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APPROXIMATION OF THE STIELTJES INTEGRAL AND APPLICATIONS IN NUMERICAL INTEGRATION

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Abstract. Some inequalities for the Stieltjes integral and applications in numerical integration are given. The Stieltjes integral is approximated by the product of the divided difference of the integrator and the Lebesgue integral of the integrand. Bounds on the approximation error are provided. Applications to the Fourier Sine and Cosine transforms on finite intervals are mentioned as well.

Keywords: Stieltjes integral, quadrature rule, Fourier Sine transform, Fourier Cosine transform

MSC 2000: 26D15, 41A55

1. Introduction

The following definitions will be used throughout the paper.

A function $w\colon [a,b] \to \mathbb{R}$ is said to be of r-H-Hölder type if for $x, y \in [a,b]$ it satisfies the conditions

$$|w(x) - w(y)| \leqslant H|x-y|^r, \quad r \in (0,1] \quad \text{and} \quad H > 0.$$

A 1-L-Hölder type function is also said to be L-Lipschitzian. A function w is said to be of bounded variation if for any division I_n of $[a,b]$, $I_n\colon a = x_0 < x_1 < \ldots < x_n = b$, the variation of w on I_n is finite, which means that

$$\sum_{i=0}^{n-1} |w(x_{i+1}) - w(x_i)| < \infty.$$

The total variation of w on $[a,b]$ is denoted by $\bigvee_a^b(w)$, where

$$\bigvee_a^b(w) := \sup\left\{\sum_{i=0}^{n-1} |w(x_{i+1}) - w(x_i)|,\ \ I_n \text{ is a division of } [a,b]\right\}.$$

In [5], [6], the authors considered the functional

$$D(f;u) := \int_a^b f(x)\,\mathrm{d}u(x) - [u(b)-u(a)]\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \tag{1.1}$$

provided that the integrals involved exist, and established various bounds for the absolute value of $D(f;u)$.

Applications to approximating the Stieltjes integral were also provided in both [5] and [6]. In [2] general results for three-point approximations of the Stieltjes integral were investigated.

In this paper we point out other similar inequalities in an effort to complete the picture and apply them in the numerical approximation of the Stieltjes integral $\int_a^b f(x)\,\mathrm{d}u(x)$. Approximations for the Fourier Sine and Cosine transforms on finite intervals are mentioned as well (see also [7] and [9]).

2. The case of Lipschitzian integrators

Throughout this section, the integrator $u\colon [a,b] \to \mathbb{R}$ in the Stieltjes integral $\int_a^b f(t)\,\mathrm{d}u(t)$ is assumed to be Lipschitzian with a constant L.

The following theorem holds.

Theorem 1. *Assume that $u\colon [a,b]\to\mathbb{R}$ is as above.*

(i) *If $f\colon [a,b]\to\mathbb{R}$ is of bounded variation, then*

$$|D(f;u)| \leqslant \frac{3}{4}L(b-a)\bigvee_a^b(f). \tag{2.1}$$

(ii) *If $f\colon [a,b]\to\mathbb{R}$ is of r-H-Hölder type, then*

$$|D(f;u)| \leqslant \frac{2HL(b-a)^{r+1}}{(r+1)(r+2)}. \tag{2.2}$$

(iii) *If $f\colon [a,b]\to\mathbb{R}$ is absolutely continuous, then*

$$
|D(f;u)| \leqslant \begin{cases} \dfrac{1}{3}L(b-a)^2\|f'\|_\infty & \text{if } f'\in L_\infty[a,b];\\[2mm] \dfrac{2^{1/q}L(b-a)^{1/q+1}\|f'\|_p}{(q+1)^{1/q}(q+2)^{1/q}} & \text{if } f'\in L_p[a,b],\\ & p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1;\\[2mm] \dfrac{3}{4}L(b-a)\|f'\|_1. \end{cases} \tag{2.3}
$$

P r o o f. First, let us observe that $D(f,u)$ defined in (1.1) satisfies the identity

$$
D(f;u)=\int_a^b\left(f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\mathrm{d}u(x). \tag{2.4}
$$

It is well known that if $p\colon [c,d]\to\mathbb{R}$ is Riemann integrable and $v\colon [c,d]\to\mathbb{R}$ is L-Lipschitzian, then the Stieltjes integral $\int_c^d p(t)\,\mathrm{d}v(t)$ exists and

$$
\left|\int_c^d p(t)\,\mathrm{d}v(t)\right|\leqslant L\int_c^d |p(t)|\,\mathrm{d}t. \tag{2.5}
$$

Taking the modulus in (2.4) and using (2.5) we get

$$
|D(f;u)|\leqslant L\int_a^b\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\mathrm{d}x. \tag{2.6}
$$

(i) In [4], the author proved the following Ostrowski type inequality for functions of bounded variation:

$$
\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\leqslant\frac{1}{b-a}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f) \tag{2.7}
$$

for any $x\in[a,b]$. Then, by (2.6), we may state that

$$
\begin{aligned}
|D(f;u)| &\leqslant \frac{L}{b-a}\int_a^b\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\mathrm{d}x\bigvee_a^b(f)\\
&=\frac{L}{b-a}\cdot\left[\frac{1}{2}(b-a)^2+\frac{1}{4}(b-a)^2\right]\bigvee_a^b(f)
\end{aligned}
$$

and the inequality (2.1) is proved.

(ii) In [1], the following inequality of Ostrowski type for r-H-Hölder type functions, f has been pointed out:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \frac{H}{r+1}\left[\Big(\frac{b-x}{b-a}\Big)^{r+1}+\Big(\frac{x-a}{b-a}\Big)^{r+1}\right](b-a)^r$$

for any $x\in[a,b]$. Then, by (2.6), we have

$$\begin{aligned}|D(f;u)| &\leqslant \frac{H}{r+1}(b-a)^r\left\{\frac{1}{(b-a)^{r+1}}\left[\int_a^b (b-x)^{r+1}\,\mathrm{d}x+\int_a^b (x-a)^{r+1}\,\mathrm{d}x\right]\right\}\\ &= \frac{2HL(b-a)^{r+1}}{(r+1)(r+2)}\end{aligned}$$

and the inequality (2.2) is proved.

(iii) Using the following set of inequalities of Ostrowski type for absolutely continuous functions [1]:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \begin{cases}\left[\frac{1}{4}+\Big(\frac{x-(a+b)/2}{b-a}\Big)^2\right](b-a)\|f'\|_\infty, & \text{if } f'\in L_\infty[a,b];\\ \frac{1}{(q+1)^{1/q}}\left[\Big(\frac{b-x}{b-a}\Big)^{q+1}+\Big(\frac{x-a}{b-a}\Big)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_p & \text{if } f'\in L_p[a,b];\\ \left[\frac{1}{2}+\left|\frac{x-(a+b)/2}{b-a}\right|\right]\|f'\|_1 \end{cases} \tag{2.8}$$

for any $x\in[a,b]$, we have from (2.6)

$$|D(f;u)| \leqslant \begin{cases} L\cdot(b-a)\|f'\|_\infty\displaystyle\int_a^b\left[\frac{1}{4}+\Big(\frac{x-(a+b)/2}{b-a}\Big)^2\right]\mathrm{d}x,\\ \dfrac{L}{(q+1)^{1/q}}(b-a)^{1/q}\|f'\|_p\displaystyle\int_a^b\left[\Big(\frac{b-x}{b-a}\Big)^{q+1}+\Big(\frac{x-a}{b-a}\Big)^{q+1}\right]^{1/q}\mathrm{d}x,\\ L\cdot\|f'\|_1\displaystyle\int_a^b\left[\frac{1}{2}+\left|\frac{x-(a+b)/2}{b-a}\right|\right]\mathrm{d}x.\end{cases} \tag{2.9}$$

Since

$$\int_a^b\left[\frac{1}{4}+\Big(\frac{x-(a+b)/2}{b-a}\Big)^2\right]\mathrm{d}x=\frac{1}{3}(b-a),$$

this proves the first part of (2.3).

Using Hölder's integral inequality, we have

$$\begin{aligned}
&\int_a^b \Big[\Big(\frac{b-x}{b-a}\Big)^{q+1} + \Big(\frac{x-a}{b-a}\Big)^{q+1}\Big]^{1/q} \,\mathrm{d}x \\
&\leqslant \Big(\int_a^b \mathrm{d}x\Big)^{1/p} \Big(\int_a^b \Big\{\Big[\Big(\frac{x-a}{b-a}\Big)^{q+1} + \Big(\frac{b-x}{b-a}\Big)^{q+1}\Big]^{1/q}\Big\}^q \,\mathrm{d}x\Big)^{1/q} \\
&= \frac{(b-a)2^{1/q}}{(q+2)^{1/q}}
\end{aligned}$$

and by (2.9) and (2.6) we deduce the second part of (2.3).

The last part of (2.3) is obvious and we omit the details. □

3. The case of integrators of bounded variation

Throughout this section, the integrator $u\colon [a,b] \to \mathbb{R}$ in the Stieltjes integral $\int_a^b f(t)\,\mathrm{d}u(t)$ is assumed to be of bounded variation. The following result holds.

Theorem 2. *Let $u\colon [a,b] \to \mathbb{R}$ be a function of bounded variation.*

(i) *If $f\colon [a,b] \to \mathbb{R}$ is continuous and of bounded variation, then*

$$|D(f;u)| \leqslant \bigvee_a^b(f) \bigvee_a^b(u). \tag{3.1}$$

(ii) *If $f\colon [a,b] \to \mathbb{R}$ is of r-H-Hölder type with $r \in (0,1]$ and $H > 0$, then*

$$|D(f;u)| \leqslant \frac{H}{r+1}(b-a)^r \bigvee_a^b(u). \tag{3.2}$$

(iii) *If $f\colon [a,b] \to \mathbb{R}$ is absolutely continuous, then*

$$|D(f;u)| \leqslant \begin{cases} \frac{1}{2}(b-a)\|f'\|_\infty \bigvee_a^b(u) & \text{if } f' \in L_\infty[a,b]; \\[2mm] \frac{1}{(q+1)^{1/q}}(b-a)^{1/q}\|f'\|_p \bigvee_a^b(u), & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2mm] \|f'\|_1 \bigvee_a^b(u). \end{cases} \tag{3.3}$$

P r o o f. It is well known that if $p\colon [c,d]\to\mathbb{R}$ is continuous and $v\colon [a,b]\to\mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)\,\mathrm{d}v(t)$ exists and

$$\left|\int_c^d p(t)\,\mathrm{d}v(t)\right| \leqslant \sup_{t\in[c,d]} |p(t)| \bigvee_c^d (v). \tag{3.4}$$

Using the identity (2.4) and taking the modulus, we get via (3.4)

$$|D(f;u)| \leqslant \sup_{x\in[a,b]} \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right| \bigvee_a^b (u). \tag{3.5}$$

The proof follows now as in Theorem 1, and we omit the details. □

4. A quadrature formula

Consider a partition of the interval $[a,b]$ given by

$$I_n\colon\ a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b. \tag{4.1}$$

Denote $h_i := x_{i+1} - x_i$ $(i = 0,\ldots,n-1)$ and define the quadrature

$$A_n(f,u;I_n) := \sum_{i=0}^{n-1} \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(t)\,\mathrm{d}t. \tag{4.2}$$

We also define the norm of the division I_n by

$$\nu_{I_n}(h) := \max\{h_i \mid i \in \{1,\ldots,n-1\}\}. \tag{4.3}$$

In [6] (see also [8, p. 468]), the authors pointed out some results in approximating the Riemann-Stieltjes integral $\int_a^b f(x)\,\mathrm{d}u(x)$ in terms of the quadrature rules defined by (4.2).

The following new result for Lipschitzian integrators holds.

Theorem 3. *Assume that I_n is a division of the interval $[a,b]$ as defined in* (4.1) *and $u\colon [a,b]\to\mathbb{R}$ is Lipschitzian with a constant L.*

(i) *If $f\colon [a,b]\to\mathbb{R}$ is of bounded variation, then the remainder $R_n(f,u;I_n)$ in the representation*

$$\int_a^b f(x)\,\mathrm{d}u(x) = A_n(f,u;I_n) + R_n(f,u;I_n) \tag{4.4}$$

satisfies the estimate

$$|R_n(f,u;I_n)| \leqslant \frac{3}{4} L\nu_{I_n}(h) \bigvee_a^b (f). \tag{4.5}$$

(ii) *If $f\colon [a,b] \to \mathbb{R}$ is of r-H-Hölder type, then we have the estimate*

$$|R_n(f,u;I_n)| \leqslant \frac{2HL}{(r+1)(r+2)} \sum_{i=0}^{n-1} h_i^{r+1} \leqslant \frac{2HL(b-a)(\nu_{I_n}(h))^r}{(r+1)(r+2)}.$$

(iii) *If $f\colon [a,b] \to \mathbb{R}$ is absolutely continuous, then*

$$|R_n(f,u;I_n)| \leqslant \begin{cases} \dfrac{1}{3} L\|f'\|_{\infty,[a,b]} \displaystyle\sum_{i=0}^{n-1} h_i^2 & \text{if } f' \in L_\infty[a,b];\\ \dfrac{2^{1/q} L\|f'\|_{p,[a,b]}}{(q+1)^{1/q}(q+2)^{1/q}} \left(\displaystyle\sum_{i=0}^{n-1} h_i^{1+q}\right)^{1/q} & \text{if } f' \in L_p[a,b],\\ & p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1;\\ \dfrac{3}{4} L\nu_{I_n}(h)\|f'\|_{1,[a,b]} \end{cases} \tag{4.6}$$

where $\nu_{I_n}(h)$ is given by (4.3) *and $A_n(f,u;I_n)$ by* (4.2).

P r o o f. The proof follows by Theorem 1 and we omit the details. □

Further, using of Theorem 2, we may point out the following result for integrators of bounded variation.

Theorem 4. *Assume that I_n is a division of the interval $[a,b]$ as defined by* (4.1) *and $u\colon [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$.*

(i) *If $f\colon [a,b] \to \mathbb{R}$ is continuous and of bounded variation on $[a,b]$, then the remainder in* (4.4) *satisfies the estimate*

$$|R_n(f,u;I_n)| \leqslant \max_{i=0,\dots,n-1} \left\{ \bigvee_{x_i}^{x_{i+1}} (f) \right\} \bigvee_a^b (u). \tag{4.7}$$

(ii) *If $f\colon [a,b] \to \mathbb{R}$ is of r-H-Hölder type with $r \in (0,1]$, $H>0$, then we have the estimate*

$$|R_n(f,u;I_n)| \leqslant \frac{H}{r+1} [\nu_{I_n}(h)]^r \bigvee_a^b (u). \tag{4.8}$$

(iii) *If* $f\colon [a,b] \to \mathbb{R}$ *is absolutely continuous, then*

(4.9) $|R_n(f,u;I_n)|$

$$\leqslant \begin{cases} \frac{1}{2}\nu_{I_n}(h)\|f'\|_{\infty,[a,b]} \bigvee_a^b(u) & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{(q+1)^{1/q}}[\nu_{I_n}(h)]^{1/q}\|f'\|_{p,[a,b]} \bigvee_a^b(u) & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \max_{i=0,\dots,n-1}\left\{\int_{x_i}^{x_{i+1}}|f'(t)|\,\mathrm{d}t\right\} \bigvee_a^b(u). \end{cases}$$

5. Approximating Fourier Sine and Cosine transforms

For a function $f\colon [0,\infty) \to \mathbb{R}$ and $0 \leqslant a < b < \infty$, consider the *Fourier Sine* and *Cosine transforms* on the finite interval $[a,b]$:

$$F_S(s) = F_S(s;a,b) := \int_a^b f(x)\sin(sx)\,\mathrm{d}x, \quad s \in [0,\infty), \tag{5.1}$$

$$F_C(s) = F_C(s;a,b) := \int_a^b f(x)\cos(sx)\,\mathrm{d}x, \quad s \in [0,\infty). \tag{5.2}$$

We also need the following trigonometric means for $p,q \in \mathbb{R}$:

$$\mathrm{SIN}(p,q) := \begin{cases} \dfrac{\sin p - \sin q}{p-q} & \text{if } p \neq q; \\ \cos q & \text{if } p = q \end{cases} \tag{5.3}$$

and

$$\mathrm{COS}(p,q) := \begin{cases} \dfrac{\cos p - \cos q}{p-q} & \text{if } p \neq q; \\ -\sin q & \text{if } p = q. \end{cases} \tag{5.4}$$

For $s \neq 0$, observe that

$$F_S(s;a,b) = -\frac{1}{s}\int_a^b f(x)\,\mathrm{d}(\cos(sx))$$

and

$$F_C(s;a,b) = \frac{1}{s}\int_a^b f(x)\,\mathrm{d}(\sin(sx)),$$

and thus (5.1) and (5.2) may be viewed as Stieltjes integrals with continuous integrators $u(x) := \cos(sx)$ and $u(x) = \sin(sx)$, respectively. Here $x \in [a,b]$ and $s > 0$.

If we consider the division (see (4.1))

$$I_n\colon\ a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b,$$

then the quadrature formula (4.2) may be written for these particular choices as

$$A_{S_n}(f, I_n, s) = -\sum_{i=0}^{n-1} \operatorname{COS}(sx_{i+1}, sx_i) \int_{x_i}^{x_{i+1}} f(t)\,\mathrm{d}t \tag{5.5}$$

and

$$A_{C_n}(f, I_n, s) = \sum_{i=0}^{n-1} \operatorname{SIN}(sx_{i+1}, sx_i) \int_{x_i}^{x_{i+1}} f(t)\,\mathrm{d}t. \tag{5.6}$$

Consider now $u\colon\ [a,b] \subset (0,\infty) \to \mathbb{R}$, $u(x) = \cos(sx)$, $s > 0$. Obviously

$$\|u'\|_{\infty,[a,b]} = \sup_{x\in[a,b]} |u'(x)| \leqslant s^2(b-a).$$

Consequently, for a given s, u as defined above is Lipschitzian on $[a,b]$ with the constant $L = s^2(b-a)$.

If $u\colon\ [a,b] \subset (0,\infty) \to \mathbb{R}$, $u(x) = \sin(sx)$, $s > 0$, then

$$\|u'\|_{\infty,[a,b]} = \sup_{x\in[a,b]} |u'(x)| \leqslant s.$$

Using Theorem 3, we may state the following result in approximating the Sine and Cosine transforms.

Proposition 1. *Let I_n be a division of the interval $[a,b]$.*

(i) *If $f\colon\ [a,b] \to \mathbb{R}$ is of bounded variation, then we have*

$$F_S(s; a, b) = A_{S_n}(f, I_n, s) + R_{S_n}(f, I_n, s), \quad s > 0 \tag{5.7}$$

and

$$F_C(s; a, b) = A_{C_n}(f, I_n, s) + R_{C_n}(f, I_n, s), \quad s > 0. \tag{5.8}$$

The remainders $R_{S_n}(f, I_n, s)$ and $R_{C_n}(f, I_n, s)$ satisfy the estimates

$$|R_{S_n}(f, I_n, s)| \leqslant \frac{3}{4} s(b-a)\nu_{I_n}(h) \bigvee_a^b (f) \tag{5.9}$$

and

$$(5.10)\qquad |R_{C_n}(f,I_n,s)| \leqslant \frac{3}{4}\nu_{I_n}(h)\bigvee_a^b(f).$$

(ii) *If $f\colon [a,b]\to\mathbb{R}$ is of r-H-Hölder type, then the remainders satisfy the bounds*

$$(5.11)\qquad |R_{S_n}(f,I_n,s)| \leqslant \frac{2Hs(b-a)}{(r+1)(r+2)}\sum_{i=0}^{n-1}h_i^{r+1} \leqslant \frac{2Hs(b-a)^2}{(r+1)(r+2)}[\nu_{I_n}(h)]^r$$

and

$$(5.12)\qquad |R_{C_n}(f,I_n,s)| \leqslant \frac{2H}{(r+1)(r+2)}\sum_{i=0}^{n-1}h_i^{r+1} \leqslant \frac{2H(b-a)}{(r+1)(r+2)}[\nu_{I_n}(h)]^r.$$

(iii) *If $f\colon [a,b]\to\mathbb{R}$ is absolutely continuous, then*

$$(5.13)\quad |R_{S_n}(f,I_n,s)| \leqslant \begin{cases} \frac{1}{3}s(b-a)\|f'\|_{\infty,[a,b]}\sum_{i=0}^{n-1}h_i^2 & \text{if } f'\in L_\infty[a,b]; \\ \frac{2^{1/q}s(b-a)\|f'\|_{p,[a,b]}}{(q+1)^{1/q}(q+2)^{1/q}}\left(\sum_{i=0}^{n-1}h_i^{q+1}\right)^{1/q} & \text{if } f'\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{3}{4}s(b-a)\nu_{I_n}(h)\|f'\|_{1,[a,b]}, \end{cases}$$

and

$$(5.14)\quad |R_{C_n}(f,I_n,s)| \leqslant \begin{cases} \frac{1}{3}\|f'\|_{\infty,[a,b]}\sum_{i=0}^{n-1}h_i^2 & \text{if } f'\in L_\infty[a,b]; \\ \frac{2^{1/q}\|f'\|_{p,[a,b]}}{(q+1)^{1/q}(q+2)^{1/q}}\left(\sum_{i=0}^{n-1}h_i^{q+1}\right)^{1/q} & \text{if } f'\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{3}{4}\nu_{I_n}(h)\|f'\|_{1,[a,b]}. \end{cases}$$

Similar bounds may be obtained from Theorem 4, but we omit the details.

R e m a r k 1. For an application in approximating the solutions of an electrical circuit, see the preprint online [3] where further details are provided. We omit them here.

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