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Applications of Mathematics, Vol. 49 (2004), No. 2, 123–140

Persistent URL: http://dml.cz/dmlcz/134563

A SEQUENCE OF MAPPINGS ASSOCIATED WITH THE HERMITE-HADAMARD INEQUALITIES AND APPLICATIONS

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(Received January 1, 2002, in revised version May 16, 2003)

Abstract. New properties for some sequences of functions defined by multiple integrals associated with the Hermite-Hadamard integral inequality for convex functions and some applications are given.

Keywords: Hermite-Hadamard inequality

MSC 2000: 26D15, 26D10, 26D99

1. INTRODUCTION

The integral inequality

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \leqslant \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f\colon [a,b] \to \mathbb{R}$, is well known in literature as the Hermite-Hadamard inequality.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the theory of special means and in information theory for divergence measures, from which we would like to refer the reader to [1]–[54].

The main aim of this paper is to consider some natural sequences of functions defined by multiple integrals and study their properties in relation to the Hermite-Hadamard inequality.

2. Properties of the sequence of mappings H_n

Let $I \subseteq \mathbb{R} \to \mathbb{R}$ be an interval of real numbers and $a, b \in I$ with $a < b$, and let $f\colon I \to \mathbb{R}$ be a mapping which is integrable on $[a, b]$. Then we can define a sequence of mappings $H_n\colon [0, 1] \to \mathbb{R}$ by

$$H_n(t) := \frac{1}{(b-a)^n} \int_a^b \dots \int_a^b f\Big(t\frac{x_1 + \dots + x_n}{n} + (1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1 \dots \mathrm{d}x_n$$

for $n \geqslant 1$ and $t \in [0, 1]$.

Some properties of this sequence of mappings are embodied in the following theorem.

Theorem 1. *Let $f\colon I \subseteq \mathbb{R}$ be a convex mapping on I and let $a, b \in I$ with $a < b$. Then*

(i) *H_n are convex on $[0, 1]$ for all $n \geqslant 1$;*

(ii) *the following inequalities hold:*

$$(2.1)\qquad f\Big(\frac{a+b}{2}\Big) \leqslant H_n(t) \leqslant \frac{1}{(b-a)^{n+1}} \times \int_a^b \dots \int_a^b f\Big(t\frac{x_1 + \dots + x_n}{n} + (1-t)x_{n+1}\Big)\,\mathrm{d}x_1 \dots \mathrm{d}x_{n+1}$$

and

$$(2.2)\qquad \begin{aligned} H_n(t) &\leqslant t\frac{1}{(b-a)^n} \int_a^b \dots \int_a^b f\Big(\frac{x_1 + \dots + x_n}{n}\Big)\,\mathrm{d}x_1 \dots \mathrm{d}x_n + (1-t)f\Big(\frac{a+b}{2}\Big) \\ &\leqslant \frac{1}{(b-a)^n} \int_a^b \dots \int_a^b f\Big(\frac{x_1 + \dots + x_n}{n}\Big)\,\mathrm{d}x_1 \dots \mathrm{d}x_n \end{aligned}$$

for all $t \in [0, 1]$;

(iii) *the mapping H_n is monotonic nondecreasing on $[0, 1]$ for all $n \geqslant 1$ and one has the bounds*

$$(2.3)\qquad \min_{t\in[0,1]} H_n(t) = f\Big(\frac{a+b}{2}\Big) = H_n(0) \quad \text{for all } n \geqslant 1$$

and

$$(2.4)\qquad \begin{aligned} \max_{t\in[0,1]} H_n(t) &= \frac{1}{(b-a)^n} \int_a^b \dots \int_a^b f\Big(\frac{x_1 + \dots + x_n}{n}\Big)\,\mathrm{d}x_1 \dots \mathrm{d}x_n \\ &= H_n(1) \quad \text{for } n \geqslant 1. \end{aligned}$$

Proof. (i) Follows by the convexity of f.

(ii) Applying Jensen's integral inequality, we obtain

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)x_{n+1}\Big)\,\mathrm{d}x_{n+1}\\
&\quad\geqslant f\left[\frac{1}{b-a}\int_a^b \Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)x_{n+1}\Big)\,\mathrm{d}x_{n+1}\right]\\
&\quad= f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)
\end{aligned}$$

for all $x_1,\ldots,x_n\in[a,b]$ and $t\in[0,1]$.

Taking an integral mean over $[a,b]^n$ we deduce the second inequality in (2.1).

By Jensen's integral inequality for multiple integrals we have

$$\begin{aligned}
&\frac{1}{(b-a)^n}\int_a^b\!\!\ldots\!\int_a^b f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\\
&\quad\geqslant f\left[\frac{1}{(b-a)^n}\int_a^b\!\!\ldots\!\int_a^b \Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\right]\\
&\quad= f\Big(\frac{a+b}{2}\Big),
\end{aligned}$$

and the inequality (2.1) is completely proved.

By the convexity of f on $[a,b]$, we can write

$$f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\leqslant tf\Big(\frac{x_1+\ldots+x_n}{n}\Big)+(1-t)f\Big(\frac{a+b}{2}\Big)$$

for all $x_1,\ldots,x_n\in[a,b]$ and $t\in[0,1]$. Taking an integral mean over $[a,b]^n$, we deduce

$$H_n(t)\leqslant t\frac{1}{(b-a)^n}\int_a^b\!\!\ldots\!\int_a^b f\Big(\frac{x_1+\ldots+x_n}{n}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n+(1-t)f\Big(\frac{a+b}{2}\Big),$$

and the first inequality in (2.2) is proved.

As we know (see for example [26]) that

$$f\Big(\frac{a+b}{2}\Big)\leqslant\frac{1}{(b-a)^n}\int_a^b\!\!\ldots\!\int_a^b f\Big(\frac{x_1+\ldots+x_n}{n}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n,$$

we obtain the last part of (2.2).

(iii) Let $0<t_1<t_2\leqslant 1$. By the convexity of H_n, which follows by (i) now proved, we have that

$$\frac{H_n(t_2)-H_n(t_1)}{t_2-t_1}\geqslant\frac{H_n(t_1)-H_n(0)}{t_1},$$

but $H_n(t_1) \geqslant H_n(0)$ (see the first inequality in (2.1)) and hence we get that $H_n(t_2) - H_n(t_1) \geqslant 0$ for all $0 \leqslant t_1 < t_2 \leqslant 1$, which shows that the mapping $H_n(\cdot)$ is monotonic nondecreasing on $[0,1]$. The bounds (2.3) and (2.4) follow by the inequalities (2.1) and (2.2). We omit the details. □

We now give another result on monotonicity which, in a sense, completes the above theorem.

Theorem 2. *Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping on I and let $a, b \in I$ with $a < b$. Then*

$$f\Big(\frac{a+b}{2}\Big) \leqslant H_{n+1}(t) \leqslant H_n(t) \leqslant \ldots \leqslant H_1(t) = H(t), \tag{2.5}$$

where

$$H(t) := \frac{1}{b-a} \int_a^b f\Big(tx + (1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x$$

for all $n \geqslant 1$ and $t \in [0,1]$. That is, the sequence of mappings $(H_n)_{n \geqslant 1}$ is monotonically nonincreasing.

Proof. Let us define real numbers belonging to $[a,b]$:

$$\begin{aligned}
y_1 &:= t\frac{x_1 + \ldots + x_n}{n} + (1-t)\frac{a+b}{2};\\
y_2 &:= t\frac{x_2 + x_3 + \ldots + x_{n+1}}{n} + (1-t)\frac{a+b}{2};\\
&\ \vdots\\
y_{n+1} &:= t\frac{x_{n+1} + x_1 + \ldots + x_{n-1}}{n} + (1-t)\frac{a+b}{2},
\end{aligned}$$

where $x_1, \ldots, x_{n+1} \in [a,b]$.

Using Jensen's discrete inequality, we may state that

$$\begin{aligned}
&\frac{1}{n+1}\Big[f\Big(t\frac{x_1 + \ldots + x_n}{n} + (1-t)\frac{a+b}{2}\Big)\\
&\qquad + f\Big(t\frac{x_2 + \ldots + x_{n+1}}{n} + (1-t)\frac{a+b}{2}\Big) + \ldots\\
&\qquad + f\Big(t\frac{x_{n+1} + x_1 + \ldots + x_{n-1}}{n} + (1-t)\frac{a+b}{2}\Big)\Big]\\
&\geqslant f\Big(t\frac{x_1 + \ldots + x_{n+1}}{n+1} + (1-t)\frac{a+b}{2}\Big)
\end{aligned}$$

for all $t \in [0,1]$ and $x_1, \ldots, x_{n+1} \in [a,b]$.

Taking an integral mean over $[a,b]^{n+1}$, we deduce

$$\frac{1}{n+1}\left[\frac{1}{(b-a)^{n+1}}\int_a^b\cdots\int_a^b f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_{n+1}\right.$$
$$+\ldots+\frac{1}{(b-a)^{n+1}}\int_a^b\cdots\int_a^b f\Big(t\frac{x_{n+1}+x_1+\ldots+x_{n-1}}{n}$$
$$\left.+(1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_{n+1}\right]$$
$$\geqslant\frac{1}{(b-a)^{n+1}}\int_a^b\cdots\int_a^b f\Big(t\frac{x_1+\ldots+x_{n+1}}{n+1}+(1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_{n+1}.$$

However, it is easy to see that

$$\frac{1}{(b-a)^{n+1}}\int_a^b\cdots\int_a^b f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_{n+1}$$
$$=\ldots=\frac{1}{(b-a)^{n}}\int_a^b\cdots\int_a^b f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_{n}$$

and thus, by the above inequality, we conclude

$$H_n(t)\geqslant H_{n+1}(t)\quad\text{for all } t\in[0,1] \text{ and } n\geqslant 1.$$

The proof is thus completed. □

It is natural to ask what happens with the difference $H_n(t)-f\big(\frac{1}{2}(a+b)\big)$ which is clearly non-negative for all $t\in[0,1]$.

The following theorem contains an upper bound for this difference.

Theorem 3. *Let $f\colon I\subseteq\mathbb{R}\to R$ be a convex mapping and f'_+ its right derivative which exists on $\mathring{I}$ and is monotonic nondecreasing on $\mathring{I}$. If $a,b\in\mathring{I}$ with $a<b$, then the inequalities*

$$0\leqslant H_n(t)-f\Big(\frac{a+b}{2}\Big)\tag{2.6}$$
$$\leqslant\frac{t}{(b-a)^n}\int_a^b\cdots\int_a^b f'_+\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)$$
$$\times\Big(x_1-\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n$$
$$\leqslant\frac{t}{\sqrt{n}}\frac{b-a}{2\sqrt{3}}\left[\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b\Big[f'_+\Big(t\frac{x_1+\ldots+x_n}{n}\right.$$
$$\left.+(1-t)\frac{a+b}{2}\Big)\Big]^2\,\mathrm{d}x_1\ldots\mathrm{d}x_n\right]^{\frac{1}{2}}$$

hold for all $n\geqslant 1$ and $t\in[0,1]$.

Proof. As f is convex on I, we can write

$$f(x) - f(y) \geqslant f'_+(y)(x-y) \quad \text{for all } x, y \in \mathring{I}.$$

Choosing in this inequality

$$x = \frac{a+b}{2} \quad \text{and} \quad y = t\frac{x_1 + \ldots + x_n}{n} + (1-t)\frac{a+b}{2},$$

we deduce the inequality

$$\begin{aligned} &f\Big(\frac{a+b}{2}\Big) - f\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big) \\ &\geqslant t f'_+\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big)\Big(\frac{a+b}{2} - \frac{x_1+\ldots+x_n}{n}\Big). \end{aligned}$$

Taking an integral mean over $[a,b]^n$, we derive that

$$\begin{aligned} (2.7) \qquad & f\Big(\frac{a+b}{2}\Big) - H_n(t) \\ & \geqslant t\bigg[\frac{1}{(b-a)^n}\int_a^b \!\!\cdots\! \int_a^b \frac{a+b}{2} f'_+\Big(t\frac{x_1+\ldots+x_n}{n} \\ &\qquad\qquad + (1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n \\ &\quad - \frac{1}{(b-a)^n}\int_a^b \!\!\cdots\! \int_a^b f'_+\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big) \\ &\qquad\qquad \times \Big(\frac{x_1+\ldots+x_n}{n}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\bigg], \end{aligned}$$

from where we deduce the second part of (2.6).

Now, let us observe that the right hand side in the inequality (2.7) is the integral

$$\begin{aligned} I = -\frac{t}{(b-a)^n}\int_a^b \!\!\cdots\! \int_a^b & f'_+\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big) \\ & \times \Big(\frac{x_1+\ldots+x_n}{n} - \frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n. \end{aligned}$$

By the well-known Cauchy-Buniakowsky-Schwartz integral inequality for multiple integrals, we deduce the last part of the inequality (2.6).

The proof of the theorem is thus completed. □

Corollary 1. *Let $f\colon I \subseteq \mathbb{R} \to R$ be a convex mapping and $a, b \in \mathring{I}$ with $a < b$. Put $M := \sup_{x\in[a,b]} |f'(x)| < \infty$. Then we have the inequality*

$$0 \leqslant H_n(t) - f\Big(\frac{a+b}{2}\Big) \leqslant \frac{t(b-a)M}{2\sqrt{3}\sqrt{n}} \tag{2.8}$$

for all $t \in [0,1]$ and $n \geqslant 1$.

In particular, we have

$$\lim_{n\to\infty} H_n(t) = f\Big(\frac{a+b}{2}\Big) \quad \textit{uniformly on } [0,1].$$

The following result also holds:

Theorem 4. *Under the assumptions as in Theorem 3, we have*

$$\begin{aligned}
&0 \leqslant tH_n(1) + (1-t)H_n(0) - H_n(t) \\
&\leqslant \frac{t(1-t)}{(b-a)^n} \int_a^b \cdots \int_a^b f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\Big(x_1 - \frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n \\
&\leqslant \frac{t(1-t)(b-a)}{2\sqrt{3}\sqrt{n}}\left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \Big[f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\Big]^2\,\mathrm{d}x_1\ldots\mathrm{d}x_n\right]^{\frac12}
\end{aligned} \tag{2.9}$$

for all $n \geqslant 1$ and $t \in [0,1]$.

Proof. By the convexity of f we can write

$$\begin{aligned}
&f\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big) - f\Big(\frac{x_1+\ldots+x_n}{n}\Big) \\
&\geqslant (1-t)f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\Big(\frac{a+b}{2} - \frac{x_1+\ldots+x_n}{n}\Big)
\end{aligned} \tag{2.10}$$

for all $t \in [0,1]$ and $x_1,\ldots,x_n \in [a,b]$.

Similarly, we have

$$\begin{aligned}
&f\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big) - f\Big(\frac{a+b}{2}\Big) \\
&\geqslant -tf'_+\Big(\frac{a+b}{2}\Big)\Big(\frac{a+b}{2} - \frac{x_1+\ldots+x_n}{n}\Big)
\end{aligned} \tag{2.11}$$

for all $t \in [0,1]$ and $x_1,\ldots,x_n \in [a,b]$.

If we multiply the inequality (2.10) by t and (2.11) by $(1-t)$ and add the obtained inequalities, we deduce

$$\begin{aligned}
&f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)-tf\Big(\frac{x_1+\ldots+x_n}{n}\Big)-(1-t)f\Big(\frac{a+b}{2}\Big)\\
&\geqslant t(1-t)\Big[f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)-f'_+\Big(\frac{a+b}{2}\Big)\Big]\Big(\frac{a+b}{2}-\frac{x_1+\ldots+x_n}{n}\Big).
\end{aligned}$$

That is,

$$\begin{aligned}
&tf\Big(\frac{x_1+\ldots+x_n}{n}\Big)+(1-t)f\Big(\frac{a+b}{2}\Big)-f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\\
&\leqslant t(1-t)\Big[f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)-f'_+\Big(\frac{a+b}{2}\Big)\Big]\Big(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\Big)
\end{aligned}$$

for all $t\in[0,1]$ and $x_1,\ldots,x_n\in[a,b]$.

Taking an integral mean over $[a,b]^n$, we have

$$\begin{aligned}
0&\leqslant tH_n(1)+(1-t)H_n(0)-H_n(t)\\
&\leqslant t(1-t)\Big[\frac{1}{(b-a)^n}\int_a^b\!\!\cdots\!\int_a^b f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)x_1\,\mathrm{d}x_1\ldots\mathrm{d}x_n\\
&\qquad-\frac{a+b}{2}\frac{1}{(b-a)^n}\int_a^b\!\!\cdots\!\int_a^b f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\Big],
\end{aligned}$$

because a simple calculation shows us that

$$\begin{aligned}
&\frac{1}{(b-a)^n}\int_a^b\!\!\cdots\!\int_a^b f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\Big(\frac{x_1+\ldots+x_n}{n}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\\
&\quad=\frac{1}{(b-a)^n}\int_a^b\!\!\cdots\!\int_a^b f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)x_1\,\mathrm{d}x_1\ldots\mathrm{d}x_n
\end{aligned}$$

and

$$\frac{1}{(b-a)^n}\int_a^b\!\!\cdots\!\int_a^b\frac{x_1+\ldots+x_n}{n}\,\mathrm{d}x_1\ldots\mathrm{d}x_n=\frac{a+b}{2}.$$

Thus, the second inequality in (2.9) is proved.

Now, applying the Cauchy-Buniakowsky-Schwartz integral inequality, we deduce the last part of the theorem. We omit the deatils. □

Corollary 2. *Under the assumptions as in Theorem 3 and provided* $M := \sup_{x\in[a,b]} |f'(x)| < \infty$, *we have the inequality*

$$0 \leqslant tH_n(1) + (1-t)H_n(0) - H_n(t) \leqslant \frac{t(1-t)}{2\sqrt{3}\sqrt{n}}M \tag{2.12}$$

for all $n \geqslant 1$ *and* $t \in [0,1]$.

In particular,

$$\lim_{n\to\infty} [tH_n(1) + (1-t)H_n(0) - H_n(t)] = 0$$

uniformly on $[0,1]$.

The following corollary is interesting as well.

Corollary 3. *Under the assumptions as in Theorem 3 and provided there exists a constant* $K > 0$ *such that*

$$|f'_+(x) - f'_+(y)| \leqslant K|x-y| \quad \textit{for all } x, y \in [a,b],$$

we have the inequality

$$0 \leqslant tH_n(1) + (1-t)H_n(0) - H_n(t) \leqslant \frac{Kt(1-t)}{12n}(b-a)^2 \tag{2.13}$$

for all $t \in [0,1]$ *and* $n \geqslant 1$.

In addition, it is natural to ask for an upper bound for the difference $H_n(1) - H_n(t)$, $n \geqslant 1$ for all $t \in [0,1]$.

Theorem 5. *Let* $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ *be a convex mapping on the interval* I *and let* $a, b \in \mathring{I}$ *with* $a < b$. *Then we have the inequalities*

$$\begin{aligned}\tag{2.14} &\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\Big(\frac{x_1+\ldots+x_n}{n}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n - H_n(t) \\ &\leqslant (1-t)\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\Big(x_1 - \frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n \\ &\leqslant \frac{(1-t)(b-a)}{2\sqrt{3}\sqrt{n}}\Big(\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \Big[f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\Big]^2\,\mathrm{d}x_1\ldots\mathrm{d}x_n\Big)^{\frac{1}{2}}\end{aligned}$$

for all $t \in [0,1]$ *and* $n \geqslant 1$.

Proof. By the convexity of f, we have that

$$\begin{aligned}
&f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)-f\Big(\frac{x_1+\ldots+x_n}{n}\Big)\\
&\geqslant(1-t)f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\Big[\frac{a+b}{2}-\frac{x_1+\ldots+x_n}{n}\Big]
\end{aligned}$$

for all $x_1,\ldots,x_n\in[a,b]$ and $t\in[0,1]$.

Now, the argument proceeds as above and we omit the details. □

Corollary 4. *Under the assumptions as in Theorem 3 and provided* $M:=\sup\limits_{x\in[a,b]}|f'(x)|<\infty$, *we have the inequality*

$$0\leqslant H_n(1)-H_n(t)\leqslant\frac{(1-t)(b-a)M}{2\sqrt{3}\sqrt{n}}. \tag{2.15}$$

In particular,

$$\lim_{n\to\infty}[H_n(1)-H_n(t)]=0$$

uniformly on $[0,1]$.

Corollary 5. *Under the assumptions as in Theorem 3 and provided there exists a constant* $K>0$ *such that*

$$|f'_+(x)-f'_+(y)|\leqslant K|x-y|\quad\text{for all } x,y\in[a,b],$$

we have the inequality

$$0\leqslant H_n(1)-H_n(t)\leqslant\frac{K(1-t)(b-a)^2}{12n} \tag{2.16}$$

for all $n\geqslant1$ *and* $t\in[0,1]$.

Now we establish an upper bound for the difference $H_n(t)-H_{n+1}(t)$, $n\geqslant1$ which is non-negative for all $t\in[0,1]$ (cf. Theorem 2):

Theorem 6. *Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I and let $a, b \in \mathring{I}$ with $a < b$. Then the inequality*

$$
\begin{aligned}
(2.17)\qquad & 0 \leqslant H_n(t) - H_{n+1}(t) \\
& \leqslant \frac{t}{n+1}\left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f'_+\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big)\right. \\
& \qquad\qquad \left. \times \Big(x_1 - \frac{a+b}{2}\Big)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n\right] \\
& \leqslant \frac{t(b-a)}{2\sqrt{3}\sqrt{n}(n+1)}\left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b \Big[f'_+\Big(t\frac{x_1+\ldots+x_n}{n}\right. \\
& \qquad\qquad \left. + (1-t)\frac{a+b}{2}\Big)\Big]^2 \,\mathrm{d}x_1 \ldots \mathrm{d}x_n\right]^{\frac{1}{2}}
\end{aligned}
$$

holds for all $t \in [0,1]$ and $n \geqslant 1$.

Proof. By the convexity of f, we have that

$$
\begin{aligned}
& f\Big(t\frac{x_1+\ldots+x_{n+1}}{n+1} + (1-t)\frac{a+b}{2}\Big) - f\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big) \\
& \geqslant \frac{t}{n(n+1)} f'_+\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big)[nx_{n+1} - (x_1+\ldots+x_n)]
\end{aligned}
$$

for all $x_1, \ldots, x_{n+1} \in [a,b]$ and $t \in [0,1]$.

Taking an integral mean on $[a,b]^{n+1}$, we derive

$$
\begin{aligned}
& 0 \leqslant H_n(t) - H_{n+1}(t) \\
& \leqslant \frac{t}{n+1}\left[\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f'_+\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big)x_1\,\mathrm{d}x_1 \ldots \mathrm{d}x_n\right. \\
& \quad \left. - \frac{a+b}{2}\frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f'_+\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n\right],
\end{aligned}
$$

and the second inequality in (2.17) is proved.

Now, let us observe that

$$\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f'_+\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big) \times\Big(x_1-\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n$$

$$=\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f'_+\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big) \times\Big(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n$$

$$\leqslant\bigg(\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b\Big[f'_+\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\Big]^2\,\mathrm{d}x_1\ldots\mathrm{d}x_n\bigg)^{\frac12} \times\bigg(\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b\Big(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\Big)^2\,\mathrm{d}x_1\ldots\mathrm{d}x_n\bigg)^{\frac12},$$

and thus the last inequality is also proved. □

Corollary 6. *Under the assumptions as in Theorem 3, given that* $M:=\sup_{x\in[a,b]}|f'(x)|<\infty$, *we have*

$$0\leqslant H_n(t)-H_{n+1}(t)\leqslant\frac{Mt(b-a)}{2\sqrt{3}\sqrt{n}(n+1)} \tag{2.18}$$

for all $t\in[0,1]$ *and* $n\geqslant 1$.

In particular,

$$\lim_{n\to\infty}[H_n(t)-H_{n+1}(t)]=0$$

uniformly on $[0,1]$.

The following theorem also holds.

Theorem 7. *Under the assumptions as in Theorem 3, we also have the bound*

$$\begin{aligned} &0\leqslant H_n(t)-H_{n+1}(t)\\ &\leqslant\frac{t(b-a)}{2\sqrt{3}\sqrt{n}(n+1)}\bigg[\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b\Big[f'_+\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\\ &\qquad-f'_+\Big(\frac{a+b}{2}\Big)\Big]^2\,\mathrm{d}x_1\ldots\mathrm{d}x_n\bigg]^{\frac12}\end{aligned} \tag{2.19}$$

for all $t\in[0,1]$ *and* $n\geqslant 1$.

Proof. The proof is similar to the one of Theorem 6 and we omit the details. □

Corollary 7. *Under the assumptions as in Theorem 3, given that there exists a $K > 0$ such that*

$$|f'_+(x) - f'_+(y)| \leqslant K|x - y| \quad \text{for all } x, y \in [a, b],$$

we have the inequality

$$0 \leqslant H_n(t) - H_{n+1}(t) \leqslant \frac{t^2(b-a)^2K^2}{12n(n+1)} \tag{2.20}$$

for all $t \in [0, 1]$ and $n \geqslant 1$.

Finally, note that, by a similar argument to that in the proof of Theorem 6, we can give the following result which completes, in a sense, the estimate in Theorem 3.

Theorem 8. *Under the above assumptions the inequality*

$$\begin{aligned} 0 &\leqslant H_n(t) - f\Big(\frac{a+b}{2}\Big) \\ &\leqslant \frac{t(b-a)}{2\sqrt{3}\sqrt{n}}\bigg[\frac{1}{(b-a)^n}\int_a^b \!\!\cdots\! \int_a^b \Big[f'_+\Big(t\frac{x_1 + \ldots + x_n}{n} + (1-t)\frac{a+b}{2}\Big) \\ &\qquad - f'_+\Big(\frac{a+b}{2}\Big)\Big]^2 \,\mathrm{d}x_1 \ldots \mathrm{d}x_n\bigg]^{\frac{1}{2}} \end{aligned} \tag{2.21}$$

holds for all $t \in [0, 1]$ and $n \geqslant 1$.

Corollary 8. *Under the above assumptions, given that there exists a $K > 0$ such that*

$$|f'_+(x) - f'_+(y)| \leqslant K|x - y| \quad \text{for all } x, y \in [a, b],$$

we have the inequality

$$0 \leqslant H_n(t) - f\Big(\frac{a+b}{2}\Big) \leqslant \frac{t^2(b-a)^2K^2}{12n}$$

for all $t \in [0, 1]$ and $n \geqslant 1$.

3. Applications to special means

Let $0 \leqslant a < b$ and $p \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$. Let us define sequence of mappings

$$h_{p,n}(t) := \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b \Big(t\frac{x_1 + \ldots + x_n}{n} + (1-t)\frac{a+b}{2}\Big)^p \, \mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

where $n \geqslant 1$, $n \in \mathbb{N}$, and $t \in [0,1]$.

By virtue of the above results, we can establish the following properties:

(i) $h_{p,n}(t)$ are convex and monotonic nondecreasing on $[0,1]$;

(ii) $h_{p,n}(t) \geqslant h_{p,n+1}(t)$ for all $n \geqslant 1$ and $t \in [0,1]$;

(iii) the inequalities

$$\begin{aligned}[A(a,b)]^p &\leqslant h_{p,n}(t) \\ &\leqslant \frac{1}{(b-a)^{n+1}} \int_a^b \cdots \int_a^b \Big(t\frac{x_1 + \ldots + x_n}{n} + (1-t)x_{n+1}\Big)^p \, \mathrm{d}x_1 \ldots \mathrm{d}x_{n+1}\end{aligned}$$

and

$$\begin{aligned}h_{p,n}(t) &\leqslant t\frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b \Big(\frac{x_1 + \ldots + x_n}{n}\Big)^p \, \mathrm{d}x_1 \ldots \mathrm{d}x_n \\ &\quad + (1-t)[A(a,b)]^p \\ &\leqslant \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b \Big(\frac{x_1 + \ldots + x_n}{n}\Big)^p \, \mathrm{d}x_1 \ldots \mathrm{d}x_n\end{aligned}$$

hold for all $n \geqslant 1$ and $t \in [0,1]$, where $A(a,b) := \frac{a+b}{2}$.

(iv) If $p \geqslant 1$, then the inequalities

$$0 \leqslant h_{p,n}(t) - [A(a,b)]^p \leqslant \frac{t(b-a)pb^{p-1}}{2\sqrt{3}\sqrt{n}}$$

and

$$\begin{aligned}0 &\leqslant \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b \Big(\frac{x_1 + \ldots + x_n}{n}\Big)^p \, \mathrm{d}x_1 \ldots \mathrm{d}x_n - h_{p,n}(t) \\ &\leqslant \frac{(1-t)(b-a)pb^{p-1}}{2\sqrt{3}\sqrt{n}}\end{aligned}$$

hold for all $n \geqslant 1$ and $t \in [0,1]$.

(v) If $p \geqslant 1$, the inequalities

$$0 \leqslant th_{p,n}(1) + (1-t)h_{p,n}(0) - h_{p,n}(t) \leqslant \frac{t(1-t)pb^{p-1}}{2\sqrt{3}\sqrt{n}}$$

and

$$0 \leqslant h_{p,n}(t) - h_{p,n+1}(t) \leqslant \frac{t(b-a)pb^{p-1}}{2\sqrt{3}\sqrt{n}(n+1)}$$

hold for all $n \geqslant 1$ and $t \in [0,1]$.

(vi) If $p \geqslant 2$, then the inequalities

$$0 \leqslant h_{p,n}(t) - [A(a,b)]^p \leqslant \frac{t^2(b-a)^2 p(p-1)b^{p-2}}{12n}$$

and

$$0 \leqslant h_{p,n}(t) - h_{p,n+1}(t) \leqslant \frac{t^2(b-a)^2 p(p-1)b^{p-2}}{12n(n+1)}$$

and

$$0 \leqslant th_{p,n}(1) + (1-t)h_{p,n}(0) - h_{p,n}(t)$$
$$\leqslant \frac{t(1-t)(b-a)^2 p(p-1)b^{p-2}}{12n}$$

and

$$0 \leqslant h_{p,n}(1) - h_{p,n}(t) \leqslant \frac{(1-t)(b-a)^2 p(p-1)b^{p-2}}{12n}$$

hold for all $n \geqslant 1$ and $t \in [0,1]$.

Acknowledgement. The author would like to thank the anonymous referee for many valuable comments and remarks that have been incorporated in the final version of the paper.

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A SEQUENCE OF MAPPINGS ASSOCIATED WITH THE HERMITE-HADAMARD INEQUALITIES AND APPLICATIONS

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(Received January 1, 2002, in revised version May 16, 2003)

Abstract. New properties for some sequences of functions defined by multiple integrals associated with the Hermite-Hadamard integral inequality for convex functions and some applications are given.

Keywords: Hermite-Hadamard inequality

MSC 2000: 26D15, 26D10, 26D99

1. Introduction

The integral inequality

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \leqslant \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f\colon [a,b] \to \mathbb{R}$, is well known in literature as the Hermite-Hadamard inequality.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the theory of special means and in information theory for divergence measures, from which we would like to refer the reader to [1]–[54].

The main aim of this paper is to consider some natural sequences of functions defined by multiple integrals and study their properties in relation to the Hermite-Hadamard inequality.

2. Properties of the sequence of mappings H_n

Let $I \subseteq \mathbb{R} \to \mathbb{R}$ be an interval of real numbers and $a, b \in I$ with $a < b$, and let $f\colon I \to \mathbb{R}$ be a mapping which is integrable on $[a, b]$. Then we can define a sequence of mappings $H_n\colon [0,1] \to \mathbb{R}$ by

$$H_n(t) := \frac{1}{(b-a)^n} \int_a^b \dots \int_a^b f\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1 \dots \mathrm{d}x_n$$

for $n \geqslant 1$ and $t \in [0,1]$.

Some properties of this sequence of mappings are embodied in the following theorem.

Theorem 1. *Let $f\colon I \subseteq \mathbb{R}$ be a convex mapping on I and let $a, b \in I$ with $a < b$. Then*

(i) *H_n are convex on $[0,1]$ for all $n \geqslant 1$;*

(ii) *the following inequalities hold:*

$$(2.1)\qquad f\Big(\frac{a+b}{2}\Big) \leqslant H_n(t) \leqslant \frac{1}{(b-a)^{n+1}} \times \int_a^b \dots \int_a^b f\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)x_{n+1}\Big)\,\mathrm{d}x_1 \dots \mathrm{d}x_{n+1}$$

and

$$(2.2)\qquad H_n(t) \leqslant t\frac{1}{(b-a)^n} \int_a^b \dots \int_a^b f\Big(\frac{x_1+\ldots+x_n}{n}\Big)\,\mathrm{d}x_1 \dots \mathrm{d}x_n + (1-t)f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{(b-a)^n} \int_a^b \dots \int_a^b f\Big(\frac{x_1+\ldots+x_n}{n}\Big)\,\mathrm{d}x_1 \dots \mathrm{d}x_n$$

for all $t \in [0,1]$;

(iii) *the mapping H_n is monotonic nondecreasing on $[0,1]$ for all $n \geqslant 1$ and one has the bounds*

$$(2.3)\qquad \min_{t\in[0,1]} H_n(t) = f\Big(\frac{a+b}{2}\Big) = H_n(0) \quad \text{for all } n \geqslant 1$$

and

$$(2.4)\qquad \max_{t\in[0,1]} H_n(t) = \frac{1}{(b-a)^n} \int_a^b \dots \int_a^b f\Big(\frac{x_1+\ldots+x_n}{n}\Big)\,\mathrm{d}x_1 \dots \mathrm{d}x_n = H_n(1) \quad \text{for } n \geqslant 1.$$

P r o o f. (i) Follows by the convexity of f.

(ii) Applying Jensen's integral inequality, we obtain

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)x_{n+1}\Big)\,\mathrm{d}x_{n+1}\\
&\quad\geqslant f\Big[\frac{1}{b-a}\int_a^b \Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)x_{n+1}\Big)\,\mathrm{d}x_{n+1}\Big]\\
&\quad= f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)
\end{aligned}$$

for all $x_1,\ldots,x_n\in[a,b]$ and $t\in[0,1]$.

Taking an integral mean over $[a,b]^n$ we deduce the second inequality in (2.1).

By Jensen's integral inequality for multiple integrals we have

$$\begin{aligned}
&\frac{1}{(b-a)^n}\int_a^b\!\!\ldots\!\int_a^b f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\\
&\quad\geqslant f\Big[\frac{1}{(b-a)^n}\int_a^b\!\!\ldots\!\int_a^b \Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\Big]\\
&\quad= f\Big(\frac{a+b}{2}\Big),
\end{aligned}$$

and the inequality (2.1) is completely proved.

By the convexity of f on $[a,b]$, we can write

$$f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\leqslant tf\Big(\frac{x_1+\ldots+x_n}{n}\Big)+(1-t)f\Big(\frac{a+b}{2}\Big)$$

for all $x_1,\ldots,x_n\in[a,b]$ and $t\in[0,1]$. Taking an integral mean over $[a,b]^n$, we deduce

$$H_n(t)\leqslant t\frac{1}{(b-a)^n}\int_a^b\!\!\ldots\!\int_a^b f\Big(\frac{x_1+\ldots+x_n}{n}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n+(1-t)f\Big(\frac{a+b}{2}\Big),$$

and the first inequality in (2.2) is proved.

As we know (see for example [26]) that

$$f\Big(\frac{a+b}{2}\Big)\leqslant\frac{1}{(b-a)^n}\int_a^b\!\!\ldots\!\int_a^b f\Big(\frac{x_1+\ldots+x_n}{n}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n,$$

we obtain the last part of (2.2).

(iii) Let $0<t_1<t_2\leqslant 1$. By the convexity of H_n, which follows by (i) now proved, we have that

$$\frac{H_n(t_2)-H_n(t_1)}{t_2-t_1}\geqslant\frac{H_n(t_1)-H_n(0)}{t_1},$$

but $H_n(t_1) \geqslant H_n(0)$ (see the first inequality in (2.1)) and hence we get that $H_n(t_2) - H_n(t_1) \geqslant 0$ for all $0 \leqslant t_1 < t_2 \leqslant 1$, which shows that the mapping $H_n(\cdot)$ is monotonic nondecreasing on $[0,1]$. The bounds (2.3) and (2.4) follow by the inequalities (2.1) and (2.2). We omit the details. $\square$

We now give another result on monotonicity which, in a sense, completes the above theorem.

Theorem 2. *Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping on I and let $a, b \in I$ with $a < b$. Then*

$$f\Big(\frac{a+b}{2}\Big) \leqslant H_{n+1}(t) \leqslant H_n(t) \leqslant \ldots \leqslant H_1(t) = H(t), \tag{2.5}$$

where

$$H(t) := \frac{1}{b-a} \int_a^b f\Big(tx + (1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x$$

for all $n \geqslant 1$ and $t \in [0,1]$. That is, the sequence of mappings $(H_n)_{n\geqslant 1}$ is monotonically nonincreasing.

Proof. Let us define real numbers belonging to $[a,b]$:

$$\begin{aligned}
y_1 &:= t\frac{x_1 + \ldots + x_n}{n} + (1-t)\frac{a+b}{2};\\
y_2 &:= t\frac{x_2 + x_3 + \ldots + x_{n+1}}{n} + (1-t)\frac{a+b}{2};\\
&\vdots\\
y_{n+1} &:= t\frac{x_{n+1} + x_1 + \ldots + x_{n-1}}{n} + (1-t)\frac{a+b}{2},
\end{aligned}$$

where $x_1, \ldots, x_{n+1} \in [a,b]$.

Using Jensen's discrete inequality, we may state that

$$\begin{aligned}
&\frac{1}{n+1}\bigg[f\Big(t\frac{x_1 + \ldots + x_n}{n} + (1-t)\frac{a+b}{2}\Big)\\
&\qquad + f\Big(t\frac{x_2 + \ldots + x_{n+1}}{n} + (1-t)\frac{a+b}{2}\Big) + \ldots\\
&\qquad + f\Big(t\frac{x_{n+1} + x_1 + \ldots + x_{n-1}}{n} + (1-t)\frac{a+b}{2}\Big)\bigg]\\
&\geqslant f\Big(t\frac{x_1 + \ldots + x_{n+1}}{n+1} + (1-t)\frac{a+b}{2}\Big)
\end{aligned}$$

for all $t \in [0,1]$ and $x_1, \ldots, x_{n+1} \in [a,b]$.

Taking an integral mean over $[a,b]^{n+1}$, we deduce

$$\frac{1}{n+1}\Bigg[\frac{1}{(b-a)^{n+1}}\int_a^b\!\!\ldots\int_a^b f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_{n+1}$$
$$+\ldots+\frac{1}{(b-a)^{n+1}}\int_a^b\!\!\ldots\int_a^b f\Big(t\frac{x_{n+1}+x_1+\ldots+x_{n-1}}{n}$$
$$+(1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_{n+1}\Bigg]$$
$$\geqslant\frac{1}{(b-a)^{n+1}}\int_a^b\!\!\ldots\int_a^b f\Big(t\frac{x_1+\ldots+x_{n+1}}{n+1}+(1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_{n+1}.$$

However, it is easy to see that

$$\frac{1}{(b-a)^{n+1}}\int_a^b\!\!\ldots\int_a^b f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_{n+1}$$
$$=\ldots=\frac{1}{(b-a)^{n}}\int_a^b\!\!\ldots\int_a^b f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_{n}$$

and thus, by the above inequality, we conclude

$$H_n(t)\geqslant H_{n+1}(t)\quad\text{for all }\ t\in[0,1]\ \text{ and }\ n\geqslant 1.$$

The proof is thus completed. $\square$

It is natural to ask what happens with the difference $H_n(t)-f\big(\frac{1}{2}(a+b)\big)$ which is clearly non-negative for all $t\in[0,1]$.

The following theorem contains an upper bound for this difference.

Theorem 3. *Let $f\colon I\subseteq\mathbb{R}\to R$ be a convex mapping and f'_+ its right derivative which exists on $\mathring{I}$ and is monotonic nondecreasing on $\mathring{I}$. If $a,b\in\mathring{I}$ with $a<b$, then the inequalities*

$$0\leqslant H_n(t)-f\Big(\frac{a+b}{2}\Big)\tag{2.6}$$
$$\leqslant\frac{t}{(b-a)^n}\int_a^b\!\!\ldots\int_a^b f'_+\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)$$
$$\times\Big(x_1-\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n$$
$$\leqslant\frac{t}{\sqrt{n}}\frac{b-a}{2\sqrt{3}}\Bigg[\frac{1}{(b-a)^n}\int_a^b\!\!\ldots\int_a^b\Big[f'_+\Big(t\frac{x_1+\ldots+x_n}{n}$$
$$+(1-t)\frac{a+b}{2}\Big)\Big]^2\,\mathrm{d}x_1\ldots\mathrm{d}x_n\Bigg]^{\frac{1}{2}}$$

hold for all $n\geqslant 1$ and $t\in[0,1]$.

Proof. As f is convex on I, we can write

$$f(x) - f(y) \geqslant f'_+(y)(x-y) \quad \text{for all } x, y \in \mathring{I}.$$

Choosing in this inequality

$$x = \frac{a+b}{2} \quad \text{and} \quad y = t\frac{x_1 + \ldots + x_n}{n} + (1-t)\frac{a+b}{2},$$

we deduce the inequality

$$\begin{aligned} & f\Big(\frac{a+b}{2}\Big) - f\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big) \\ \geqslant\, & t f'_+\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big)\Big(\frac{a+b}{2} - \frac{x_1+\ldots+x_n}{n}\Big). \end{aligned}$$

Taking an integral mean over $[a,b]^n$, we derive that

$$\begin{aligned} (2.7) \qquad & f\Big(\frac{a+b}{2}\Big) - H_n(t) \\ & \geqslant t\Big[\frac{1}{(b-a)^n}\int_a^b \!\!\ldots\! \int_a^b \frac{a+b}{2} f'_+\Big(t\frac{x_1+\ldots+x_n}{n} \\ & \qquad\qquad + (1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n \\ & \quad - \frac{1}{(b-a)^n}\int_a^b \!\!\ldots\! \int_a^b f'_+\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big) \\ & \qquad\qquad \times \Big(\frac{x_1+\ldots+x_n}{n}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\Big], \end{aligned}$$

from where we deduce the second part of (2.6).

Now, let us observe that the right hand side in the inequality (2.7) is the integral

$$\begin{aligned} I = -\frac{t}{(b-a)^n}\int_a^b \!\!\ldots\! \int_a^b & f'_+\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big) \\ & \times\Big(\frac{x_1+\ldots+x_n}{n} - \frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n. \end{aligned}$$

By the well-known Cauchy-Buniakowsky-Schwartz integral inequality for multiple integrals, we deduce the last part of the inequality (2.6).

The proof of the theorem is thus completed. □

Corollary 1. *Let $f\colon I \subseteq \mathbb{R} \to R$ be a convex mapping and $a, b \in \mathring{I}$ with $a < b$. Put $M := \sup\limits_{x\in[a,b]} |f'(x)| < \infty$. Then we have the inequality*

$$0 \leqslant H_n(t) - f\Big(\frac{a+b}{2}\Big) \leqslant \frac{t(b-a)M}{2\sqrt{3}\sqrt{n}} \tag{2.8}$$

for all $t \in [0,1]$ and $n \geqslant 1$.

In particular, we have

$$\lim_{n\to\infty} H_n(t) = f\Big(\frac{a+b}{2}\Big) \quad \textit{uniformly on } [0,1].$$

The following result also holds:

Theorem 4. *Under the assumptions as in Theorem 3, we have*

$$\begin{aligned} &0 \leqslant tH_n(1) + (1-t)H_n(0) - H_n(t) \\ &\leqslant \frac{t(1-t)}{(b-a)^n} \int_a^b \cdots \int_a^b f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\Big(x_1 - \frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n \\ &\leqslant \frac{t(1-t)(b-a)}{2\sqrt{3}\sqrt{n}}\bigg[\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b \Big[f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\Big]^2\,\mathrm{d}x_1\ldots\mathrm{d}x_n\bigg]^{\frac12} \end{aligned} \tag{2.9}$$

for all $n \geqslant 1$ and $t \in [0,1]$.

P r o o f. By the convexity of f we can write

$$\begin{aligned} &f\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big) - f\Big(\frac{x_1+\ldots+x_n}{n}\Big) \\ &\geqslant (1-t)f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\Big(\frac{a+b}{2} - \frac{x_1+\ldots+x_n}{n}\Big) \end{aligned} \tag{2.10}$$

for all $t \in [0,1]$ and $x_1, \ldots, x_n \in [a,b]$.

Similarly, we have

$$\begin{aligned} &f\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big) - f\Big(\frac{a+b}{2}\Big) \\ &\geqslant -tf'_+\Big(\frac{a+b}{2}\Big)\Big(\frac{a+b}{2} - \frac{x_1+\ldots+x_n}{n}\Big) \end{aligned} \tag{2.11}$$

for all $t \in [0,1]$ and $x_1, \ldots, x_n \in [a,b]$.

If we multiply the inequality (2.10) by t and (2.11) by $(1-t)$ and add the obtained inequalities, we deduce

$$
\begin{aligned}
&f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)-tf\Big(\frac{x_1+\ldots+x_n}{n}\Big)-(1-t)f\Big(\frac{a+b}{2}\Big)\\
&\geqslant t(1-t)\Big[f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)-f'_+\Big(\frac{a+b}{2}\Big)\Big]\Big(\frac{a+b}{2}-\frac{x_1+\ldots+x_n}{n}\Big).
\end{aligned}
$$

That is,

$$
\begin{aligned}
&tf\Big(\frac{x_1+\ldots+x_n}{n}\Big)+(1-t)f\Big(\frac{a+b}{2}\Big)-f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\\
&\leqslant t(1-t)\Big[f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)-f'_+\Big(\frac{a+b}{2}\Big)\Big]\Big(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\Big)
\end{aligned}
$$

for all $t\in[0,1]$ and $x_1,\ldots,x_n\in[a,b]$.

Taking an integral mean over $[a,b]^n$, we have

$$
\begin{aligned}
0&\leqslant tH_n(1)+(1-t)H_n(0)-H_n(t)\\
&\leqslant t(1-t)\Big[\frac{1}{(b-a)^n}\int_a^b\!\!\cdots\int_a^b f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)x_1\,\mathrm{d}x_1\ldots\,\mathrm{d}x_n\\
&\quad-\frac{a+b}{2}\frac{1}{(b-a)^n}\int_a^b\!\!\cdots\int_a^b f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\,\mathrm{d}x_1\ldots\,\mathrm{d}x_n\Big],
\end{aligned}
$$

because a simple calculation shows us that

$$
\begin{aligned}
&\frac{1}{(b-a)^n}\int_a^b\!\!\cdots\int_a^b f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\Big(\frac{x_1+\ldots+x_n}{n}\Big)\,\mathrm{d}x_1\ldots\,\mathrm{d}x_n\\
&\quad=\frac{1}{(b-a)^n}\int_a^b\!\!\cdots\int_a^b f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)x_1\,\mathrm{d}x_1\ldots\,\mathrm{d}x_n
\end{aligned}
$$

and

$$
\frac{1}{(b-a)^n}\int_a^b\!\!\cdots\int_a^b \frac{x_1+\ldots+x_n}{n}\,\mathrm{d}x_1\ldots\,\mathrm{d}x_n=\frac{a+b}{2}.
$$

Thus, the second inequality in (2.9) is proved.

Now, applying the Cauchy-Buniakowsky-Schwartz integral inequality, we deduce the last part of the theorem. We omit the deatils. $\square$

Corollary 2. *Under the assumptions as in Theorem 3 and provided* $M := \sup_{x\in[a,b]} |f'(x)| < \infty$, *we have the inequality*

$$0 \leqslant tH_n(1) + (1-t)H_n(0) - H_n(t) \leqslant \frac{t(1-t)}{2\sqrt{3}\sqrt{n}} M \tag{2.12}$$

for all $n \geqslant 1$ *and* $t \in [0,1]$.

In particular,

$$\lim_{n\to\infty} [tH_n(1) + (1-t)H_n(0) - H_n(t)] = 0$$

uniformly on $[0,1]$.

The following corollary is interesting as well.

Corollary 3. *Under the assumptions as in Theorem 3 and provided there exists a constant* $K > 0$ *such that*

$$|f'_+(x) - f'_+(y)| \leqslant K|x-y| \quad \text{for all } x, y \in [a,b],$$

we have the inequality

$$0 \leqslant tH_n(1) + (1-t)H_n(0) - H_n(t) \leqslant \frac{Kt(1-t)}{12n}(b-a)^2 \tag{2.13}$$

for all $t \in [0,1]$ *and* $n \geqslant 1$.

In addition, it is natural to ask for an upper bound for the difference $H_n(1) - H_n(t)$, $n \geqslant 1$ for all $t \in [0,1]$.

Theorem 5. *Let* $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ *be a convex mapping on the interval* I *and let* $a, b \in \mathring{I}$ *with* $a < b$. *Then we have the inequalities*

$$\begin{aligned} &\frac{1}{(b-a)^n} \int_a^b \!\!\cdots\! \int_a^b f\Big(\frac{x_1+\ldots+x_n}{n}\Big)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n - H_n(t) \\ &\leqslant (1-t)\frac{1}{(b-a)^n} \int_a^b \!\!\cdots\! \int_a^b f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\Big(x_1 - \frac{a+b}{2}\Big)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n \\ &\leqslant \frac{(1-t)(b-a)}{2\sqrt{3}\sqrt{n}} \bigg(\frac{1}{(b-a)^n} \int_a^b \!\!\cdots\! \int_a^b \Big[f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\Big]^2 \mathrm{d}x_1 \ldots \mathrm{d}x_n\bigg)^{\frac{1}{2}} \end{aligned} \tag{2.14}$$

for all $t \in [0,1]$ *and* $n \geqslant 1$.

P r o o f. By the convexity of f, we have that

$$
\begin{aligned}
&f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)-f\Big(\frac{x_1+\ldots+x_n}{n}\Big)\\
&\geqslant(1-t)f'_+\Big(\frac{x_1+\ldots+x_n}{n}\Big)\Big[\frac{a+b}{2}-\frac{x_1+\ldots+x_n}{n}\Big]
\end{aligned}
$$

for all $x_1,\ldots,x_n\in[a,b]$ and $t\in[0,1]$.

Now, the argument proceeds as above and we omit the details. □

Corollary 4. *Under the assumptions as in Theorem* 3 *and provided* $M:=\sup\limits_{x\in[a,b]}|f'(x)|<\infty$, *we have the inequality*

$$0\leqslant H_n(1)-H_n(t)\leqslant\frac{(1-t)(b-a)M}{2\sqrt{3}\sqrt{n}}. \tag{2.15}$$

In particular,

$$\lim_{n\to\infty}[H_n(1)-H_n(t)]=0$$

uniformly on $[0,1]$.

Corollary 5. *Under the assumptions as in Theorem* 3 *and provided there exists a constant* $K>0$ *such that*

$$|f'_+(x)-f'_+(y)|\leqslant K|x-y|\quad\text{for all }\ x,y\in[a,b],$$

we have the inequality

$$0\leqslant H_n(1)-H_n(t)\leqslant\frac{K(1-t)(b-a)^2}{12n} \tag{2.16}$$

for all $n\geqslant 1$ *and* $t\in[0,1]$.

Now we establish an upper bound for the difference $H_n(t)-H_{n+1}(t)$, $n\geqslant 1$ which is non-negative for all $t\in[0,1]$ (cf. Theorem 2):

Theorem 6. *Let $f\colon I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function on the interval I and let $a,b\in\mathring{I}$ with $a<b$. Then the inequality*

$$
\begin{aligned}
(2.17)\qquad & 0\leqslant H_n(t)-H_{n+1}(t)\\
&\leqslant\frac{t}{n+1}\Bigg[\frac{1}{(b-a)^n}\int_a^b\!\!\dots\!\int_a^b f'_+\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\\
&\qquad\times\Big(x_1-\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\Bigg]\\
&\leqslant\frac{t(b-a)}{2\sqrt{3}\sqrt{n}(n+1)}\Bigg[\frac{1}{(b-a)^n}\int_a^b\!\!\dots\!\int_a^b\Big[f'_+\Big(t\frac{x_1+\ldots+x_n}{n}\\
&\qquad+(1-t)\frac{a+b}{2}\Big)\Big]^2\,\mathrm{d}x_1\ldots\mathrm{d}x_n\Bigg]^{\frac12}
\end{aligned}
$$

holds for all $t\in[0,1]$ and $n\geqslant 1$.

Proof. By the convexity of f, we have that

$$
\begin{aligned}
&f\Big(t\frac{x_1+\ldots+x_{n+1}}{n+1}+(1-t)\frac{a+b}{2}\Big)-f\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\\
&\geqslant\frac{t}{n(n+1)}f'_+\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)[nx_{n+1}-(x_1+\ldots+x_n)]
\end{aligned}
$$

for all $x_1,\ldots,x_{n+1}\in[a,b]$ and $t\in[0,1]$.

Taking an integral mean on $[a,b]^{n+1}$, we derive

$$
\begin{aligned}
&0\leqslant H_n(t)-H_{n+1}(t)\\
&\leqslant\frac{t}{n+1}\Bigg[\frac{1}{(b-a)^n}\int_a^b\!\!\dots\!\int_a^b f'_+\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)x_1\,\mathrm{d}x_1\ldots\mathrm{d}x_n\\
&\quad-\frac{a+b}{2}\frac{1}{(b-a)^n}\int_a^b\!\!\dots\!\int_a^b f'_+\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\Bigg],
\end{aligned}
$$

and the second inequality in (2.17) is proved.

Now, let us observe that

$$
\begin{aligned}
&\frac{1}{(b-a)^n}\int_a^b\!\!\dots\int_a^b f'_+\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\\
&\qquad\qquad\qquad\qquad\times\Big(x_1-\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\\
&=\frac{1}{(b-a)^n}\int_a^b\!\!\dots\int_a^b f'_+\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\\
&\qquad\qquad\qquad\qquad\times\Big(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\Big)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\\
&\leqslant\Big(\frac{1}{(b-a)^n}\int_a^b\!\!\dots\int_a^b\Big[f'_+\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\Big]^2\,\mathrm{d}x_1\ldots\mathrm{d}x_n\Big)^{\frac12}\\
&\qquad\times\Big(\frac{1}{(b-a)^n}\int_a^b\!\!\dots\int_a^b\Big(\frac{x_1+\ldots+x_n}{n}-\frac{a+b}{2}\Big)^2\,\mathrm{d}x_1\ldots\mathrm{d}x_n\Big)^{\frac12},
\end{aligned}
$$

and thus the last inequality is also proved. □

Corollary 6. *Under the assumptions as in Theorem 3, given that* $M:=\sup\limits_{x\in[a,b]}|f'(x)|<\infty$, *we have*

$$0\leqslant H_n(t)-H_{n+1}(t)\leqslant\frac{Mt(b-a)}{2\sqrt3\sqrt n(n+1)} \tag{2.18}$$

for all $t\in[0,1]$ *and* $n\geqslant1$.

In particular,

$$\lim_{n\to\infty}[H_n(t)-H_{n+1}(t)]=0$$

uniformly on $[0,1]$.

The following theorem also holds.

Theorem 7. *Under the assumptions as in Theorem 3, we also have the bound*

$$
\begin{aligned}
&0\leqslant H_n(t)-H_{n+1}(t)\\
&\quad\leqslant\frac{t(b-a)}{2\sqrt3\sqrt n(n+1)}\Big[\frac{1}{(b-a)^n}\int_a^b\!\!\dots\int_a^b\Big[f'_+\Big(t\frac{x_1+\ldots+x_n}{n}+(1-t)\frac{a+b}{2}\Big)\\
&\qquad\qquad\qquad\qquad-f'_+\Big(\frac{a+b}{2}\Big)\Big]^2\,\mathrm{d}x_1\ldots\mathrm{d}x_n\Big]^{\frac12}
\end{aligned}
\tag{2.19}
$$

for all $t\in[0,1]$ *and* $n\geqslant1$.

P r o o f. The proof is similar to the one of Theorem 6 and we omit the details. □

Corollary 7. *Under the assumptions as in Theorem 3, given that there exists a $K > 0$ such that*

$$|f'_+(x) - f'_+(y)| \leqslant K|x - y| \quad \textit{for all } x, y \in [a, b],$$

we have the inequality

$$0 \leqslant H_n(t) - H_{n+1}(t) \leqslant \frac{t^2(b-a)^2K^2}{12n(n+1)} \tag{2.20}$$

for all $t \in [0, 1]$ and $n \geqslant 1$.

Finally, note that, by a similar argument to that in the proof of Theorem 6, we can give the following result which completes, in a sense, the estimate in Theorem 3.

Theorem 8. *Under the above assumptions the inequality*

$$\begin{aligned} 0 &\leqslant H_n(t) - f\Big(\frac{a+b}{2}\Big) \\ &\leqslant \frac{t(b-a)}{2\sqrt{3}\sqrt{n}}\bigg[\frac{1}{(b-a)^n}\int_a^b\!\!\dots\int_a^b\Big[f'_+\Big(t\frac{x_1+\ldots+x_n}{n} + (1-t)\frac{a+b}{2}\Big) \\ &\qquad - f'_+\Big(\frac{a+b}{2}\Big)\Big]^2 \,\mathrm{d}x_1\ldots\mathrm{d}x_n\bigg]^{\frac{1}{2}} \end{aligned} \tag{2.21}$$

holds for all $t \in [0, 1]$ and $n \geqslant 1$.

Corollary 8. *Under the above assumptions, given that there exists a $K > 0$ such that*

$$|f'_+(x) - f'_+(y)| \leqslant K|x - y| \quad \textit{for all } x, y \in [a, b],$$

we have the inequality

$$0 \leqslant H_n(t) - f\Big(\frac{a+b}{2}\Big) \leqslant \frac{t^2(b-a)^2K^2}{12n}$$

for all $t \in [0, 1]$ and $n \geqslant 1$.

3. Applications to special means

Let $0 \leqslant a < b$ and $p \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$. Let us define sequence of mappings

$$h_{p,n}(t) := \frac{1}{(b-a)^n} \int_a^b \ldots \int_a^b \Big(t \frac{x_1 + \ldots + x_n}{n} + (1-t) \frac{a+b}{2} \Big)^p \, \mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

where $n \geqslant 1$, $n \in \mathbb{N}$, and $t \in [0,1]$.

By virtue of the above results, we can establish the following properties:

(i) $h_{p,n}(t)$ are convex and monotonic nondecreasing on $[0,1]$;

(ii) $h_{p,n}(t) \geqslant h_{p,n+1}(t)$ for all $n \geqslant 1$ and $t \in [0,1]$;

(iii) the inequalities

$$\begin{aligned} [A(a,b)]^p &\leqslant h_{p,n}(t) \\ &\leqslant \frac{1}{(b-a)^{n+1}} \int_a^b \ldots \int_a^b \Big(t \frac{x_1 + \ldots + x_n}{n} + (1-t) x_{n+1} \Big)^p \, \mathrm{d}x_1 \ldots \mathrm{d}x_{n+1} \end{aligned}$$

and

$$\begin{aligned} h_{p,n}(t) &\leqslant t \frac{1}{(b-a)^n} \int_a^b \ldots \int_a^b \Big(\frac{x_1 + \ldots + x_n}{n} \Big)^p \, \mathrm{d}x_1 \ldots \mathrm{d}x_n \\ &\quad + (1-t)[A(a,b)]^p \\ &\leqslant \frac{1}{(b-a)^n} \int_a^b \ldots \int_a^b \Big(\frac{x_1 + \ldots + x_n}{n} \Big)^p \, \mathrm{d}x_1 \ldots \mathrm{d}x_n \end{aligned}$$

hold for all $n \geqslant 1$ and $t \in [0,1]$, where $A(a,b) := \frac{a+b}{2}$.

(iv) If $p \geqslant 1$, then the inequalities

$$0 \leqslant h_{p,n}(t) - [A(a,b)]^p \leqslant \frac{t(b-a)pb^{p-1}}{2\sqrt{3}\sqrt{n}}$$

and

$$\begin{aligned} 0 &\leqslant \frac{1}{(b-a)^n} \int_a^b \ldots \int_a^b \Big(\frac{x_1 + \ldots + x_n}{n} \Big)^p \, \mathrm{d}x_1 \ldots \mathrm{d}x_n - h_{p,n}(t) \\ &\leqslant \frac{(1-t)(b-a)pb^{p-1}}{2\sqrt{3}\sqrt{n}} \end{aligned}$$

hold for all $n \geqslant 1$ and $t \in [0,1]$.

(v) If $p \geqslant 1$, the inequalities

$$0 \leqslant th_{p,n}(1) + (1-t)h_{p,n}(0) - h_{p,n}(t) \leqslant \frac{t(1-t)pb^{p-1}}{2\sqrt{3}\sqrt{n}}$$

and

$$0 \leqslant h_{p,n}(t) - h_{p,n+1}(t) \leqslant \frac{t(b-a)pb^{p-1}}{2\sqrt{3}\sqrt{n}(n+1)}$$

hold for all $n \geqslant 1$ and $t \in [0,1]$.

(vi) If $p \geqslant 2$, then the inequalities

$$0 \leqslant h_{p,n}(t) - [A(a,b)]^p \leqslant \frac{t^2(b-a)^2p(p-1)b^{p-2}}{12n}$$

and

$$0 \leqslant h_{p,n}(t) - h_{p,n+1}(t) \leqslant \frac{t^2(b-a)^2p(p-1)b^{p-2}}{12n(n+1)}$$

and

$$0 \leqslant th_{p,n}(1) + (1-t)h_{p,n}(0) - h_{p,n}(t)$$

$$\leqslant \frac{t(1-t)(b-a)^2p(p-1)b^{p-2}}{12n}$$

and

$$0 \leqslant h_{p,n}(1) - h_{p,n}(t) \leqslant \frac{(1-t)(b-a)^2p(p-1)b^{p-2}}{12n}$$

hold for all $n \geqslant 1$ and $t \in [0,1]$.

Acknowledgement. The author would like to thank the anonymous referee for many valuable comments and remarks that have been incorporated in the final version of the paper.

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