PERIODIC SOLUTIONS OF A NONLINEAR EVOLUTION PROBLEM

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Abstract. In this paper we prove existence of periodic solutions to a nonlinear evolution system of second order partial differential equations involving the pseudo-Laplacian operator. To show the existence of periodic solutions we use Faedo-Galerkin method with a Schauder fixed point argument.

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1. Introduction

The motion of charged mesons in an electromagnetic field can be described by the following system of nonlinear Klein-Gordon equations:

$$\Box u + \alpha^2 u + a v^2 u = 0, \quad \Box v + \beta^2 v + b u^2 v = 0, \tag{$*$}$$

where $\Box = \partial^2/\partial t^2 - \Delta$, u and v are scalar fields of masses α and β, respectively, and a, b are interaction constants. This model was proposed by I. Segal [7].

In 1987, Medeiros and Miranda [6] considered the following generalization of $(*)$:

$$\begin{cases} \Box u + |v|^{\varrho+2}|u|^{\varrho}u = f \\ \Box v + |u|^{\varrho+2}|v|^{\varrho}v = g. \end{cases} \tag{$**$}$$

Here ϱ is a real number, $\varrho > -1$, and f, g are given functions. The authors [6] proved existence of solutions to $(**)$ with initial and homogeneous Dirichlet boundary conditions for any dimension n and uniqueness for $n \leqslant 3$.

More recently, in 1997, N. N. O. Castro [2] studied the system

$$(***) \qquad \begin{cases} u'' + Au - \Delta u' + |v|^{\varrho+2}|u|^{\varrho}u = f_1 \\ v'' + Av - \Delta v' + |u|^{\varrho+2}|v|^{\varrho}v = f_2 \end{cases}$$

where $z' = \partial z/\partial t$ and A is the pseudo-Laplacian operator given by

$$Aw = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial w}{\partial x_i}\right|^{p-2}\frac{\partial w}{\partial x_i}\right)$$

and $p > 2$.

This system can be viewed as a mathematical generalization of $(**)$. Castro [2] obtained results on the existence of global solutions and their decay rates as $t \to \infty$ with a given initial condition and considering homogeneous Dirichlet boundary conditions. For related problems see [4], [8]. The question of existence of periodic solutions to system $(***)$ is a significant one and to the best of our knowledge it has not yet been studied. So the purpose of this paper is to show that under some conditions the answer to this question is affirmative.

The plan of the paper is as follows: in Section 2 we fix some notation and give (without proofs) some results that will be used in the paper. References are given at the end. In Section 3 we state the theorem on existence of periodic solutions to system $(***)$ and in the remaining sections the proof is carried out.

2. Notation and auxiliary results

A. Let Ω be a bounded regular domain in $\mathbb{R}^n$. We will use the following notation for Sobolev spaces and its norms:

$\|\cdot\|_0$—norm in $W_0^{1,p}(\Omega)$;

$\|\cdot\|_{1,p} = \|\cdot\|_{W^{1,p}(\Omega)}$—denotes norm in $W^{1,p}(\Omega)$;

$\|\cdot\|$, $((\cdot))$—is the norm and inner product in $H_0^1(\Omega)$, respectively;

$|\cdot|$, $(,)$—is the norm and inner product in $L^2(\Omega)$, respectively.

The absolute value of a real number is also denoted by $|\cdot|$ and it will be clear from the context.

$\|\cdot\|_m$—denotes the Euclidean norm in $\mathbb{R}^m$,

V' means the topological dual space of the linear space V, and p' denotes the conjugate number of p, that is $1/p + 1/p' = 1$, $p > 1$.

$C_T^k(\mathbb{R})$ denotes the linear subspace of all periodic real functions with period T in $C^k(\mathbb{R})$, $T > 0$.

We represent by I_T any real interval of the form $[a, a+T]$, $a \in \mathbb{R}$. Integration over I_T will be denoted by $\int_T$.

Let X be a Banach space. Then $L^p(T, X)$ will be the linear space of all T-periodic functions $u\colon \mathbb{R} \to X$ such that $\|u(t)\|_X \in L^p(I_T)$, where $\|\cdot\|_X$ means the norm in X.

If $1 \leqslant p < \infty$, then $\|u\|_{L^p(T,X)} = \left(\int_T \|u(t)\|_X^p \,\mathrm{d}t\right)^{1/p}$ defines a norm in $L^p(T, X)$ in which it is a Banach space.

If $p = \infty$, $L^\infty(T, X)$ is a Banach space with respect to the norm defined by $\|u\|_{L^\infty(T,X)} = \operatorname{ess\,sup} \|u(t)\|_X$.

$W_{2m} = C_T^1(\mathbb{R}) \times \ldots \times C_T^1(\mathbb{R})$ denotes the Cartesian product of $2m$ copies of $C_T^1(\mathbb{R})$ and by $K(W_{2m})$ we mean the linear space of all compact operators from W_{2m} into W_{2m}.

B. Now we list, without proofs, a result and a Lemma that will be used in the sequel.

I. If $u, v \in W_0^{1,p}(\Omega)$ then

(i) $uv \in L^{\varrho+2}(\Omega)$;

(ii) $|u|^{\varrho+2}|v|^\varrho v$, $|v|^{\varrho+2}|u|^\varrho u \in L^\theta(\Omega)$ where θ is related to ϱ, p and n.

II. Lemma. Suppose that $s > n\left(\frac{1}{2} - \frac{1}{p}\right) + 1$, $p > 2$. Then $H_0^s(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$ with a continuous and dense injection.

The proof of I is straightforward and just makes use of some well known properties of L^p spaces and of algebraic calculus. The proof of the Lemma is a consequence of Sobolev embedding theorems and of a result of Browder & Bui An Ton [1].

3. Statement of the result

Now we consider system $(***)$ with periodic conditions (PS), that is, given functions f_1, f_2 periodic with period T we look for functions u, v that are solutions, in a weak sense, of the following problem:

$$(1) \qquad (\mathrm{PS}) \begin{cases} u'' + Au - \Delta u' + |v|^{\varrho+2}|u|^\varrho u = f_1 & \text{on } Q = \Omega \times (0,T), \\ v'' + Av - \Delta v' + |u|^{\varrho+2}|v|^\varrho v = f_2 & \text{on } Q = \Omega \times (0,T), \\ u(x,0) = u(x,T);\ \ u'(x,0) = u'(x,T) & \text{on } \Omega, \\ v(x,0) = v(x,T);\ \ v'(x,0) = v'(x,T) & \text{on } \Omega, \\ u = 0;\ \ v = 0 & \text{on } \Sigma = \Gamma \times (0,T) \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a regular bounded set and Γ is its boundary. We assume that $n > p$ and $0 \leqslant \varrho \leqslant 4(n-1+p)/\big(2(n-p-1)+np\big)$.

We will prove the following result:

Theorem. *Given* $f_i \in L^2(T, L^2(\Omega))$, $i=1,2$ *there exist* u *and* v *in* $L^\infty(T, W_0^{1,p}(\Omega))$ *and* u', v' *in* $L^2(T, H_0^1(\Omega))$ *such that*

$$\tag{2} \begin{aligned} -\int_T (u'(t), w)\varphi' \,\mathrm{d}t + \int_T \langle Au(t), w\rangle\varphi \,\mathrm{d}t + \int_T ((u'(t), w))\varphi \,\mathrm{d}t \\ + \int_T \langle |v(t)|^{\varrho+2}|u(t)|^{\varrho}u(t), w\rangle\varphi \,\mathrm{d}t = \int_T (f_1(t), w)\varphi \,\mathrm{d}t, \end{aligned}$$

$$\tag{3} \begin{aligned} -\int_T (v'(t), w)\varphi' \,\mathrm{d}t + \int_T \langle Av(t), w\rangle\varphi \,\mathrm{d}t + \int_T ((v'(t), w))\varphi \,\mathrm{d}t \\ + \int_T \langle |u(t)|^{\varrho+2}|v(t)|^{\varrho}v(t), w\rangle\varphi \,\mathrm{d}t = \int_T (f_2(t), w)\varphi \,\mathrm{d}t \\ \forall w \in W_0^{1,p}(\Omega),\ \ \forall \varphi \in C_T^1(\mathbb{R}). \end{aligned}$$

The theorem will be proved by constructing an "approximated system" in a finite-dimensional space with the Faedo-Galerkin method and then proving that this approximated problem has periodic solutions. The proof goes through several steps. The scheme is the following:

First step: We show that there exists a sequence of approximate solutions $((u_n)_n, (v_n)_n)$. We introduce a parameter μ, $0 \leqslant \mu \leqslant 1$, and consider an equivalent μ-parametrized system. Making use of Green's function properties and Leray-Schauder theory we prove the existence of periodic approximate solutions to the original system.

Second step: We look for and obtain estimates on the "approximate solutions" sufficient to pass to the limit.

Third step: The passage to the limit is carried out using compactness arguments and monotonicity properties of the operator A.

4. The approximated problem

So let $\{w_j\}_j$ be a spectral basis of $H_0^s(\Omega)$, $s > n\left(\frac{1}{2} - \frac{1}{p}\right) + 1$ which is an orthonormal complete system in $L^2(\Omega)$. Let $\mathrm{V}_m = [w_1, \ldots, w_m]$ be a finite-dimensional linear subspace of $H_0^s(\Omega)$ generated by the first m vectors $\{w_1, \ldots, w_m\}$ in $\{w_j\}_j$.

Then the approximated problem consists in finding $u_m(t)$, $v_m(t)$ in V_m, solution to the system:

$$
\text{(4)} \qquad \begin{cases} (u''_m(t), w_j) + \langle Au_m(t), w_j\rangle + (\!(u'_m(t), w_j)\!) \\ \qquad +\langle |v_m(t)|^{\varrho+2}|u_m(t)|^{\varrho}u_m(t), w_j\rangle = (f_1(t), w_j), \\ (v''_m(t), w_j) + \langle Av_m(t), w_j\rangle + (\!(v'_m(t), w_j)\!) \\ \qquad +\langle |u_m(t)|^{\varrho+2}|v_m(t)|^{\varrho}v_m(t), w_j\rangle = (f_2(t), w_j), \\ u_m(t) = u_m(t+T);\ \ u'_m(t) = u'_m(t+T), \\ v_m(t) = v_m(t+T);\ \ v'_m(t) = v'_m(t+T). \end{cases}
$$

We look for solutions u_m, v_m in the form

$$
\text{(5)} \qquad u_m(t) = \sum_{j=1}^{m} c_{jm}(t)w_j; \quad v_m(t) = \sum_{j=1}^{m} d_{jm}(t)w_j
$$

where $c_{jm}(t) = c_{jm}(t+T)$, $c'_{jm}(t) = c'_{jm}(t+T)$, $d_{jm}(t) = d_{jm}(t+T)$, and $d'_{jm}(t) = d'_{jm}(t+T)$.

We substitute $u_m(t)$, $v_m(t)$ into (3.1) to get the ordinary differential system

$$
\text{(6)} \qquad \begin{cases} Y''_m(t) + F(Y_m(t)) + H(Y_m(t), Y'_m(t)) = P(t), \\ Y_m(t) = Y_m(t+T);\ \ Y'_m(t) = Y'_m(t+T) \end{cases}
$$

where $Y_m(t) = (c_{1m}(t), \dots, c_{mm}(t), d_{1m}(t), \dots, d_{mm}(t))^*$,

$$
F(Y_m(t)) = (\langle Au_m(t), w_1\rangle, \dots, \langle Au_m(t), w_m\rangle, \langle Av_m(t), w_1\rangle, \dots, \langle Av_m(t), w_m\rangle)^*,
$$

$$
\begin{aligned} H(Y_m(t), Y'_m(t)) = (&\langle |v_m(t)|^{\varrho+2}|u_m(t)|^{\varrho}u_m(t), w_1\rangle, \dots, \langle |v_m(t)|^{\varrho+2}|u_m(t)|^{\varrho}u_m(t), w_m\rangle, \\ &\langle |u_m(t)|^{\varrho+2}|v_m(t)|^{\varrho}v_m(t), w_1\rangle, \dots, \langle |u_m(t)|^{\varrho+2}|v_m(t)|^{\varrho}v_m(t), w_m\rangle)^* \\ &+\big((\!(u'_m(t), w_1)\!), \dots, (\!(u'_m(t), w_m)\!), (\!(v'_m(t), w_1)\!), \dots, (\!(v'_m(t), w_m)\!)\big)^* \end{aligned}
$$

and

$$
P(t) = \big((f_1(t), w_1), \dots, (f_1(t), w_m), (f_2(t), w_1), \dots, (f_2(t), w_m)\big)^*
$$

and the symbol * denotes the transpose of a vector in $\mathbb{R}^{2m}$.

Now let β, δ be positive real numbers such that the homogeneous system

$$
\text{(7)} \qquad \begin{cases} Y''_m(t) + \delta Y'_m(t) + \beta Y_m(t) = 0 \\ Y_m(0) = Y_m(T);\ \ Y'_m(0) = Y'_m(T) \end{cases}
$$

has a unique solution $Y_m(t)$.

If we consider the Green function $G_m(t,s)$ associated with (7) (see [9]) then $\mathrm{Y}_m(t) = \int_0^T G_m(t,s)P(s)\,\mathrm{d}s$ will be the unique solution of the non-homogeneous system

$$\tag{8} \begin{cases} Y_m''(t) + \delta Y_m'(t) + \beta Y_m(t) = P(t) \\ Y_m(0) = Y_m(T);\;\; Y_m'(0) = Y_m'(T). \end{cases}$$

Of course we can extend the solution $\mathrm{Y}_m(t)$ of (8) by T-periodicity to the whole set of real numbers.

Next we consider the μ-parametrized system, $\mu \in [0,1]$,

$$\tag{9} \begin{gathered} Y_m''(t) + \delta Y_m'(t) + \beta Y_m(t) + \mu\big[F(Y_m(t)) + H\big(Y_m(t), (Y_m'(t))\big) \\ - \delta Y_m'(t) - \beta Y_m(t)\big] = P(t), \\ Y_m(t) = Y_m(t+T);\;\; Y_m'(t) = Y_m'(t+T). \end{gathered}$$

In a similar way it follows that

$$\tag{10} \begin{aligned} T_m(\mu)Y_m(t) = & \int_0^T \mu[-F(Y_m(s)) - H(Y_m(s), Y_m'(s))]G_m(t,s)\,\mathrm{d}s \\ & + \int_0^T \mu[\delta Y_m'(s) + \beta Y_m(s)]G_m(t,s)\,\mathrm{d}s \\ & + \int_0^T P(s)G_m(t,s)\,\mathrm{d}s \end{aligned}$$

is the solution of (9) that satisfies the periodic conditions

$$T_m(\mu)Y_m(0) = T_m(\mu)Y_m(T);\;\; (T_m(\mu)Y_m)'(0) = (T_m(\mu)Y_m)'(T).$$

Note that if $\mu = 0$ then $T_m(0)Y_m(t)$ is the unique solution of (8) and $\mathrm{T}_m(0)Y_m(t) = Y_m(t)$. So $\mathrm{T}_m(0)$ has a fixed point.

What we have in mind now is to show that the operator $T_m(1)$ has a fixed point in some Banach space so we could apply this result to demonstrate that system (6), which is equivalent to system (4), has a solution.

We will use the following well known results given in the four lemmas below.

Lemma 1. *Let $\mu \in [0,1]$, $Y_m \in W_{2m}$ and $T_m(\mu)Y_m$ be as given in* (10). *Then $T_m(\mu)Y_m \in W_{2m}$.*

Lemma 2. *The operator $T_m(\mu)\colon W_{2m} \to W_{2m}$, is a continuous operator.*

Lemma 3. *The continuous operator $T_m(\mu)\colon W_{2m} \to W_{2m}$ is a compact operator.*

Lemma 4. *Let $B = \{Y_m \in W_{2m}\,;\ \|Y_m\| \leqslant 1\}$. Then $\|T_m(\mu)Y_m - T_m(\lambda)Y_m\| \leqslant C|\mu-\lambda|\ \ \forall Y_m \in B$.*

The proofs of these lemmas are straightforward computations using Green function properties.

As a consequence of these lemmas the operator $T_m\colon\ [0,1] \to K(W_{2m})$, $\mu \mapsto T_m(\mu)$, where $T_m(\mu)$ is defined by (10) is a compact operator and it is known that the Leray-Schauder degree of $T_m(0)$ is equal to one and we claim that the Leray-Schauder degree of the application $T_m(1)$ is also equal to one. So we will also make use of the following result:

Theorem 5. *Let $T_m\colon\ [0,1] \to K(W_{2m})$ be a compact homotopy operator. Suppose that there exist $M>0$ independent of μ such that $\|Y_m\| \leqslant M$ whenever $(I-T_m(\mu))Y_m = 0$, for all $\mu \in [0,1]$. Let $B = \{Y \in W_{2m}\,;\ \|Y\| \leqslant rM,\ r>1\}$. Then $d(I-T_m(\mu),B,0)$ is well defined and has the same value for any $\mu \in [0,1]$. See* [3], [10] *for a proof and meaning of the terms involved.*

Suppose that there exists a function $Y_m \in W_{2m}$ that depends on μ, $\mu \in [0,1]$, such that $(I-T_m(\mu))Y_m = 0$. Of course Y_m is a μ-dependent solution to the system (9). Let us multiply $(9)_1$ by $Y_m'(t)$ to obtain

$$\begin{aligned}(11)\quad &\big(Y_m''(t),Y_m'(t)\big)+\delta\big(Y_m'(t),Y_m'(t)\big)+\beta\big(Y_m(t),Y_m'(t)\big)+\mu\big(F(Y_m(t),Y_m'(t)\big)\\ &\quad+\big(H(Y_m(t),Y_m'(t)),Y_m'(t)\big)-\mu\delta\big(Y_m'(t),Y_m'(t)\big)\\ &\quad-\mu\beta\big(Y_m(t),Y_m'(t)\big)=\big(P(t),Y_m'(t)\big).\end{aligned}$$

However, $\big(F(Y_m(t)),Y_m'(t)\big) = \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|v_m(t)\|_0^p + \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|_0^p$ and

$$\begin{aligned}\big(H(Y_m(t),Y_m'(t)),Y_m'(t)\big) &= 1/(\varrho+2)\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2}\\ &\quad+\|u_m'(t)\|^2+\|v_m'(t)\|^2.\end{aligned}$$

So (11) becomes

$$\begin{aligned}(12)\qquad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y_m'(t)\|_{2m}^2+\delta\|Y_m'(t)\|_{2m}^2+\frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y_m(t)\|_{2m}^2+\frac{\mu}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|_0^p\\ &\quad+\frac{\mu}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|v_m(t)\|_0^p+\frac{\mu}{\varrho+2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2}\\ &\quad+\mu\|u_m'(t)\|^2+\mu\|v_m'(t)\|^2\\ &\leqslant\mu\delta\|Y_m'(t)\|_{2m}^2+\frac{\mu}{\beta}\frac{\mathrm{d}}{\mathrm{d}t}\|Y_m(t)\|+\|P(t)\|_{2m}\|Y_m'(t)\|_{2m}.\end{aligned}$$

Integrating this last inequality over any I_T-interval and using the periodic condition we obtain

$$
\begin{aligned}
(13)\qquad \delta \int_T \|Y'_m(t)\|^2_{2m}\,\mathrm{d}t + \mu \int_T (\|u'_m(t)\|^2 + \|v'_m(t)\|^2)\,\mathrm{d}t \\
\leqslant \mu\delta \int_T \|Y'_m(t)\|^2_{2m}\,\mathrm{d}t + \int_T \|P(t)\|_{2m}\|Y'_m(t)\|_{2m}\,\mathrm{d}t.
\end{aligned}
$$

Taking into account that $\mathrm{H}^1_0(\Omega) \hookrightarrow L^2(\Omega)$ we obtain from (13) that

$$
\begin{aligned}
\delta \int_T \|Y'_m(t)\|^2_{2m}\,\mathrm{d}t + \frac{\mu}{\delta_0^2} \int_T (|u'_m(t)|^2 + |v'_m(t)|^2)\,\mathrm{d}t \\
\leqslant \mu\delta \int_T \|Y'_m(t)\|^2_{2m}\,\mathrm{d}t + \int_T \|P(t)\|_{2m}\|Y'_m(t)\|_{2m}\,\mathrm{d}t
\end{aligned}
$$

where $|\cdot| \leqslant \delta_0\|\cdot\|$.

So we have

$$
\begin{aligned}
\delta \int_T \|Y'_m(t)\|^2_{2m}\,\mathrm{d}t + \frac{\mu}{\delta_0^2} \int_T (|u'_m(t)|^2 + |v'_m(t)|^2)\,\mathrm{d}t \\
\leqslant \mu\delta \int_T \|Y'_m(t)\|^2_{2m}\,\mathrm{d}t + \int_T \|P(t)\|_{2m}\|Y'_m(t)\|_{2m}\,\mathrm{d}t.
\end{aligned}
$$

Choosing δ such that $0 < \delta < \frac{1}{\delta_0^2}$, it follows from the last inequality that

$$
\delta \int_T \|Y'_m(t)\|^2_{2m}\,\mathrm{d}t \leqslant \int_T \|P(t)\|_{2m}\|Y'_m(t)\|_{2m}\,\mathrm{d}t
$$

and so

$$
(14)\qquad \delta^2 \int_T \|Y'_m(t)\|^2_{2m}\,\mathrm{d}t \leqslant C
$$

where C is a constant that does not depend on $\mu \in [0,1]$.

Now let us return to the system (9). We take $\mu = 1$ and then multiply by $Y_m(t)$ to obtain

$$
\begin{aligned}
(15)\quad \frac{\mathrm{d}}{\mathrm{d}t}\bigl(Y'_m(t), Y_m(t)\bigr) - \|Y'_m(t)\|^2_{2m} + \|u_m(t)\|^p_0 + \|v_m(t)\|^p_0 \\
+ 2\int_\Omega |v_m(t)u_m(t)|^{\varrho+2}\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_m(t)\|^2 \\
= \bigl(P(t), Y_m(t)\bigr).
\end{aligned}
$$

Integrating over an I_T-interval and using periodic conditions we obtain from (15)

$$
\begin{aligned}
-\int_T \|Y'_m(t)\|^2_{2m}\,\mathrm{d}t + \int_T (\|u_m(t)\|^p_0 + \|v_m(t)\|^p_0)\,\mathrm{d}t + 2\int_T \|u_m(t)v_m(t)\|^{\varrho+2}_{L^{\varrho+2}(\Omega)} \\
= \int_T (P(t), Y_m(t))\,\mathrm{d}t.
\end{aligned}
$$

But $(P(t), Y_m(t)) = (f_1(t), u_m(t)) + (f_2(t), v_m(t))$ and as $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ it follows that

$$\int_T \|Y_m(t)\|_{2m}^2 \,\mathrm{d}t \leqslant C \tag{16}$$

where once again C is a positive constant that does not depend on $\mu \in [0,1]$.

Now let us take $s, t \in I_T$, $s < t$, and integrate (12) from s to t, to obtain

$$\begin{aligned} &\frac{1}{2}\|Y_m'(t)\|_{2m}^2 + \frac{\mu}{p}(\|u_m(t)\|_0^p + \|v_m(t)\|_0^p) + \frac{\mu}{\varrho+2}\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} \\ &\quad\leqslant \frac{1}{2}\|Y_m'(s)\|_{2m}^2 + \frac{\beta}{2}\|Y_m(s)\|_{2m}^2 - \frac{\mu\beta}{2}\|Y_m(s)\|_{2m}^2 + \frac{\mu}{p}(\|u_m(s)\|_0^p \\ &\qquad + \|v_m(s)\|_0^p) + \int_s^t \|P(\sigma)\|_{2m}\|Y_m'(\sigma)\|_{2m}\,\mathrm{d}\sigma \\ &\qquad + \frac{\mu}{\varrho+2}\|u_m(s)v_m(s)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2}. \end{aligned} \tag{17}$$

From (17) it follows that

$$\begin{aligned} &\frac{1}{2}\|Y_m'(t)\|_{2m}^2 + \frac{\mu}{p}(\|u_m(t)\|_0^p + \|v_m(t)\|_0^p) + \frac{\mu}{\varrho+2}\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} \\ &\quad\leqslant \frac{1}{2}\|Y_m'(s)\|_{2m}^2 + \frac{1}{p}(\|u_m(s)\|_0^p + \|v_m(s)\|_0^p) \\ &\qquad + \frac{1}{\varrho+2}\|u_m(s)v_m(s)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} + \beta\|Y_m(s)\|_{2m}^2 + C \end{aligned} \tag{18}$$

where C is a positive constant independent of μ.

If we integrate (18) with respect to s on I_T we have

$$\begin{aligned} &\frac{1}{2}\|Y_m'(t)\|_{2m}^2 + \frac{\mu}{p}(\|u_m(t)\|_0^p + \|v_m(t)\|_0^p)\frac{\mu}{\varrho+2}\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} \\ &\quad\leqslant \frac{1}{T}\left\{\frac{1}{2}\int_T \|Y_m'(s)\|_{2m}^2\,\mathrm{d}s + \frac{1}{p}\int_T(\|u_m(s)\|_0^p + \|v_m(s)\|_0^p)\,\mathrm{d}s\right\} \\ &\qquad + \frac{1}{T}\left\{\frac{1}{\varrho+2}\int_T \|u_m(s)v_m(s)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2}\,\mathrm{d}s\right\} \\ &\qquad + \frac{\beta}{T}\int_T \|Y_m(s)\|_{2m}^2\,\mathrm{d}s + C. \end{aligned} \tag{19}$$

Inserting (14) and (16) into (19) we get

$$\frac{1}{2}\|Y_m'(t)\|_{2m}^2 + \frac{\mu}{p}(\|u_m(t)\|_0^p + \|v_m(t)\|_0^p) + \frac{\mu}{\varrho+2}\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} \leqslant C \tag{20}$$

where C is again a positive constant that does not depend on μ. Hence (20) implies that

$$\begin{cases} \|Y_m'(t)\|_{2m} \leqslant C \\ \|u_m(t)\|_0, \ \ \|v_m(t)\|_0 \leqslant C. \end{cases}$$

Since $\|Y_m(t)\|_{2m}^2 = |u_m(t)|^2 + |v_m(t)|^2$ and $\mathrm{W}_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ we have

$$\|Y_m(t)\|_{2m} \leqslant C \tag{21}$$

where C does not depend on μ.

Now consider the set $\omega_{2m} = \{Y_m \in W_{2m}\, ;\ \|Y_m\| \leqslant rC,\ r > 1\}$. Then $\mathrm{d}(I - T_m(\mu), \omega_{2m}, 0)$ does exist and has the same value for all $\mu \in [0,1]$. But we know that $\mathrm{d}(I - T_m(0)) = 1$ and this means that the system (4) has a solution as we have claimed.

5. Estimates of the approximated solutions

The next step is to obtain estimates of the approximate solution sufficient to pass to the limit.

Estimate I.

We replace w_j by $u_m'(t)$ and w_j by $v_m'(t)$ in the equations $(4)_1$ and $(4)_2$ respectively so that we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_m'(t)|^2 &+ \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|_0^p + \|u_m'(t)\|^2 \\ &+ \frac{1}{\varrho+2}\int_\Omega |v_m(t)|^{\varrho+2}\frac{\mathrm{d}}{\mathrm{d}t}|u_m(t)|^{\varrho+2}\,\mathrm{d}x = (f_1(t), u_m'(t))\end{aligned} \tag{22}$$

and

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v_m'(t)|^2 &+ \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|v_m(t)\|_0^p + \|v_m'(t)\|^2 \\ &+ \frac{1}{\varrho+2}\int_\Omega |u_m(t)|^{\varrho+2}\frac{\mathrm{d}}{\mathrm{d}t}|v_m(t)|^{\varrho+2}\,\mathrm{d}x = (f_2(t), v_m'(t)).\end{aligned} \tag{23}$$

Integrating equations (23) on I_T, taking into account the periodic conditions and Gronwall's inequality we obtain by direct calculation that

$$\int_T (\|u_m'(t)\|^2 + \|v_m'(t)\|^2)\,\mathrm{d}t \leqslant C \tag{24}$$

and as before the constant C does not depend on μ.

From the last inequality we deduce that

$$\left\{\begin{array}{l}(u'_m)_m \ \text{ is bounded in } \ L^2(T, H_0^1(\Omega))\\ (v'_m)_m \ \text{ is bounded in } \ L^2(T, H_0^1(\Omega)).\end{array}\right. \tag{25}$$

Estimate II.

Setting $w_j = u_m(t)$ and $w_j = v_m(t)$ in equations $(4)_1$ and $(4)_2$, respectively, we have

$$\begin{aligned}&(u''_m(t), u_m(t)) + \|u_m(t)\|_0^p + (\!(u'_m(t), u_m(t))\!) + \|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2}\\ &\qquad = (f_1(t), u_m(t)),\\ &(v''_m(t), v_m(t)) + \|v_m(t)\|_0^p + (\!(v'_m(t), v_m(t))\!) + \|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2}\\ &\qquad = (f_2(t), v_m(t)).\end{aligned}$$

From the above we obtain

$$\left\{\begin{array}{l}\dfrac{\mathrm{d}}{\mathrm{d}t}(u'_m(t), u_m(t)) - |u'_m(t)|^2 + \|u_m(t)\|_0^p + \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|^2\\ \qquad +\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} = (f_1(t), u_m(t)),\\ \dfrac{\mathrm{d}}{\mathrm{d}t}(v'_m(t), v_m(t)) - |v'_m(t)|^2 + \|v_m(t)\|_0^p + \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|v_m(t)\|^2\\ \qquad +\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} = (f_2(t), v_m(t)).\end{array}\right. \tag{26}$$

From the first equation in (26), integrating and taking into account the periodic conditions we conclude that

$$\begin{aligned}&\int_T \|u_m(t)\|_0^p \,\mathrm{d}t + \int_T \|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} \,\mathrm{d}t\\ &\qquad\qquad \leqslant \int_T |u'_m(t)|^2 \,\mathrm{d}t + C\int_T |f_1(t)|\,\|u_m(t)\|_0 \,\mathrm{d}t,\end{aligned}$$

which in turn yields

$$\begin{aligned}&\int_T \|u_m(t)\|_0^p \,\mathrm{d}t + \int_T \|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} \,\mathrm{d}t\\ &\qquad\qquad \leqslant C\int_T \|u'_m(t)\|^2 \,\mathrm{d}t + C\int_T |f_1(t)|^{p'} \,\mathrm{d}t\\ &\qquad\qquad\quad + \frac{1}{p}\int_T \|u_m(t)\|_0^p \,\mathrm{d}t.\end{aligned}$$

Recalling that $L^2(\Omega) \hookrightarrow L^{p'}(\Omega)$ with continuous injection and $f_1 \in L^2(T, L^2(\Omega))$ we get

$$\int_T \|u_m(t)\|_0^p \, \mathrm{d}t + p' \int_T \|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} \, \mathrm{d}t \leqslant C \tag{27}$$

where C is independent of μ.

In the same way we obtain from the second equation in (26) that

$$\int_T \|v_m(t)\|_0^p \, \mathrm{d}t + p' \int_T \|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} \, \mathrm{d}t \leqslant C \tag{28}$$

where C does not depend on μ.

From (27) and (28) we conclude that

$$(u_m)_m, (v_m)_m \text{ are bounded sequences in } L^p(T, W_0^{1,p}(\Omega)), \tag{29}$$

$$(u_m v_m)_m \text{ is a bounded sequence in } L^{\varrho+2}(T, L^{\varrho+2}(\Omega)). \tag{30}$$

Estimate III.

Now summing the equations in (22) and (23) we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}|u_m'(t)|^2 + |v_m'(t)|^2\right\} + \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{p}\|u_m(t)\|_0^p + \frac{1}{p}\|v_m(t)\|_0^p\right\} \\
&\quad + \frac{1}{\varrho+2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} + \|u_m'(t)\|^2 + \|v_m'(t)\|^2 \\
&= (f_1(t), u_m'(t)) + (f_2(t), v_m'(t)) \\
&\leqslant \frac{C^2}{2}|f_1(t)|^2 + \frac{1}{2}\|u_m'(t)\|^2 + \frac{C^2}{2}|f_2(t)|^2 + \frac{1}{2}\|v_m'(t)\|^2.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}|u_m'(t)|^2 + \frac{1}{2}|v_m'(t)|^2\right\} + \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{p}\|u_m(t)\|_0^p + \frac{1}{p}\|v_m(t)\|_0^p\right\} \\
&\qquad + \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\varrho+2}\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2} + \frac{1}{2}(\|u_m'(t)\|^2 + \|v_m'(t)\|^2) \\
&\qquad \leqslant \frac{C^2}{2}|f_1(t)|^2 + \frac{C^2}{2}|f_2(t)|^2.
\end{aligned}$$

Let $s,t\in I_T$, $s<t$. Integrate from s to t to obtain

$$
\begin{aligned}
(31)\qquad &\frac{1}{2}|u_m'(t)|^2+\frac{1}{2}|v_m'(t)|^2+\frac{1}{p}\|u_m(t)\|_0^p+\frac{1}{p}\|v_m(t)\|_0^p\\
&\quad+\frac{1}{\varrho+2}\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2}+\frac{1}{2}\int_s^t(\|u_m'(\sigma)\|^2+\|v_m'(\sigma)\|^2)\,\mathrm{d}\sigma\\
&\leqslant\frac{C^2}{2}\int_s^t(|f_1(\sigma)|^2+|f_2(\sigma)|^2)\,\mathrm{d}\sigma+\frac{1}{2}|u_m'(s)|^2+\frac{1}{2}|v_m'(s)|^2\\
&\quad+\frac{1}{p}\|u_m(s)\|_0^p+\frac{1}{p}\|v_m(s)\|_0^p+\frac{1}{\varrho+2}\|u_m(s)v_m(s)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2}\\
&\leqslant\frac{1}{2}|u_m'(s)|^2+\frac{1}{2}|v_m'(s)|^2+\frac{1}{p}\|u_m(s)\|_0^p+\frac{1}{p}\|v_m(s)\|_0^p\\
&\quad+\frac{1}{\varrho+2}\|u_m(s)v_m(s)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2}\leqslant C.
\end{aligned}
$$

Taking into account estimates I–II and integrating with respect to s the last inequality changes to

$$
\begin{aligned}
(32)\qquad &\frac{1}{2}|u_m'(t)|^2+\frac{1}{2}|v_m'(t)|^2+\frac{1}{p}\|u_m(t)\|_0^p+\frac{1}{p}\|v_m(t)\|_0^p\\
&\quad+\frac{1}{\varrho+2}\|u_m(t)v_m(t)\|_{L^{\varrho+2}(\Omega)}^{\varrho+2}\leqslant C
\end{aligned}
$$

where as before the constant C does not depend on μ.

From (32) we have

$$
\begin{aligned}
(33)\qquad &(u_m),(v_m)\ \text{ are bounded sequences in }\ L^\infty(T,W_0^{1,p}(\Omega)),\\
&(u_m'),(v_m')\ \text{ are bounded sequences in }\ L^\infty(T,L^2(\Omega)),\\
&(u_mv_m)\ \text{ is a bounded sequence in }\ L^\infty(T,L^{\varrho+2}(\Omega)).
\end{aligned}
$$

Moreover, A takes bounded sets in $\mathrm{L}^\infty(T,W_0^{1,p}(\Omega))$ into bounded sets in $L^\infty(T, W^{-1,p'}(\Omega))$. So we have

$$
(34)\qquad (Au_m)_m,(Av_m)_m\ \text{ are bounded sequences in }\ L^\infty(T,W^{-1,p'}(\Omega)),
$$

$$
(35)\qquad \begin{cases}(|u_m|^{\varrho+2}|v_m|^{\varrho}v_m)_m\ \text{ is a bounded sequence in }\ L^\infty(T,L^\theta(\Omega))\\ (|v_m|^{\varrho+2}|u_m|^{\varrho}u_m)_m\ \text{ is a bounded sequence in }\ L^\infty(T,L^\theta(\Omega)).\end{cases}
$$

This result is a consequence of the fact that $\mathrm{W}_0^{1,p}(\Omega)\hookrightarrow L^\gamma(\Omega)$ where $\frac{1}{\gamma}+\frac{1}{\theta}=1$.

Estimate IV.

Now we make use of the projection method to obtain an estimate for $(u_m'')_m$ and $(v_m'')_m$. This follows in a standard and straightforward way. See [2] or [5]. One obtains that

$$(36)\qquad (u''_m)_m \text{ and } (v''_m)_m \text{ are bounded sequences in } L^2(T, H^{-s}(\Omega)).$$

6. Passage to the limit

From estimates I–IV one can extract subsequences $(u_\nu)_\nu$ and $(v_\nu)_\nu$ of $(u_m)_m$ and $(v_m)_m$, respectively, such that

$$(37)\qquad \begin{aligned} u_\nu \to u,\ \ v_\nu \to v &\text{ weak star in } L^\infty(T, W_0^{1,p}(\Omega)),\\ u'_\nu \to u',\ \ v'_\nu \to v &\text{ weak star in } L^\infty(T, L^2(\Omega)),\\ u'_\nu \to u',\ \ v'_\nu \to v' &\text{ weak } \quad\text{ in } L^2(T, H_0^1(\Omega)),\\ u_\nu v_\nu \to \zeta &\text{ weak star in } L^\infty(T, L^{\varrho+2}(\Omega)),\\ Au_\nu \to \chi,\ \ Av_\nu \to \eta &\text{ weak star in } L^\infty(T, W_0^{-1,p'}),\\ |u_\nu|^{\varrho+2}|v_\nu|^\varrho v_\nu \to \vartheta,\ \ |v_\nu|^{\varrho+2}|u_\nu|^\varrho u_\nu \to \lambda &\text{ weak star in } L^\infty(T, L^\theta(\Omega)). \end{aligned}$$

By the well known Lions-Aubin compactness theorem [5] there exists another subsequence, still denoted by $(u_\nu)_\nu$, such that

$$(38)\qquad \begin{aligned} &u_\nu \to u, \quad v_\nu \to v \ \text{ strong in } L^2(I_T \times \Omega),\\ &u_\nu \to u, \quad v_\nu \to v \ \text{ a.e. in } \ I_T \times \Omega,\\ &u'_\nu \to u', \ \ v'_\nu \to v' \ \text{ strong in } L^2(I_T \times \Omega). \end{aligned}$$

We easily deduce from (35) that

$$(39)\qquad \begin{aligned} &\|\,|u_\nu|^{\varrho+2}|v_\nu|^\varrho v_\nu\|_{L^\theta(I_T\times\Omega)} \leqslant C, \ \ \forall \nu \in N,\\ &\|\,|v_\nu|^{\varrho+2}|u_\nu|^\varrho u_\nu\|_{L^\theta(I_T\times\Omega)} \leqslant C, \ \ \forall \nu \in N. \end{aligned}$$

By (37), (38), (39) and a Lemma of Lions (see [5], Lemma 1.3) we get that

$$(40)\qquad \begin{cases} |u_\nu|^{\varrho+2}|v_\nu|^\varrho v_\nu \to |u|^{\varrho+2}|v|^\varrho v & \text{weak in } L^\theta(I_T \times \Omega)\\ |v_\nu|^{\varrho+2}|u_\nu|^\varrho u_\nu \to |v|^{\varrho+2}|u|^\varrho u & \text{weak in } L^\theta(I_T \times \Omega). \end{cases}$$

We are now in a position to pass to the limit as $\nu \to \infty$. So let us consider the equation $(4)_1$ of the approximated problem in the form

$$(41)\qquad \begin{aligned} (u''_\nu(t), w)\langle Au_\nu(t), w\rangle + ((u'_\nu(t), w)) + \langle |v_\nu(t)|^{\varrho+2}|u_\nu(t)|^\varrho u_\nu(t), w\rangle\\ = (f_1(t), w), \ \ \nu > m, \ \ w \in V_m. \end{aligned}$$

Multiplying (41) by $\varphi \in C_T^1(\mathbb{R})$, integrating on I_T and passing to the limit as $\nu \to \infty$, $m < \nu$, we deduce from convergences in (37)–(40) that

$$
\begin{aligned}
-\int_T (u'(t), w)\varphi' \, \mathrm{d}t + \int_T \langle \chi(t), w\rangle \varphi \, \mathrm{d}t + \int_T ((u'(t), w))\varphi \, \mathrm{d}t& \\
+ \int_T \langle |v(t)|^{\varrho+2} |u(t)|^{\varrho} u(t), w\rangle \varphi \, \mathrm{d}t& \\
= \int_T (f_1(t), w)\varphi \, \mathrm{d}t, \ \ \forall w \in V_m, \ \ \forall \varphi \in C_T^1(\mathbb{R}).&
\end{aligned}
$$

By density arguments it follows from this last equation that

$$
\begin{aligned}
-\int_T (u'(t), w)\varphi' \, \mathrm{d}t + \int_T \langle \chi(t), w\rangle \varphi \, \mathrm{d}t + \int_T ((u'(t), w))\varphi \, \mathrm{d}t& \\
+ \int_T \langle |v(t)|^{\varrho+2} |u(t)|^{\varrho} u(t), w\rangle \varphi \, \mathrm{d}t& \\
= \int_T (f_1(t), w)\varphi \, \mathrm{d}t, \ \ \forall w \in W_0^{1,p}(\Omega), \ \ \forall \varphi \in C_T^1(\mathbb{R}).&
\end{aligned} \tag{42}
$$

In a similar way we have

$$
\begin{aligned}
-\int_T (v'(t), w)\varphi' \, \mathrm{d}t + \int_T \langle \eta(t), w\rangle \varphi \, \mathrm{d}t + \int_T ((u'(t), w))\varphi \, \mathrm{d}t& \\
+ \int_T \langle |u(t)|^{\varrho+2} |v(t)|^{\varrho} v(t), w\rangle \varphi \, \mathrm{d}t& \\
= \int_T (f_1(t), w)\varphi \, \mathrm{d}t, \ \ \forall w \in W_0^{1,p}(\Omega), \ \ \forall \varphi \in C_T^1(\mathbb{R}).&
\end{aligned} \tag{43}
$$

It remains to prove that u, v satisfy the periodic conditions and that $\chi(t) = Au(t)$, $\eta(t) = Av(t)$.

That u, v (u', v') are periodic, with period T, is a consequence of the fact that the "approximate solution" u_m, v_m $(u_m'$, $v_m')$ is periodic with period T and of the convergences in (5.1).

Now recalling that A is a hemicontinuous and monotone operator and taking into account the estimates (36) for $(u_m'')_m$, $(v_m'')_m$ it is not difficult to show that $\chi(t) = Au(t)$, $\eta(t) = Av(t)$. This is carried out in a standard way so we omit the details (see [2], [5]). The theorem is proved.

Remark. We have assumed that $n > p$ and

$$0 \leqslant \varrho \leqslant \frac{4(1-n-p)}{2(n-p-1)+np}.$$

In the case $n \leqslant p$ it is sufficient to take $\varrho \geqslant 0$. The proof is similar.

References

[1] *F. E. Browder, Bui An Ton*: Nonlinear functional equations in Banach spaces and elliptic super regularization. Math. Z. *105* (1968), 177–195.

[2] *N. N. Castro*: Existence and asymptotic behaviour of solutions of a nonlinear evolution problem. Appl. Math. *42* (1997), 411–420.

[3] *J. Cronin*: Fixed Point and Topological Degree in Nonlinear Analysis. Mathematical Surveys, No. 11. Amer. Math. Soc., Providence, 1964.

[4] *T. Kakita*: On the existence of time periodic solutions of some nonlinear evolution equations. Appl. Anal. *4* (1974), 63–76.

[5] *J. L. Lions*: Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires. Dunod, Paris, 1969.

[6] *L. A. Medeiros, M. M. Miranda*: Weak solutions of nonlinear Klein-Gordon equations. Ann. Math. Pura Appl. IV, ser. 146 (1987), 173–183.

[7] *I. Segal*: Nonlinear partial differential equations in quantum field theory. Proc. Sympos. Appl. Math. Amer. Math. Soc. *17* (1965), 210–226.

[8] *M. Tsutsumi*: Some nonlinear evolution equations of second order. Proc. Japan. Acad. Ser. A Math. Sci. *47* (1971), 950–955.

[9] *K. Yosida*: Equations Différentielles et Intégrales. Dunod, Paris, 1971.

[10] *E. Zeidler*: Nonlinear Functional Analysis and its Applications: I—Fixed-Point Theorems. Springer-Verlag, New York, 1986.

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