view of (3.6),

$$J_i(v^i)\leqslant J_i(\bar{u}^{i,s})+\nu_0\varphi_i+2\sigma_i. \tag{3.7}$$

Using (2.25) and the inequality

$$\delta_i-\frac{1}{2}\varepsilon_i>(\nu_0\varphi_i+2\sigma_i)^{1/2}$$

which follows from (3.4) and (3.5), in the case $s < s(i)$ one can conclude that

$$\begin{aligned}\|\mathcal{P}\bar{u}^{i,s} - \mathcal{P}u^{i,s-1}\|_H &> \|\mathcal{P}u^{i,s} - \mathcal{P}u^{i,s-1}\|_H \\ &- \|\mathcal{P}\bar{u}^{i,s} - \mathcal{P}u^{i,s}\|_H > (\nu_0\varphi_i + 2\sigma_i)^{1/2}.\end{aligned}$$

By virtue of inequality (3.7), application of Lemma 2.3 with

$$G = K_i,\ \bar{j} = j_i,\ \sigma = \sigma_i,\ z^0 = u^{i,s-1} \text{ and } u = v^i$$

leads to

$$|\bar{u}^{i,s} - v^i| < |u^{i,s-1} - v^i| + \left(2|u^{i,s-1} - v^i|\right)^{-1}\left(\nu_0\varphi_i + 2\sigma_i - \left(\delta_i - \frac{1}{2}\varepsilon_i\right)^2\right) \tag{3.8}$$

for $1 \leqslant s < s(i)$. If $s = s(i)$, Lemma 2.3 immediately gives

$$|\bar{u}^{i,s(i)} - v^i| \leqslant |u^{i,s(i)-1} - v^i| + (\nu_0\varphi_i + 2\sigma_i)^{1/2}. \tag{3.9}$$

Hence, in view of

$$|u^{i,s} - \bar{u}^{i,s}| \leqslant \frac{1}{2}\varepsilon_i,$$

the inequalities

$$|u^{i,s} - v^i| < |u^{i,s-1} - v^i| + \left(2|u^{i,s-1} - v^i|\right)^{-1}\left(\nu_0\varphi_i + 2\sigma_i - \left(\delta_i - \frac{1}{2}\varepsilon_i\right)^2\right) + \frac{1}{2}\varepsilon_i \tag{3.10}$$

are true for $1 \leqslant s < s(i)$, and

$$|u^{i,s(i)} - v^i| \leqslant |u^{i,s(i)-1} - v^i| + (\nu_0\varphi_i + 2\sigma_i)^{1/2} + \frac{1}{2}\varepsilon_i \tag{3.11}$$

holds, too. Using the assumptions $u^{1,0} \in S_{r^\star/4}(u^{\star\star})$ and $r_0 \leqslant \frac{r^\star}{8}$, in the case $s(1) > 1$ we obtain from (3.8), (3.10) and (3.5) the estimate

$$\begin{aligned}\max\{|\bar{u}^{1,1} - v^1|, |u^{1,1} - v^1|\} &< |u^{1,0} - v^1| \\ &+ \frac{1}{2r^\star}\left(\nu_0\varphi_1 + 2\sigma_1 - \left(\delta_1 - \frac{1}{2}\varepsilon_1\right)^2\right) + \frac{1}{2}\varepsilon_1 < |u^{1,0} - v^1| < \frac{r^\star}{2}.\end{aligned}$$

Analogously, for $1 < s < s(1)$ one can conclude

$$\max\{|\bar{u}^{1,s} - v^1|, |u^{1,s} - v^1|\} < |u^{1,0} - v^1| < \frac{r^\star}{2} \tag{3.12}$$

and

$$|u^{1,s} - v^1| < |u^{1,s-1} - v^1| + \frac{1}{r^\star}\left(\nu_0\varphi_1 + 2\sigma_1 - \left(\delta_1 - \frac{1}{2}\varepsilon_1\right)^2\right) + \frac{1}{2}\varepsilon_1. \tag{3.13}$$

Summing up the inequalities (3.13) with $s = 1, \dots, \bar{s} < s(1)$, we obtain

$$|u^{1,\bar{s}} - v^1| < |u^{1,0} - v^1| + \bar{s}\Big[\frac{1}{r^\star}\Big(\nu_0\varphi_1 + 2\sigma_1 - \Big(\delta_1 - \frac{1}{2}\varepsilon_1\Big)^2\Big) + \frac{1}{2}\varepsilon_1\Big], \tag{3.14}$$

and because of (3.5),

$$\bar{s} < -|u^{1,0} - v^1|\Big[\frac{1}{r^\star}\Big(\nu_0\varphi_1 + 2\sigma_1 - \Big(\delta_1 - \frac{1}{2}\varepsilon_1\Big)^2\Big) + \frac{1}{2}\varepsilon_1\Big]^{-1}.$$

Thus, $s(1) < \infty$ is obvious.

Due to (3.9), (3.11), (3.12) and (3.4), we get also the estimate

$$\begin{aligned} &\max\{|\bar{u}^{1,s(1)} - v^1|, |u^{1,s(1)} - v^1|\} \\ &\quad \leqslant |u^{1,0} - v^1| + (\nu_0\varphi_1 + 2\sigma_1)^{1/2} + \frac{1}{2}\varepsilon_1 < r^\star, \end{aligned} \tag{3.15}$$

which is true for $s(1) = 1$, too. In view of $|v^1 - u^{\star\star}| \leqslant \varphi_1$, for $1 \leqslant s \leqslant s(1)$ the relations (3.4), (3.12) and (3.15) lead to

$$\begin{aligned} &\max\{|\bar{u}^{1,s} - u^{\star\star}|, |u^{1,s} - u^{\star\star}|\} \\ &\quad \leqslant |u^{1,0} - u^{\star\star}| + (\nu_0\varphi_1 + 2\sigma_1)^{1/2} + \frac{1}{2}\varepsilon_1 + 2\varphi_1 < r^\star. \end{aligned}$$

Now, for the starting point $u^{2,0}$ on the iteration level $i = 2$ the inequality

$$|u^{2,0} - v^2| \leqslant |u^{1,0} - u^{\star\star}| + (\nu_0\varphi_1 + 2\sigma_1)^{1/2} + \frac{1}{2}\varepsilon_1 + 2\varphi_1 + \varphi_2 < r^\star$$

is valid. Continuation of this procedure with $i = 2, 3, \dots$ gives step by step the following estimates a)–e):

a) $$\begin{aligned} &\max\{|\bar{u}^{i,s} - v^i|, |u^{i,s} - v^i|\} < |u^{i,s-1} - v^i| \\ &\quad + \frac{1}{2r^\star}\Big(\nu_0\varphi_i + 2\sigma_i - \Big(\delta_i - \frac{1}{2}\varepsilon_i\Big)^2\Big) + \frac{1}{2}\varepsilon_i \quad \text{for } 1 \leqslant s < s(i); \end{aligned}$$

b) $s(i) < \infty$;

c) $$\begin{aligned} &\max\{|\bar{u}^{i,s} - v^i|, |u^{i,s} - v^i|\} \leqslant |u^{1,0} - u^{\star\star}| + \sum_{k=1}^{i-1}\Big[(\nu_0\varphi_k + 2\sigma_k)^{1/2} + \frac{1}{2}\varepsilon_k + 2\varphi_k\Big] \\ &\quad + (\nu_0\varphi_i + 2\sigma_i)^{1/2} + \frac{1}{2}\varepsilon_i + \varphi_i \quad \text{for } 1 \leqslant s \leqslant s(i); \end{aligned}$$

d) $$\begin{aligned} &\max\{|\bar{u}^{i,s} - u^{\star\star}|, |u^{i,s} - u^{\star\star}|\} \\ &\quad \leqslant |u^{1,0} - u^{\star\star}| + \sum_{k=1}^{i}\Big[(\nu_0\varphi_k + 2\sigma_k)^{1/2} + \frac{1}{2}\varepsilon_k + 2\varphi_k\Big] < r^\star \quad \text{for } 1 \leqslant s \leqslant s(i); \end{aligned}$$

e) $$\max\{|\bar{u}^{i+1,0}-v^{i+1}|,|u^{i+1,0}-v^{i+1}|\}$$
$$\leqslant |u^{1,0}-u^{\star\star}|+\sum_{k=1}^{i}\Big[(\nu_0\varphi_k+2\sigma_k)^{1/2}+\frac{1}{2}\varepsilon_k+2\varphi_k\Big]+\varphi_{i+1}<r^\star.$$

Hence we can conclude that $\bar{u}^{i,s}\in\operatorname{int} S_{r^\star}(u^{\star\star})$ and $u^{i,s}\in\operatorname{int} S_{r^\star}(u^{\star\star})$ for all (i,s).
$\square$

R e m a r k 3.3. Due to the convexity of the sets K_i and the functions $\Psi_{i,s}$, it is not difficult to show now that the statement of Lemma 3.2 is preserved, if instead of (3.6) we use

(3.16) $$\sigma_i\geqslant\sup_{u\in S_r(u^{\star\star})}|j(u)-j_i(u)|\ \text{ with } r\geqslant r^\star \text{ fixed.}$$

Theorem 3.4. *Let $r\geqslant r^\star$ and suppose that the following conditions are fulfilled:*
the assumptions of Lemma 3.2 *(with σ_i defined by* (3.6) *or* (3.16)*);*
$\varrho(Q',Q_i)\leqslant\overline{\varphi_i}$, $i=1,2,\dots$,
where $\bar\varphi_i\leqslant c_0\varphi_i$ holds with some constant c_0, and
$Q'=U^\star\cap S_{r^\star}(u^{\star\star})$, $Q_i=K_i\cap S_r(u^{\star\star})$;

(3.17) $$\frac{1}{4r}\Big(\nu(r)\bar\varphi_i+2\sigma_i-\Big(\delta_i-\frac{1}{2}\varepsilon_i\Big)^2\Big)+\frac{1}{2}\varepsilon_i<0.$$

Then the sequence $\{u^{i,s}\}$, generated by Method (2.9)–(2.12) *with the starting point $u^{1,0}\in S_{r^\star/4}(u^{\star\star})$, converges weakly to a solution $u^\star$ of Problem* (2.1) *and $\{J(u^{i,s})\}$ converges to $J(u^*)$. If, moreover, the subspace Y_1 is finite dimensional, then $\{u^{i,s}\}$ converges to $u^\star$ in the norm of the space Y.*

P r o o f. In view of $\nu_0=\nu(r_0)\leqslant\nu(r)$ condition (3.5) is an evident consequence of (3.17). Let $w\in U^\star\cap S_{r^\star}(u^{\star\star})$ be arbitrarily chosen and let a point $\xi^i\in Q_i$ be defined such that

(3.18) $$|\xi^i-w|\leqslant\bar\varphi_i.$$

Then, due to (3.2) and $J(w)\leqslant J(\bar{u}^{i,s})$,

$$J(\xi^i)\leqslant J(\bar{u}^{i,s})+\nu(r)\bar\varphi_i$$

holds and from (3.16) we obtain that

(3.19) $$J_i(\xi^i)\leqslant J_i(\bar{u}^{i,s})+\nu(r)\bar\varphi_i+2\sigma_i.$$

Lemma 3.2 ensures $u^{i,s} \in \operatorname{int} S_{r^\star}(u^{\star\star})$ for all pairs (i,s), consequently,

$$|u^{i,s} - \xi^i| \leqslant |u^{i,s} - u^{\star\star}| + |\xi^i - u^{\star\star}| < 2r.$$

For a fixed index i, using (3.17) and Lemma 2.3 with

$$G = Q_i,\ \bar{j} = j_i,\ \sigma = \sigma_i,\ u = \xi^i,\ z^0 = u^{i,s-1},$$

we obtain (as at the beginning of the proof of Lemma 3.2)

$$|\bar{u}^{i,s} - \xi^i| < |u^{i,s-1} - \xi^i| + \frac{1}{4r}\Big(\nu(r)\overline{\varphi}_i + 2\sigma_i - \Big(\delta_i - \frac{1}{2}\varepsilon_i\Big)^2\Big) \text{ if } 1 \leqslant s < s(i),$$

and

$$|\bar{u}^{i,s(i)} - \xi^i| \leqslant |u^{i,s(i)-1} - \xi^i| + (\nu(r)\overline{\varphi}_i + 2\sigma_i)^{1/2}.$$

In order to complete the proof we may repeat, starting with formula (2.29), the corresponding part of the proof of Theorem 2.5. □

R e m a r k 3.5. The conditions for the controlling parameters $\{\varphi_i\}$, $\{\sigma_i\}$ and $\{\varepsilon_i\}$ in [28] and also in Theorem 2.5 of the present paper are essentially stronger than in Theorem 3.4. Especially, if the estimates for the value

$$\sup_{u \in S_r(u^{\star\star})} |j(u) - j_i(u)|$$

do not depend on r (cf. below (5.16) for Problem (4.18), (4.19) and $j = j_i \equiv 0$ for Problem (4.5), (4.6)), then it is sufficient to require convergence of the series $\Sigma\varphi_i^{1/2}, \Sigma\sigma_i^{1/2}$ and $\Sigma\varepsilon_i$. With such $\{\varphi_i\}$, $\{\sigma_i\}$ and $\{\varepsilon_i\}$, the Lipschitz constant ν_0 can be defined and then $r^\star$ and $r \geqslant r^\star$ must be chosen such that

$$\max\{2\sum_{i=1}^{\infty}((\nu_0\varphi_i + 2\sigma_i)^{1/2} + \frac{1}{2}\varepsilon_i + 2\varphi_i), 8r_0\} < r^\star. \tag{3.20}$$

After that, $\{\delta_i\}$ has to be determined according to inequality (3.17). In contrast to condition (2.23), which was used in [28], the left hand side of (3.20) does not depend on r and $r^\star$.

We restrict ourselves here to the mathematical description of the variational problems considered; concerning their mechanical interpretation and the corresponding boundary value problems we refer to Hlaváček et al. in [22]. In order to describe scalar products and norms in special spaces of vector functions we use symbols $((\cdot,\cdot))$ and $\|\cdot\|$ marked with the corresponding indices.

4.1. Variational formulation of a two-body contact problem

Let $\Omega' \subset \mathbb{R}^2$ and $\Omega'' \subset \mathbb{R}^2$ be two open bounded domains with Lipschitz-continuous boundaries, and $\Omega = \Omega' \cup \Omega''$. In the sequel the superscripts $'$ and $''$ correspond to $\overline{\Omega}'$ and $\overline{\Omega}''$, respectively. We suppose that Ω' and Ω'' have a common boundary $\Gamma_c = \partial\Omega' \cap \partial\Omega''$. The partitions of the boundaries

$$\partial\Omega' = \Gamma_u \cup \Gamma'_\tau \cup \Gamma_c \quad (\text{ mes } \Gamma_u > 0, \quad \text{mes } \Gamma_c > 0),$$

$$\partial\Omega'' = \Gamma_0 \cup \Gamma''_\tau \cup \Gamma_c$$

and the functions $F \in [L_2(\Omega)]^2$, $P' \in [L_2(\Gamma'_\tau)]^2$, $P'' \in [L_2(\Gamma''_\tau)]^2$ are assumed to be known.

We denote by ν, ν' and ν'' the unit outward normals to Γ_0, Γ'_τ and Γ''_τ respectively, and for Γ_c the symbols ν' and ν'' mean the unit normals pointed outside of Ω' and Ω'', respectively.

The case of $\Gamma_0 = \emptyset$ is permitted.

The given functions a_{klpm} $(k,l,p,m = 1,2)$ are assumed to be measurable and bounded on Ω. Moreover, symmetry

$$a_{klpm} = a_{lkpm} = a_{pmkl} \tag{4.1}$$

is supposed as well as the existence of a positive constant c_0 such that

$$a_{klpm}(x)\sigma_{kl}\sigma_{pm} \geqslant c_0\sigma_{kl}\sigma_{kl} \tag{4.2}$$

holds for all symmetric matrices $[\sigma_{kl}]_{k,l=1,2}$ and almost every $x \in \Omega$ (ellipticity property). Here and in the sequel we follow Einstein's summation convention, i.e., the summation is performed over terms with repeating indices.

Denote $u''_\nu = u''_k\nu_k$ and define the space

$$\begin{aligned} V = \{u = (u', u'') \in [H^1(\Omega')]^2 \times [H^1(\Omega'')]^2 \colon\ & u' = 0 \ \text{ on } \Gamma_u, \\ & u''_\nu = 0 \text{ on } \Gamma_0\} \end{aligned} \tag{4.3}$$

endowed with the norm

$$\|u\|_{1,\Omega} = \Big(\|u'\|^2_{1,\Omega'} + \|u''\|^2_{1,\Omega''}\Big)^{1/2}.$$

In the sequel we shall also use the spaces $[H^s(\Omega')]^2 \times [H^s(\Omega'')]^2$ with s integer (including $s = 0$) and

$$\|u\|_{s,\Omega} = \Big(\|u'\|^2_{s,\Omega'} + \|u''\|^2_{s,\Omega''}\Big)^{1/2}.$$

Denoting

$$\varepsilon_{kl}(u) = \frac{1}{2}\Big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\Big),\ k,l = 1,2,\ u'_{\nu'} = u'_k \nu'_k,\ u''_{\nu''} = u''_k \nu''_k, \tag{4.4}$$

we will consider the following variational formulation of the two-body contact problem:

Minimize the functional

$$J(u) = \frac{1}{2} a(u,u) - \ell(u) \tag{4.5}$$

on the set

$$K \equiv \{u \in V\colon\ u'_{\nu'} + u''_{\nu''} \leqslant 0 \text{ on } \Gamma_c\}, \tag{4.6}$$

where

$$a(u,v) = \int_\Omega a_{klpm} \varepsilon_{kl}(u) \varepsilon_{pm}(v)\, \mathrm{d}\Omega, \tag{4.7}$$

$$\ell(u) = \int_\Omega F_k u_k\, \mathrm{d}\Omega + \int_{\Gamma_\tau} P_k u_k\, \mathrm{d}\Gamma, \quad \Gamma_\tau = \Gamma'_\tau \cup \Gamma''_\tau. \tag{4.8}$$

The kernel $\mathcal{K}$ of the bilinear form (4.7) on the space $W = [H^1(\Omega')]^2 \times [H^1(\Omega'')]^2$ consists of elements $z = (z', z'')$, where z' and z'' are vector functions with components

$$\begin{aligned} z'_1(x) &= a'_1 - b' x_2, \qquad & z'_2(x) &= a'_2 + b' x_1, \\ z''_1(x) &= a''_1 - b'' x_2, \qquad & z''_2(x) &= a''_2 + b'' x_1, \end{aligned}$$

with arbitrary coefficients a'_1, a'_2, b' and a''_1, a''_2, b''.

It is easy to see that

$$\ell(y) \leqslant 0 \ \text{ for all } \ y \in K \cap \mathcal{K}$$

is a necessary condition for the existence of a solution of Problem (4.5), (4.6).

Theorem 4.1. (see HLAVÁČEK et al. [22]) *If the conditions*

$$\ell(y) \leqslant 0 \quad \textit{for all } y \in K \cap \mathcal{K}$$

and

$$\ell(y) < 0 \quad \textit{for all } y \in K \cap \mathcal{K} \textit{ with } \inf_{x \in \Gamma_c} (y''_{\nu''}(x) + y'_{\nu'}(x)) < 0$$

hold, then Problem (4.5), (4.6) *has at least one solution* $u^\star$. *Moreover, the solution set has the structure*

$$U^\star = \{u^\star + y\colon\ y \in V \cap \mathcal{K},\ u^\star + y \in K,\ \ell(y) = 0\}$$

with $u^\star$ *fixed.*

If $\Gamma_0 = \emptyset$, then $\dim(V \cap \mathcal{K}) = 3$ and, in the case $\operatorname{mes} \Gamma_0 > 0$, $\dim(V \cap \mathcal{K}) \leqslant 1$.

4.2. Finite element approximation of two-body contact problems

We describe the application of finite element methods to Problem (4.5), (4.6), following HLAVÁČEK et al. [22]. Let us assume that Ω' and Ω'' are bounded polyhedral domains. Then, triangulations $\mathcal{T}'_h$, $\mathcal{T}''_h$ of Ω' and Ω'' are performed so that

$$\bigcup_{T \in \mathcal{T}'_h} T = \bar{\Omega}', \qquad \bigcup_{T \in \mathcal{T}''_h} T = \bar{\Omega}'' \tag{4.9}$$

with h a parameter characterizing the maximal side of the triangles $\mathcal{T}'_h$ and $\mathcal{T}''_h$. With respect to $h \to 0$ regular systems $\{\mathcal{T}'_h\}$ and $\{\mathcal{T}''_h\}$ of triangulations are considered, i.e., the areas of the triangles in $\mathcal{T}'_h$ and $\mathcal{T}''_h$ are bounded from below by $d_0 h^2$ ($d_0 > 0$) if $h \to 0$. Moreover, we assume that the following conditions are fulfilled for each value of h:

(i) the points where the type of the boundary condition changes belong to the set of nodes of the corresponding triangulations $\mathcal{T}'_h$ and $\mathcal{T}''_h$;

(ii) for a fixed representation of the contact boundary

$$\Gamma_c = \bigcup_{j=1}^{m} \Gamma_{c,j} \tag{4.10}$$

with $\Gamma_{c,j}$ closed straight-line segments, the end points of the segments $\Gamma_{c,j}$ are common nodes of both the triangulations $\mathcal{T}'_h$ and $\mathcal{T}''_h$. The nodes lying on Γ_c must also be common nodes of $\mathcal{T}'_h$ and $\mathcal{T}''_h$.

The corresponding sequence $\{\mathcal{T}_h\}$ with $\mathcal{T}_h = \mathcal{T}_h' \cup \mathcal{T}_h''$ is a regular system of triangulations of the domain $\Omega = \Omega' \cup \Omega''$.

The index sets I_h', I_h'', I_h^u, $I_h^{(0)}$ and $I_h^{c,j}$ indicate that the nodes π_k belong to the sets $\bar{\Omega}', \bar{\Omega}''$, Γ_u, Γ_0 and $\Gamma_{c,j}$, respectively. In order to simplify the description, we assume that $\Gamma_0 = \emptyset$ or that Γ_0 is a straight line segment. In the latter case ν^0 denotes a unit outward normal to Γ_0. By ν^j we denote the unit normal to $\Gamma_{c,j}$, pointed outside to Ω'.

For a fixed pair of the triangulations $\mathcal{T}_h'$ and $\mathcal{T}_h''$ a finite element space V_h of vector functions

$$v_h = (v_h', v_h'') \in ([C(\bar{\Omega}')]^2 \times [C(\bar{\Omega}'')^2]) \cap V$$

is defined with

$$v_h'(x) = \sum_{k \in I_h' \setminus I_h^u} \alpha^k \varphi_k(x),\ \alpha^k \in \mathbb{R}^2, \tag{4.11}$$

$$v_h''(x) = \sum_{k \in I_h''} \beta^k \eta_k(x),\ \beta^k \in \mathbb{R}^2. \tag{4.12}$$

Here $\varphi_k \in C(\bar{\Omega}')$ and $\eta_k \in C(\bar{\Omega}'')$ are affine functions on each element of the corresponding triangulation and

$$\begin{aligned} \varphi_k(\pi_k) &= 1, \quad \varphi_k(\pi_j) = 0 \quad \text{for } j \neq k\ (k \in I_h'), \\ \eta_k(\pi_k) &= 1, \quad \eta_k(\pi_j) = 0 \quad \text{for } j \neq k\ (k \in I_h''). \end{aligned}$$

To satisfy $u_\nu'' = 0$ on Γ_0, the coefficients β^k must be chosen such that

$$(\nu^0, \beta^k)_{\mathbb{R}^2} = 0 \text{ for } k \in I_h^{(0)}. \tag{4.13}$$

The set K_h which approximates K on the space

$$V_h = \{v_h = (v_h', v_h'')\colon\ (\nu^0, \beta^k)_{\mathbb{R}^2} = 0,\ k \in I_h^{(0)}\} \tag{4.14}$$

can be expressed by

$$K_h = \{v_h \in V_h\colon\ (\nu^j, v_h'(\pi_k) - v_h''(\pi_k))_{\mathbb{R}^2} \leqslant 0,\ k \in I_h^{c,j},\ j = 1, \ldots, m\} \tag{4.15}$$

or

$$K_h = \{v_h \in V_h\colon\ (\nu^j, \alpha^k - \beta^k)_{\mathbb{R}^2} \leqslant 0,\ k \in I_h^{c,j},\ j = 1, \ldots, m\}. \tag{4.16}$$

So, it is not difficult to show that the inclusion $K_h \subset K$ is true.

Finally, the approximate problem in the space V_h can be represented by

$$\min\{J(u_h)\colon\ u_h \in K_h\}, \tag{4.17}$$

where J is defined according to (4.5), (4.7), (4.8) and the feasible set K_h is given by relation (4.15).

If sufficient conditions of solvability of the studied problem (cf. Theorem 4.1) are valid, then solvability of the approximate problems follows from the inclusion $K_h \subset K$.

For the case that Γ_c is not piecewise affine, but is a graph of a convex function, Hlaváček et al. [22] have used curved triangles T_j along the contact boundary such that

$$\bigcup_{T \in \mathcal{T}_h'} \overline{T} = \bar{\Omega}', \qquad \bigcup_{T \in \mathcal{T}_h''} \overline{T} = \bar{\Omega}''.$$

Note that in this case the inclusion $K_h \subset K$ is not valid in general.

Applying the above finite element approximation to the two-body contact problem, convergence of the corresponding minimizing sequence in the norm of the space V has been established only in the following special case (i) – (ii) (see [22]):

(i) $\dim(V \cap \mathcal{K}) \leqslant 1$,

(ii) either Korn's inequality (see Fichera [14], Chapt. 1) is fulfilled on V, or a subspace $V^1 \subset V$ can be chosen easily such that $U^\star \cap (K \cap V^1) \neq \emptyset$ and an analogue of Korn's inequality holds on V^1.

In the case $\dim(V \cap \mathcal{K}) = 3$ finite element approximations of the dual problem are commonly applied. However, the description of the feasible set of the dual problem is rather complicated so that special approximation techniques, based on equilibrium models of finite elements, have to be applied [22].

4.3. Static problems of linear elasticity with given friction

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with a Lipschitz-continuous boundary $\partial\Omega$. We assume further that a partition

$$\partial\Omega = \Gamma_\tau \cup \Gamma_c$$

of $\partial\Omega$ and vector functions $P \in [L_2(\Gamma_\tau)]^2$, $S \in [L_\infty(\Gamma_c)]^2$ and $F \in [L_2(\Omega)]^2$ are given. With measurable functions a_{klpm} $(k,l,p,m = 1,2)$ satifying (4.1), (4.2), ε_{kl} defined by (4.4) and a positive function $\mu \in C^2(\Gamma_c)$, the problem under consideration is as follows:

Minimize the functional

$$J(u) = \frac{1}{2}a(u,u) - \ell(u) + j(u) \tag{4.18}$$

on the space

$$V = [H^1(\Omega)]^2, \tag{4.19}$$

with

$$a(u,v) = \int_\Omega a_{klpm}\varepsilon_{kl}(u)\varepsilon_{pm}(v)\,\mathrm{d}\Omega, \tag{4.20}$$

$$\ell(u) = \int_\Omega F_k u_k\,\mathrm{d}\Omega + \int_{\Gamma_\tau} P_k u_k\,\mathrm{d}\Gamma + \int_{\Gamma_c} S_\nu u_k \nu_k\,\mathrm{d}\Gamma, \tag{4.21}$$

$$j(u) = \int_{\Gamma_c} \mu |S_\nu||u_t|\,\mathrm{d}\Gamma, \tag{4.22}$$

$u_t = u - (u_k\nu_k)\nu$, $S_\nu = S_k\nu_k$ and ν the unit outward normal to $\partial\Omega$.

The kernel $\mathcal{K}$ of the bilinear form (4.20) on the space V has the structure

$$\mathcal{K} = \{z = (z_1, z_2)\colon\ z_1 = a_1 - bx_2, z_2 = a_2 + bx_1\}$$

with arbitrary coefficients a_1, a_2 and b. The uniqueness of the solution of this problem is probably not known, only the following result has been established.

Theorem 4.2. (see Duvaut/Lions [13], Chapt. 3) *The condition*

$$|\ell(y)| \leqslant j(y) \quad \textit{for } y \in \mathcal{K}$$

is necessary for the solvability of Problem (4.18), (4.19). *A solution exists if*

$$|\ell(y)| < j(y) \quad \textit{for all } y \in \mathcal{K},\ \ y \neq 0.$$

The following fact is obvious: if u_1 and u_2 are solutions of Problem (4.18), (4.19), then $u_1 - u_2 \in \mathcal{K}$.

A finite element approximation of Problem (4.18), (4.19) can be performed in a standard way, using piecewise linear basic functions and a regular system $\{\mathcal{T}_h\}$ of triangulations. The arising auxiliary problems consist in the unconstrained minimization of convex non-smooth functions. In general, unique solvability of these problems as well as strong convergence of their solutions (if $h \to 0$) are not guaranteed.

R e m a r k 4.3. Taking into account the properties of the bilinear forms (4.7) and (4.20) and the structure of the solution sets of both model problems, we may expect the following. If a minimizing sequence has one of the properties

(i) the projections of its elements onto the kernel $\mathcal{K}$ are convergent, or

(ii) the sequence converges in the norm of $[L_2(\Omega)]^2$,

then this sequence converges in the norm of the space V (see (4.3) and (4.19)) to an element of the optimal set.

5. Application to elliptic variational inequalities

5.1. MSR-methods for two-body contact problems

Identifying Problem (2.1) with the contact problem (4.5), (4.6) and assuming that the sets K_i are obtained by means of a finite element approximation of K, we can specify the MSR-methods considered in Section 2 as follows.

We put $Y = V$, where V is defined by (4.3). Because the objective functional (4.5) has the form (2.2) with $j \equiv 0$, it makes sense to take $j_i \equiv 0$ for all i. Recall that mes $\Gamma_u > 0$ in (4.3).

If $Y_1 = V$, $H = V$, $\mathcal{P} = I$ (identity operator in V), we obtain the *basic variant of the MSR-method.* Taking $b(u,v) \equiv 0$, one may put $\varepsilon_i' = \varepsilon_i$. Obviously, in this case inequality (2.3) is fulfilled with $\beta = 1$.

Regularization on the kernel occurs in the case $H = [L_2(\Omega)]^2$, $Y_1 = V_1$ with

$$V_1 = \{u = (u', u'') \in V\colon\ u' \equiv 0;$$

$$u_1''(x) = a_1 - bx_2, \quad u_2''(x) = a_2 + bx_1 \text{ on } \Omega''\} \subset \mathcal{K} \tag{5.1}$$

and $\mathcal{P}\colon\ V \to V_1$ an orthoprojector onto V_1 corresponding to the norm $\|\cdot\|_{0,\Omega}$. Obviously, the calculation of $\mathcal{P}z$ is not complicated. We further suppose that

$$b(u,v) = c_0 \int_\Omega \varepsilon_{kl}(u)\varepsilon_{kl}(v)\,\mathrm{d}\Omega \tag{5.2}$$

with c_0 defined in (4.2).

Finally, the choice of $Y_1 = \{u = (u', u'') \in V\colon\ u' \equiv 0\}$, $H = [L_2(\Omega)]^2$ and

$$\mathcal{P}(u) = \begin{cases} 0 & \text{for } x \in \Omega' \\ u''(x) & \text{for } x \in \Omega'' \end{cases}$$

corresponds to the *method with weak regularization.* Again we choose $b(\cdot,\cdot)$ as in (5.2).

In order to extend the results of convergence from Section 3 to the last two MSR-methods, we need the following statement.

Theorem 5.1. *In the case of the method with regularization on the kernel and the method with weak regularization the spaces Y, Y_1, H and the projector $\mathcal{P}$ fulfil Assumption* 2.2(ii).

P r o o f. We use the inequality

$$\int_{\Omega'} \varepsilon_{kl}(u)\varepsilon_{kl}(u)\,\mathrm{d}\Omega \geqslant c_1 \|u\|^2_{1,\Omega'} \quad (c_1 > 0), \tag{5.3}$$

which reflects the equivalence between the seminorm

$$[u] = \left[\int_{\Omega'} \varepsilon_{kl}(u)\varepsilon_{kl}(u)\,\mathrm{d}\Omega\right]^{1/2}$$

and the norm $\|u\|^2_{1,\Omega'}$ in the space $\{u \in [H^1(\Omega')]^2 \colon u|_{\Gamma_u} = 0\}$ if $\operatorname{mes}\Gamma_u > 0$ (cf. Ciarlet [12], Sect. 1.2).

In the case the method with weak regularization is considered, the second Korn inequality (see Fichera [14], Chapt. 1)

$$\int_{\Omega''} \varepsilon_{kl}(u)\varepsilon_{kl}(u)\,\mathrm{d}\Omega + \int_{\Omega''} u_k u_k\,\mathrm{d}\Omega \geqslant c_2 \|u\|^2_{1,\Omega''} \quad (c_2 > 0) \tag{5.4}$$

and (5.3) immediately enable us to establish the validity of condition (2.3) with

$$\beta = \min[c_1, c_2] \min\left[\frac{1}{2}c_0, 1\right].$$

Indeed, we get

$$\begin{aligned}|u|^2 = \frac{1}{2}b(u,u) + \|\mathcal{P}u\|^2_H &= \frac{1}{2}c_0 \int_{\Omega} \varepsilon_{kl}(u)\varepsilon_{kl}(u)\,\mathrm{d}\Omega + \int_{\Omega''} u_k u_k\,\mathrm{d}\Omega \\ &\geqslant \frac{1}{2}c_0 c_1 \|u\|^2_{1,\Omega'} + \min\left\{\frac{1}{2}c_0, 1\right\} c_2 \|u\|^2_{1,\Omega''} \geqslant \beta \|u\|^2_{1,\Omega}.\end{aligned}$$

Now we turn to the method with regularization on the kernel. Let Θ_1 be the orthoprojector, mapping from $[L_2(\Omega'')]^2$ onto the linear set of functions

$$\mathcal{K}'' = \{z \colon z_1(x) = a_1 - bx_2, z_2(x) = a_2 + bx_1 \text{ on } \Omega''\}$$

with a_1, a_2, b arbitrary real numbers, and put $\Theta = I - \Theta_1$ with the identity operator I on $[L_2(\Omega'')]^2$. For each element $u \in [H^1(\Omega'')]^2$ we get

$$\int_{\Omega''} \varepsilon_{kl}(u)\varepsilon_{kl}(u)\,\mathrm{d}\Omega = \int_{\Omega''} \varepsilon_{kl}(\Theta u)\varepsilon_{kl}(\Theta u)\,\mathrm{d}\Omega. \tag{5.5}$$

Let us show that for some $c_3 > 0$ and each $u \in [H^1(\Omega'')]^2$ the inequality

$$\int_{\Omega''} \varepsilon_{kl}(u)\varepsilon_{kl}(u)\,\mathrm{d}\Omega \geqslant c_3 \|\Theta u\|^2_{1,\Omega''} \tag{5.6}$$

is satisfied. If this is wrong, then, due to (5.5), there exists a sequence $\{u^j\} \in [H^1(\Omega'')]^2$ such that with $v^j = \Theta u^j$ the relations

$$\|v^j\|_{1,\Omega''} = 1, \tag{5.7}$$

$$\lim_{j\to\infty} \int_{\Omega''} \varepsilon_{kl}(v^j)\varepsilon_{kl}(v^j)\,\mathrm{d}\Omega = 0 \tag{5.8}$$

and

$$\lim_{j\to\infty} \int_{\Omega''} \varepsilon_{kl}(u^j)\varepsilon_{kl}(u^j)\,\mathrm{d}\Omega = 0$$

are true.

Without loss of generality, we assume here that the sequence $\{v^j\}$ converges weakly to an element $\bar{v}$ in $[H^1(\Omega'')]^2$. Then, with regard to the compact embedding of $[H^1(\Omega'')]^2$ into $[L_2(\Omega'')]^2$, $\{v^j\}$ converges to $\bar{v}$ in the norm of the space $[L_2(\Omega'')]^2$. On account of inequality (5.4) the estimate

$$\|v^{j+p} - v^j\|^2_{1,\Omega''} \leqslant c_2^{-1}\Big[\int_{\Omega''} \varepsilon_{kl}(v^{j+p} - v^j)\varepsilon_{kl}(v^{j+p} - v^j)\,\mathrm{d}\Omega + \|v^{j+p} - v^j\|^2_{0,\Omega''}\Big] \tag{5.9}$$

is fulfilled for all indices j and p. But, due to (5.8) and the strong convergence of $\{v^j\}$ in $[L_2(\Omega'')]^2$, inequality (5.9) implies that

$$\lim_{i\to\infty} \|v^i - \bar{v}\|_{1,\Omega''} = 0.$$

Hence, according to (5.7), we get

$$\|\bar{v}\|^2_{1,\Omega''} = 1, \tag{5.10}$$

and the relation

$$\int_{\Omega''} \varepsilon_{kl}(\bar{v})\varepsilon_{kl}(\bar{v})\,\mathrm{d}\Omega = 0,$$

following from (5.8), means that $\bar{v} \in \mathcal{K}''$ (cf. NEČAS/HLAVÁČEK [35]).

But $\lim \|v^j - \bar{v}\|_{0,\Omega''} = 0$ and

$$((v^j, z))_{0,\Omega''} = 0 \quad \text{for all} \quad z \in \mathcal{K}'',$$

therefore we obtain

$$((\bar{v}, z))_{0,\Omega''} = 0 \quad \text{for all} \quad z \in \mathcal{K}''.$$

Hence $\bar{v} = 0$, and this contradicts (5.10).

The definition of the projectors $\mathcal{P}$ (observe regularization on the kernel) and Θ_1 implies

$$\mathcal{P}u\big|_{\Omega''} = \Theta_1 u'', \quad \mathcal{P}u\big|_{\Omega'} = 0,$$

and with regard to (5.6), for each $u \in [H^1(\Omega'')]^2$ the inequality

$$\begin{aligned}
&\frac{1}{4}c_0 \int_{\Omega''} \varepsilon_{kl}(u)\varepsilon_{kl}(u)\,\mathrm{d}\Omega + \|\mathcal{P}u\|^2_{0,\Omega''} \\
&\quad \geqslant \frac{1}{4}c_0 c_3 \|\Theta u\|^2_{0,\Omega''} + \|\Theta_1 u\|^2_{0,\Omega''} \\
&\quad \geqslant \min\left\{\frac{1}{4}c_0 c_3, 1\right\} \|u\|^2_{0,\Omega''}
\end{aligned}$$

holds true. Therefore

$$\begin{aligned}
&\frac{1}{2}c_0 \int_{\Omega''} \varepsilon_{kl}(u)\varepsilon_{kl}(u)\,\mathrm{d}\Omega + \|\mathcal{P}u\|^2_{0,\Omega''} \\
&\quad \geqslant \frac{1}{4}c_0 \int_{\Omega''} \varepsilon_{kl}(u)\varepsilon_{kl}(u)\,\mathrm{d}\Omega + \min\left\{\frac{1}{4}c_0 c_3, 1\right\} \|u\|^2_{0,\Omega''}
\end{aligned}$$

and, in view of (5.4), we finally obtain (with $c_4 = c_2 \min\left\{\frac{1}{4}c_0, \frac{1}{4}c_0 c_3, 1\right\}$)

$$\frac{1}{2}c_0 \int_{\Omega''} \varepsilon_{kl}(u)\varepsilon_{kl}(u)\,\mathrm{d}\Omega + \|\mathcal{P}u\|^2_{0,\Omega''} \geqslant c_4 \|u\|^2_{1,\Omega''}.$$

Now, from the last inequality and (5.3) relation (2.3) with

$$\beta = \min\left\{\frac{1}{2}c_0 c_1, c_4\right\}$$

follows immediately.

By virtue of Theorem 5.1, in the case of Problem (4.5), (4.6) Assumptions 2.1 and 2.2 hold true for all variants of MSR-methods considered. Therefore, in order to guarantee convergence of these methods, the crucial point is the choice of the sequence $\{\varphi_i\}$. Recall that, in the general case, $\{\varphi_i\}$ must satisfy conditions (2.20), (2.22) and (2.23). However, if $K_i \subset K$ for each i, then $\{\varphi_i\}$ has to be chosen according to (3.1), (3.4), (3.5) and (ii), (iii) in Theorem 3.4.

Now we show for the basic variant of the MSR-method how the sequence $\{\varphi_i\}$ can be chosen in the case $K_i \subset K$. The corresponding analysis of the other two MSR-methods can be carried out analogously.

Because we take $b(u,v) \equiv 0$, the norms $|\cdot|$ and $\|\cdot\|_{1,\Omega}$ coincide, hence $\mathcal{Y} = V$. We assume that a solution $\bar{u}$ of the two-body contact problem belongs to the space $[H^2(\Omega')]^2 \times [H^2(\Omega'')]^2$. Then, according to the structure of the solution set (cf. Theorem 4.1), each other solution $\bar{\bar{u}}$ is also contained in this space, moreover,

$$\|\bar{u} - \bar{\bar{u}}\|_{1,\Omega} = \|\bar{u} - \bar{\bar{u}}\|_{2,\Omega}. \tag{5.11}$$

Therefore, for any r_1 the set $U^\star \cap S_{r_1}(u^{\star\star})$ is also bounded in the space $[H^2(\Omega')]^2 \times [H^2(\Omega'')]^2$ and

$$\|u\|_{2,\Omega} \leqslant \|u^{\star\star}\|_{2,\Omega} + r_1$$

for each function $u \in U^\star \cap S_{r_1}(u^{\star\star})$. Thus, by virtue of Theorem 3.2.1 in Ciarlet [12], the linear interpolant $u_{I,h}$ of each function $u \in U^\star \cap S_{r_1}(u^{\star\star})$ on the triangulation $\mathcal{T}_h$ yields

$$\|u - u_{I,h}\|_{1,\Omega} \leqslant c(\|u^{\star\star}\|_{2,\Omega} + r_1)h, \tag{5.12}$$

with c independent of u and r_1.

But for $h = h_i$, due to (4.6), (4.15) and the construction of $\mathcal{T}_h$, one can conclude that $u_{I,h_i} \in K_i \equiv K_{h_i}$. This enables us immediately to define $\{\varphi_i\}$ and $\{\bar{\varphi}_i\}$ according to Theorem 3.4.

Indeed, if h_i is chosen such that

$$\sup h_i \leqslant r_1(c(c_5 + r_1))^{-1}$$

with r_1 fixed and $c_5 \geqslant \|u^{\star\star}\|_{2,\Omega}$, then the sequence

$$\varphi_i = c(c_5 + r_1)h_i, \quad i = 1, 2, \ldots \tag{5.13}$$

satisfies the inequality $\varphi_i \leqslant r_1$. Thus we may identify the radius r_0 with r_1 (see Assumption 3.1(iii)), and $\Sigma h_i^{1/2} < \infty$ ensures that $\Sigma \varphi_i^{1/2} < \infty$.

Now, determine $r^\star$ according to (3.4) and $r^\star > 8r_0$ and choose

$$\bar{\varphi}_i = c(c_5 + r^\star)h_i, \; i = 1, 2, \dots \tag{5.14}$$

Then, using (5.12) with $r_1 = r^\star$, $h = h_i$, we obtain

$$|\!|\!| u - u_{I,h_i} |\!|\!|_{1,\Omega} \leqslant \bar{\varphi}_i \tag{5.15}$$

and, because of $u_{I,h_i} \in K_i$, the estimate $\varrho(u, K_i) \leqslant \bar{\varphi}_i$ is valid for each $u \in U^\star \cap S_{r^\star}(u^{\star\star}) = Q'$. □

Remark 5.2. Let the sequences $\{h_i\}$, $\{\varepsilon_i\}$ be chosen such that $\Sigma h_i^{1/2} < \infty$ and $\Sigma\varepsilon_i < \infty$ and let $\{\delta_i\}$ be defined by (3.17) with

$$r = r^\star + \bar{r} \quad \text{and} \quad \bar{r} \geqslant \sup \bar{\varphi}_i.$$

Then, resuming the analysis above, $\varrho(u, K_i) \leqslant \bar{\varphi}_i$ leads to $\varrho(Q', Q_i) \leqslant \bar{\varphi}_i$. Finally, Theorem 3.4 guarantees that $\{u^{i,s}\}$, calculated by means of the regularization on the kernel, converges in the norm of the space V to a solution of the two-body contact problem.

Relation (5.11) is the point of this analysis. The consideration may be extended to other elliptic variational inequalities provided their solution sets possess a similar property.

However, it should be remarked that the verification of the condition

$$u^\star \in [H^2(\Omega')]^2 \times [H^2(\Omega'')]^2 \quad \text{for some} \quad u^\star \in U^\star$$

is complicated.

5.2. Weak regularization for a model problem with given friction

In order to apply the results of Sections 2 and 3 to Problem (4.18), (4.19) we have to consider Problem (2.1) in the space $Y = V \equiv [H^1(\Omega)]^2$. For the approximation of the non-smooth functional (4.22) it is convenient to use convex functionals

$$j_i(u) = \int_{\Gamma_c} \mu |S_\nu| \sqrt{u_t^2 + \kappa_i} \, \mathrm{d}\Gamma$$

with $\{\kappa_i\}$ a positive sequence converging to 0. Obviously, for all $u \in V$ the estimate

$$|j(u) - j_i(u)| \leqslant \sqrt{\kappa_i} \int_{\Gamma_c} \mu |S_\nu| \, \mathrm{d}\Gamma \tag{5.16}$$

is true.

In the sequel our consideration is concentrated on the method with weak regularization. Therefore, in Method (2.9)–(2.12) we put $Y = V$, $H = [L_2(\Omega)]^2$, and $\mathcal{P}\colon V \to H$ is the embedding operator. Since the model problem is unconstrained, i.e., $K = V$, the auxiliary problems (2.9) have the form

$$\min\{J_i(u) + \|u - u^{i,s-1}\|^2_{0,\Omega}\colon\ u \in V_i\} \tag{5.17}$$

with $J_i(u) = \frac{1}{2}a(u,u) - \ell(u) + j_i(u)$. The data a and ℓ are given according to (4.20), (4.21), and $V_i = V_{h_i}$ is a linear span of piecewise linear basis functions corresponding to the triangulation $\mathcal{T}_{h_i}$. Here the domain Ω is not supposed to be polygonal. Choosing

$$b(u,v) = c_0 \int\limits_\Omega \varepsilon_{kl}(u)\varepsilon_{kl}(v)\,\mathrm{d}\Omega,$$

where c_0 satisfies condition (4.2) with the data a_{kplm} from (4.20), Assumption 2.2(ii) immediately follows from the second Korn inequality.

Hence, the norm $|\cdot|_\Omega$ introduced according to (2.3),

$$|u|^2_\Omega = \frac{1}{2}b(u,u) + \|u\|^2_{0,\Omega},$$

and the norm $\|\cdot\|_{1,\Omega}$ are equivalent. Therefore, in the sequel we consider Problem (4.18), (4.19) in the space $\mathcal{V}$, where $\mathcal{V}$ is the vector space V endowed with the norm $|\cdot|_\Omega$.

If a solution $\bar{u}$ of Problem (4.18), (4.19) belongs to $[H^2(\Omega)]^2$, then, due to the structure of $U^\star$, $\bar{\bar{u}} \in [H^2(\Omega)]^2$ holds for any $\bar{\bar{u}} \in U^\star$ and

$$\|\bar{u} - \bar{\bar{u}}\|_{1,\Omega} = \|\bar{u} - \bar{\bar{u}}\|_{2,\Omega}.$$

Thus, for fixed $u^{\star\star} \in U^\star$ and each $u \in U^\star \cap S_{r_1}(u^{\star\star})$ the inequality

$$\|u\|_{2,\Omega} \leqslant \|u^{\star\star}\|_{2,\Omega} + c_6 r_1$$

is satisfied with c_6 independent of r_1 and $S_{r_1}(u^{\star\star})$ a sphere in $\mathcal{V}$. For an interpolant $v_{I,h}$ of any function $v \in [H^2(\Omega)]^2$ the estimate

$$\|v - v_{I,h}\|_{1,\Omega} \leqslant c\|v\|_{2,\Omega} h$$

is usually true in a non-polygonal domain Ω, too (see Scarpini/Vivaldi [42]). In view of the equivalence between $\|\cdot\|_{1,\Omega}$ and $|\cdot|_\Omega$ this leads to

$$|v - v_{I,h}|_\Omega \leqslant \bar{c}\|v\|_{2,\Omega} h \tag{5.18}$$

with $\bar{c}$ independent of v, hence

$$|u^{\star\star} - u^{\star\star}_{I,h_i}|_\Omega \leqslant \bar{c}\|u^{\star\star}\|_{2,\Omega} h_i.$$

Let

$$M \geqslant \sup_{u \neq 0} \frac{a(u,u)}{|u|^2_\Omega}$$

and let M_1 be the Lipschitz constant of the functional

$$-\ell(\cdot) + j(\cdot) \quad \text{on} \quad \mathcal{V}.$$

Then, for arbitrary $r_1 > 0$,

$$\nu(r_1) = M(|u^{\star\star}|_\Omega + r_1) + M_1$$

is the Lipschitz constant of the functional (4.18) on $S_{r_1}(u^{\star\star})$. Choosing $c_7 \geqslant \bar{c}\|u^{\star\star}\|_{2,\Omega}$ and $r_0 = c_7 h_1$, we put

$$\varphi_i = c_7 h_i, \quad i = 1, 2, \ldots \tag{5.19}$$

and

$$\nu_0 = M\Big(|u^{\star\star}|_\Omega + r_0\Big) + M_1.$$

Analogously, let $c_8 \geqslant \bar{c}(\|u^{\star\star}\|_{2,\Omega} + c_6 r^\star)$ ($r^\star$ will be specified by (5.21) below) and

$$\bar{\varphi}_i = c_8 h_i, \quad i = 1, 2, \ldots \tag{5.20}$$

Using the results obtained in Section 3 (cf. Lemma 3.2 and Theorem 3.4) for Problem (4.18), (4.19), we get the following statement.

Theorem 5.3. *Assume that for the triangulation parameter the relation $h_i \geqslant h_{i+1}$ is valid for all i and that the control parameters $\varphi_i, \sigma_i, \varepsilon_i$ satisfy the conditions*

$$\Sigma\varphi_i^{1/2} < \infty, \ \Sigma\sigma_i^{1/2} < \infty, \ \Sigma\varepsilon_i < \infty.$$

Moreover, let $r^\star$ and r be chosen so that

$$r^\star > \max\Big\{8r_0, 2\sum_{i=1}^{\infty}\Big[(\nu_0\varphi_i + 2\sigma_i)^{1/2} + \frac{1}{2}\varepsilon_i + 2\varphi_i\Big]\Big\}, \tag{5.21}$$

$$r \geqslant r^\star + \bar{c}\Big(\|u^{\star\star}\|_{2,\Omega} + c_6 r^\star\Big)h_1$$

and let the parameters $\{\delta_i\}$ fulfil the inequalities

$$\frac{1}{4r}\Big[\nu(r)\bar{\varphi}_i + 2\sigma_i - \Big(\delta_i - \frac{1}{2}\varepsilon_i\Big)^2\Big] + \frac{1}{2}\varepsilon_i < 0, \quad i = 1, 2, \ldots$$

Then, starting with $u^{1,0} \in S_{r^\star/4}(u^{\star\star})$, the method with weak regularization (where the auxiliary problems have the form (5.17)) converges to a solution of Problem (4.18), (4.19) in the norm of $[H^1(\Omega)]^2$.

Using (5.19), (5.20) together with

$$\sigma_i = \sqrt{\kappa_i} \int\limits_{\Gamma_c} \mu |S_\nu| \, \mathrm{d}\Gamma,$$

one can reformulate this statement in terms of the original controlling parameters h_i, κ_i, ε_i and δ_i.

6. Concluding remarks

An implementation of the methods described requires to translate the non-constructive criterion (2.10) into a practicable stopping rule for the computation of $u^{i,s}$. This transformation depends essentially on the properties of the original problem and on the algorithm solving the auxiliary problems. For our model problems no difficulties occur: in the first case we deal with quadratic programming problems, for which finite numerical algorithms exist, and in the second case, we have to solve approximately unconstrained minimization problems with strongly convex, differentiable functions.

The choice of the discretization parameter according to Theorems 2.5 and 3.4 does not contradict the usual application of finite element methods in the case of well-posed elliptic variational inequalities or linear problems.

The use of two parameters r and $r^\star$ in the statements on convergence of the MSR-methods seems to be superfluous, because $r = r^\star$ is possible and less restrictive for the choice of the controlling parameters. Nevertheless, we have to pay attention to $r > r^\star$, taking into account the technique for estimating the values $\varrho(Q', Q_i)$. Upper bounds for $\varrho(\bar{u}, Q_i)$ with $\bar{u} \in U^\star$ are obtained usually by estimating the distance between $\bar{u}$ and its interpolant $\bar{u}_{I,h_i}$. But the norm of $\bar{u}_{I,h_i}$ in the space $\mathcal{Y}$ may be larger than the norm of $\bar{u}$. Thus, we cannot guarantee that the interpolant of an arbitrary function $u \in U^\star \cap S_{r^\star}$ belongs to $S_{r^\star}$. The choice of a suitable combination of $r^\star$ and r ensures that $u_{I,h_i} \in Q_i$ for $u \in U^\star \cap S_{r^\star}$ under the condition that $u_{I,h_i} \in K_i$.

Concerning the case $K_i \not\subset K$ the question on estimating $\varrho(K_i, K)$ was considered by Kaplan/Tichatschke [26] for the problem of a persistent fluid stream in a domain bounded by a half-permeable membrane.

The present paper is dedicated first of all to the theoretical analysis of iterative prox-regularization methods for solving variational inequalities with weakly coercive operators. Efficiency of the numerical treatment depends substantially on the adapted choice of the controlling parameters in the framework of the statements of convergence. In the special case of OSR-methods numerical experiments were tested by KUSTOVA [29] for a number of two-body contact problems.

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