Hans-Görg Roos

Error estimates for linear finite elements on Bakhvalov-type meshes

Applications of Mathematics, Vol. 51 (2006), No. 1, 63–72

Persistent URL: http://dml.cz/dmlcz/134630

ERROR ESTIMATES FOR LINEAR FINITE ELEMENTS ON BAKHVALOV-TYPE MESHES

Hans-Görg Roos, Dresden

(Received May 26, 2004, in revised version August 22, 2004)

Abstract. For convection-diffusion problems with exponential layers, optimal error estimates for linear finite elements on Shishkin-type meshes are known. We present the first optimal convergence result in an energy norm for a Bakhvalov-type mesh.

Keywords: numerical analysis, convection-diffusion problem, boundary layer, uniform convergence

MSC 2000: 65N30

1. Introduction

Let us consider the boundary value problem

$$Lu := -\varepsilon u'' - b(x)u' + c(x)u = f(x) \quad \text{in } (0,1),\ u(0) = u(1) = 0 \tag{1.1}$$

under the conditions $0 < \varepsilon \ll 1$ and

$$b(x) \geqslant \beta \quad \text{with } \beta > 1, \quad c + \tfrac{1}{2}b' \geqslant \gamma > 0. \tag{1.2}$$

The latter condition ensures that the bilinear form in the variational formulation of the given problem is coercive. Assuming b, c and f to be sufficiently smooth, the boundary value problem has a unique solution with a boundary layer at $x = 0$.

We discretize the problem with linear conforming finite elements and are interested a priori to construct such meshes that for the finite element approximation u^N using N subintervals for the mesh we can prove

$$\|u - u^N\|_\varepsilon \leqslant CN^{-1}g(N). \tag{1.3}$$

Here, C is a constant independent of ε and $\|\cdot\|_\varepsilon$ denotes the ε-weighted H^1-norm:

$$\|v\|_\varepsilon^2 := \varepsilon|v|_1^2 + |v|_0^2.$$

We call the method *almost optimal uniformly convergent* with respect to the $\|\cdot\|_\varepsilon$-norm if $g(N) = o(N^m)$ for every $m > 0$. In the case $g = O(1)$ we have an *optimal* method.

In 1997 Stynes and O'Riordan [8] proved for a Shishkin mesh (see Section 2) the almost optimal estimate

$$\|u - u^N\|_\varepsilon \leqslant CN^{-1}\ln N. \tag{1.4}$$

Two years later the class of S-type meshes (see Section 2) was introduced in [5]. For that class one has

$$\|u - u^N\|_\varepsilon \leqslant CN^{-1}\max|\psi'|. \tag{1.5}$$

For some well-known meshes, the mesh characterizing function ψ has the desired property $\max|\psi'| \leqslant C$ leading to optimal error estimates.

In a survey paper on layer adapted meshes Linss [2] stated recently: "We are not aware of any uniform convergence results for the Galerkin method on Bakhvalov-type meshes." Further, in the recent book [7] the authors announce an optimal error estimate for a B-mesh, but the proof is incomplete. The aim of this paper is to prove an optimal uniform convergence result on a Bachvalov-type mesh. Surprisingly, our proof uses ingredients never used so far in similar proofs for singularly perturbed problems, the so-called quasi-interpolants. In fact our efforts to work with standard interpolants were not successful.

A remark that although we work in $1D$, it should be possible to generalize the results to $2D$-problems with exponential layers on tensor-product meshes of the type considered here. But our use of quasi-interpolants requires a more sophisticated study as usual.

2. Layer-adapted meshes and the analysis for an S-mesh of Stynes and O'Riordan

Because a boundary layer of the type $\mathrm{e}^{-x/\varepsilon}$ is located at $x = 0$, we want to use a fine mesh in the subinterval $[0,\lambda]$ and a uniform mesh in $[\lambda, 1]$. Therefore, we define the following class of layer adapted meshes:

$$x_i = \begin{cases} \sigma\varepsilon\varphi(t_i) \text{ with } t_i = i/N & \text{for } i = 0,1,\ldots,\frac{1}{2}N,\\ 1-(1-x_{N/2})\cdot 2(N-i)/N & \text{for } i = \frac{1}{2}N+1,\ldots,N. \end{cases} \tag{2.1}$$

The mesh generating function φ is supposed to be monotone increasing and to satisfy $\varphi(0) = 0$, $\varphi(\frac{1}{2}) = \lambda/(\sigma\varepsilon)$. The mesh characterizing function mentioned in Section 1 is defined by $\varphi = -\ln\psi$.

For the transition point of the fine to the rough part of the mesh we assume with some parameter σ

$$x_{N/2} = \lambda = \sigma\varepsilon \ln\frac{1}{\varepsilon} \quad \text{(B-type mesh)}. \tag{2.2}$$

Then, $\exp(-x_{N/2}/\varepsilon) = \varepsilon^\sigma$ and the parameter σ is chosen in such a way that ε^σ is as small as we want.

Note that an S-type mesh is characterized by

$$x_{N/2} = \sigma\varepsilon \ln N, \quad \text{thus} \quad \exp(-x_{N/2}/\varepsilon) = N^{-\sigma}.$$

We want to compare linear finite elements on a Shishkin-mesh, characterized by

$$\varphi(t) = 2(\ln N)t \tag{2.3}$$

(that means, the mesh is equidistant in $[0, \lambda]$ as well as in $[\lambda, 1]$ with $\lambda = \sigma\varepsilon\ln N$) and on a Bakhvalov-type mesh with the transition point (2.2) and

$$\varphi(t) = -\ln[1 - 2(1-\varepsilon)t]. \tag{2.4}$$

We call the mesh Bakhvalov-type mesh, because in the original B-mesh the mesh generating function is continuously differentiable and the transition point is only implicitly given, but of the same order as in (2.2).

We shall assume throughout the paper that

$$\varepsilon \leqslant CN^{-1}$$

as is generally the case for discretizations of convection-dominated problems.

Next, we sketch the analysis of Stynes and O'Riordan for an S-mesh with $\sigma = 2$ assuming that the reader knows interpolation error estimates for such meshes (see [2], for instance). Let us introduce (v, w) for the L_2-scalar product and define the bilinear form of the variational formulation:

$$a(v, w) := \varepsilon(v', w') + (bv, w') + ((c + b')v, w).$$

Further, we denote by u^I the linear interpolant of u and introduce

$$\eta = u^I - u, \quad \chi = u^I - u^N.$$

Since $u - u^N = \chi - \eta$, we estimate χ instead of $u - u^N$: With $\alpha = \min(1, \gamma)$ we obtain

$$\alpha \|\chi\|_\varepsilon^2 \leqslant a(\chi, \chi) = a(\eta, \chi) = \varepsilon(\eta', \chi') + (b\eta, \chi') + ((c + b')\eta, \chi).$$

To estimate the first and the third term we simply use the Cauchy-Schwarz inequality. For the crucial convective term Stynes and O'Riordan use the following tricks:

(i) Apply the inverse inequality—the mesh is uniform—on $[\lambda, 1]$:

$$|(b\eta, \chi')_{(\lambda,1)}| \leqslant CN \|\eta\|_{0,(\lambda,1)} \, \|\chi\|_{0,(\lambda,1)}.$$

This is sufficient, because $\|\eta\|_{0,(\lambda,1)} \leqslant CN^{-2}$.

(ii) Apply an L_∞/L_1 estimate combined with the Cauchy-Schwarz inequality on $[0, \lambda]$:

$$|(b\eta, \chi')|_{(0,\lambda)} \leqslant C\|\eta\|_{\infty,(0,\lambda)} \, \|\chi\|_{L_1} \leqslant C\|\eta\|_{\infty,(0,\lambda)} \, \|\chi\|_\varepsilon \frac{\lambda^{1/2}}{\varepsilon^{1/2}}.$$

For a Shishkin mesh we have

$$\|\eta\|_{\infty,(0,\lambda)} \leqslant C(N^{-1} \ln N)^2 \quad \text{and} \quad \frac{\lambda^{1/2}}{\varepsilon^{1/2}} = O((\ln N)^{1/2}).$$

But for a B-type mesh the application of (ii) yields the factor $(\ln 1/\varepsilon)^{1/2}$ in the error estimate. Practically, it may not be important, but it is interesting from the theoretical point of view: Is it possible to avoid this dependence on ε?

3. A modification of the proof and a negative result

It is well known [2] that the solution of problem (1.1) can be decomposed into a smooth part S and a layer part E with $u = S + E$ and (for every given positive integer l)

$$|E^{(k)}(x)| \leqslant C\varepsilon^{-k} \mathrm{e}^{-x/\varepsilon} \quad \text{for } k = 0, 1, \ldots, l. \tag{3.1}$$

Here S satisfies $LS = f$, whereas $LE = 0$. This allows us to estimate $S - S^N$ and $E - E^N$ separately.

On the Shishkin mesh we can now estimate the crucial convective part as follows. For the smooth part S we use

$$|(S - S^I, \chi')_{(x_{i-1},x_i)}| \leqslant Ch_i \|\chi\|_{0,(x_{i-1},x_i)}.$$

For the layer part E on $[\lambda, 1]$ we use the smallness of E. More precisely: Assuming $\sigma \geqslant 2$, we have $\|E\|_\infty \leqslant CN^{-2}$ on $[\lambda, 1]$. As a consequence, it holds for $\eta_E = E - E^I$

$$|(\eta_E, \chi')|_{(\lambda,1)} \leqslant C\|\eta_E\|_{\infty,(\lambda,1)} N \,\|\chi\|_{0,(\lambda,1)}.$$

On $(0, \lambda)$, however, we estimate

$$|(\eta_E, \chi')|_{(0,\lambda)} \leqslant \varepsilon^{-1/2}\|\eta_E\|_{0,(0,\lambda)}\varepsilon^{1/2}\,\|\chi'\|_{0,(0,\lambda)}.$$

Since $|E|_2 \leqslant C\varepsilon^{-3/2}$, we get for the L_2 norm of the interpolation error

$$\|\eta_E\|_{0,(0,\lambda)} \leqslant Ch^2|E|_{2,(0,\lambda)} \leqslant C\varepsilon^{1/2}(N^{-1}\ln N)^2.$$

The additional factor $\varepsilon^{1/2}$ allows to complete the proof of (1.4).

Therefore, we ask the following question: what optimal error estimates can be proved for the layer part E in the layer region $[0, \lambda]$ of a Bakhvalov-type mesh? Do we have

$$\|\eta_E\|_{0,(0,\lambda)} \leqslant C\varepsilon^{1/2}N^{-2}? \tag{3.2}$$

Unfortunately, it turns out that in general we do not have (3.2) for a Bakhvalov-type mesh.

To see this fact we compute $\int_{x_{N/2-1}}^{x_{N/2}} \eta_E^2$ explicitly for $E = \exp(-x/\varepsilon)$. We have

$$h_{N/2} = x_{N/2} - x_{N/2-1} = \sigma\varepsilon\ln\Bigl(1 + 2\frac{1-\varepsilon}{\varepsilon N}\Bigr) \tag{3.3}$$

and for the interpolation error η_i on (x_{i-1}, x_i)

$$\eta_i = \frac{x - x_{i-1}}{h_i}D_* + \mathrm{e}^{-x/\varepsilon} - \mathrm{e}^{-x_{i-1}/\varepsilon} \quad \text{with} \quad D_* = \mathrm{e}^{-x_{i-1}/\varepsilon} - \mathrm{e}^{-x_i/\varepsilon}.$$

A direct computation yields

$$\int_{x_{i-1}}^{x_i} \eta_i^2 = \frac{h_i}{3}D_*^2 + 2\frac{\varepsilon^2}{h_i}D_*^2 + h_i\,\mathrm{e}^{-(x_{i-1}+x_i)/\varepsilon} - \frac{3}{2}\varepsilon(\mathrm{e}^{-2x_{i-1}/\varepsilon} - \mathrm{e}^{-2x_i/\varepsilon}).$$

We want to extract the factor ε. With some terms we have no difficulties but the first term with the factor h_i causes troubles, mostly on the interval $(x_{N/2-1}, x_{N/2})$. The mesh size $h_{N/2}$ contains the factor $\ln\bigl(1 + 2(1-\varepsilon)/\varepsilon N\bigr)$ which grows logarithmically as $\varepsilon \to 0$. This factor cannot be compensated by $\mathrm{e}^{-x_{N/2-1}/\varepsilon}$, because

$$\mathrm{e}^{-x_{N/2-1}/\varepsilon} = (\varepsilon + 2(1-\varepsilon)N^{-1})^\sigma$$

is not uniformly small with respect to ε. Thus, we do not have for a Bakhvalov-type mesh

$$\|\eta_E\|_{0,(0,\lambda)} \leqslant C\varepsilon^{1/2} g(N)$$

with C independent of ε.

To prove

$$\|u - u^N\|_\varepsilon \leqslant CN^{-1}$$

for a Bakhvalov-type mesh, we propose to use a quasi-interpolant instead of the usual interpolant in the next section.

4. The optimal energy norm estimate on a Bakhvalov-type mesh

Let us start to fix some more or less known properties of a B-mesh which we want to use. It is obvious that the mesh sizes

$$h_i = \sigma\varepsilon \ln \frac{1 - 2(1-\varepsilon)(i-1)N^{-1}}{1 - 2(1-\varepsilon)iN^{-1}}, \quad i = 1, \dots, \tfrac{1}{2}N,$$

form an increasing sequence. We have

$$\mathrm{e}^{h_i/\varepsilon} \leqslant C, \quad h_i \leqslant CN^{-1}, \quad \text{and} \quad \frac{h_i}{h_{i-1}} \leqslant C \ \text{ for } i \leqslant \tfrac{1}{2}N - 1, \tag{4.1}$$

the final mesh size $h_{N/2}$ of the fine mesh plays an exceptional role. Note that the middle inequality is also valid for $i = \frac{1}{2}N$.

Due to (3.1) the layer function E satisfies

$$|E(x_{N/2})| \leqslant C\varepsilon^\sigma \quad \text{and} \quad |E(x_{N/2-1})| \leqslant C(\varepsilon + N^{-1})^\sigma. \tag{4.2}$$

In the following, we shall estimate $S - S^N$ and $E - E^N$ separately. For the smooth part S the diffusion and the reaction part are no problem at all. The convective term is estimated as follows:

$$\left| \int_0^1 \eta_S \chi' \right| \leqslant \|\eta_S\|_0 \, \|\chi'\|_0.$$

Application of the estimate for the L_2-error of the standard linear interpolation and the inverse estimate result in

$$\left| \int_0^1 \eta_S \chi' \right| \leqslant CN^{-1} \|\chi\|_0.$$

Therefore, it follows that

$$\|S - S^N\|_\varepsilon \leqslant CN^{-1}.$$

To estimate $E - E^N$ we recall that E satisfies

$$LE = 0, \quad E(0) = e_0, \;\; E(1) = e_1 \quad \text{with } \; |e_1| \leqslant C\mathrm{e}^{-1/\varepsilon}.$$

Instead of the usual interpolation operator in the finite element space we use a quasi-interpolant with improved stability properties. Such a quasi-interpolant is often used in a posteriori error analysis, since there one needs an interpolant which is defined for H^1-functions (and not only for continuous functions). Let us sketch the construction of that interpolant and note its basic properties, following [3], page 37.

For every mesh point x_i we denote by Δ_i the macroelement (x_{i-1}, x_{i+1}). Let $P_1(\Delta_i)$ be the space of linear polynomials on Δ_i. To a given function $v \in L_1$ we define the local L_2-projection $p_i \in P_1(\Delta_i)$ by

$$\int_{\Delta_i} p_i r = \int_{\Delta_i} vr \quad \text{for all } \; r \in P_1(\Delta_i).$$

Denoting the usual nodal basis of V^N by $\{\varphi_i\}$, our projection operator is defined by

$$v^\pi(x) := \sum_i p_i(x_i)\varphi_i(x). \tag{4.3}$$

Let e be an element, that means some interval (x_{i-1}, x_i). We denote by Δ the union of the two corresponding macroelements Δ_{i-1} and Δ_i and by H the diameter of Δ. Then, in contrast to the usual interpolation operator now we have the stability property

$$\|v^\pi\|_{L^q(e)} \leqslant C\|v\|_{L^q(\Delta)} \quad \text{for } \; 1 \leqslant q \leqslant \infty. \tag{4.4}$$

Denoting by D the operator of differentiation, the approximation error can be estimated by

$$\|v - v^\pi\|_{L^q(e)} \leqslant CH^l\|D^l v\|_{L^q(\Delta)} \quad \text{for } \; l = 1, 2 \; \text{ and } \; 1 \leqslant q \leqslant \infty. \tag{4.5}$$

Estimates for Dv^π and $D(v - v^\pi)$ in some norm depend more sensitively on the mesh used. Therefore, we shall later present such estimates only in a special situation necessary for our proof.

In principle, now we want to apply the technique of Section 2. Therefore we need estimates for $E - E^\pi$ on our Bakhvalov mesh. We start with the L_2-error.

Let $e = (x_{i-1}, x_i)$ be some subinterval of the fine mesh. Using first the triangle inequality and (4.4), then (4.5), both with $q = 2$, and combining the resulting estimates we obtain

$$\|E - E^\pi\|_{0,e}^2 \leqslant C\varepsilon \min\left[\left(\frac{h_{i+1}}{\varepsilon}\right)^5, 1\right] \mathrm{e}^{-2x_{i-2}/\varepsilon}.$$

Using $\mathrm{e}^{h_i/\varepsilon} \leqslant C$ twice, see (4.1), we obtain for $i \leqslant \frac{1}{2}N - 1$

$$\|E - E^\pi\|_{0,e}^2 \leqslant C\varepsilon \min\left[\left(\frac{h_{i+1}}{\varepsilon}\right)^5, 1\right] \mathrm{e}^{-2x_i/\varepsilon}.$$

Because for $\sigma \geqslant 1/\gamma$

$$\min\left[\left(\frac{h_i}{\varepsilon}\right), 1\right] \mathrm{e}^{-\gamma x_{i-1}/\varepsilon} \leqslant CN^{-1}$$

(see [4], proof of Corollary 2), we obtain for $\sigma \geqslant 5/2$

$$\sum_{i \leqslant N/2-1} \|E - E^\pi\|_{0,e}^2 \leqslant C\varepsilon N^{-4}.$$

On the other hand, the stability estimate (4.4) for $q = 2$ yields

$$\|E - E^\pi\|_{0,e}^2 \leqslant C\varepsilon\, \mathrm{e}^{-2x_{i-2}/\varepsilon}.$$

Thus for $i \geqslant \frac{1}{2}N$ every term has the order $O(\varepsilon N^{-2\sigma})$. Summation gives us the desired estimate:

$$\|E - E^\pi\|_0 \leqslant C\varepsilon^{1/2} N^{-2}. \tag{4.6}$$

Remark 1. For the standard interpolation operator in the next step it would be useful to prove approximation error estimates in the L_∞-norm, because then we obtain automatically estimates in $\varepsilon^{1/2}|\cdot|_1$ as well. But for the quasi-interpolant this is not so easy, since $(v - v^\pi)(x_i) \neq 0$, in general.

Therefore, we just mention that similarly as above we could prove for $\sigma \geqslant 2$

$$\|E - E^\pi\|_\infty \leqslant CN^{-2}.$$

In the next step we estimate the interpolation error in the weighted H^1-seminorm explicitly. Again we choose some subinterval of the fine mesh $e = (x_{i-1}, x_i)$ but assume $i \leqslant \frac{1}{2}N - 2$. Then, the boundedness of h_{i+1}/h_i allows the stability estimate

$$|E^\pi|_{1,e} \leqslant C|E|_{1,\Delta}$$

and the approximation error estimate

$$|E - E^\pi|_{1,e} \leqslant CH|E|_{2,\Delta}.$$

The combination of both estimates results in

$$|E - E^\pi|^2_{1,e} \leqslant C\varepsilon^{-1} \min\left[\Big(\frac{h_{i+1}}{\varepsilon}\Big)^3, 1\right] \mathrm{e}^{-2x_{i-2}/\varepsilon}.$$

Summation gives similarly as above

$$\sum_{i\leqslant N/2-2} |E - E^\pi|^2_{1,e} \leqslant C\varepsilon^{-1}N^{-2}.$$

For the remaining subintervals the smallness of E is used. First the triangle inequality and an inverse inequality result in

$$|E - E^\pi|_{1,(x_{i-1},x_i)} \leqslant |E|_{1,(x_{i-1},x_i)} + \frac{1}{h_i}|E^\pi|_{0,(x_{i-1},x_i)}.$$

For $i \geqslant \frac{1}{2}N - 1$ we have $h_i \geqslant C\varepsilon$, further we use the L_2-stability of the projection operator:

$$|E - E^\pi|_{1,(x_{i-1},x_i)} \leqslant |E|_{1,(x_{i-1},x_i)} + \frac{C}{\varepsilon^{1/2}}\,\mathrm{e}^{-x_{i-2}/\varepsilon}.$$

Summation over all subintervals finally results in

$$\varepsilon^{1/2}|E - E^\pi|_1 \leqslant CN^{-1}. \tag{4.7}$$

As mentioned in Remark 1 the interpolant E^π does not satisfies the boundary conditions for E. Therefore, a small modification in the estimation of $\|E - E^N\|_\varepsilon$ in comparison to Section 3 is necessary. We start from

$$\alpha\|E - E^N\|^2_\varepsilon \leqslant a(E - E^N, E - E^N) = a(E - E^N, E - E^\pi + \kappa^N).$$

Here κ^N is piecewise linear on the given mesh and corrects the boundary conditions:

$$\kappa^N(x_i) = \begin{cases} -(E - E^\pi) & \text{for } i = 0, N, \\ 0 & \text{otherwise.} \end{cases}$$

Now we estimate $a(E - E^N, E - E^\pi + \kappa^N)$ as in Section 3 using the smallness of E and E^π on $[\lambda, 1]$ and estimates (4.6) and (4.7). In addition, we need similar estimates for the correction term κ^N.

The correction κ^N lives only on $[0,x_1]$ and $[x_{N-1},1]$ and is exponentially small on the last subinterval. On the first subinterval we get

$$\left(\int_0^{x_1}(\kappa^N)^2\right)^{1/2} \leqslant CN^{-2}(\varepsilon N^{-1})^{1/2} = C\varepsilon^{1/2}N^{-5/2}.$$

An inverse estimate then results in

$$\left(\int_0^{x_1}\left((\kappa^N)'\right)^2\right)^{1/2} \leqslant C\varepsilon^{1/2}N^{-5/2}/(\varepsilon N^{-1}) = C\varepsilon^{-1/2}N^{-3/2}.$$

Thus we have all ingredients to conclude that $\|E-E^N\|_\varepsilon \leqslant CN^{-1}$. Together with the estimate for the smooth part S this gives our main result:

Theorem 1. *The error of the finite element method with linear elements on a Bakhvalov-type mesh with $\sigma \geqslant 5/2$ satisfies*

$$\|u-u^N\|_\varepsilon \leqslant CN^{-1}.$$

References

[1] *A. S. Bakhvalov*: On the optimization of methods for solving boundary value problems with boundary layers. Zh. Vychisl. Mat. Mat. Fiz. *9* (1969), 841–859. (In Russian.)
[2] *T. Linss*: Layer-adapted meshes for convection-diffusion problems. Comput. Methods Appl. Mech. Engrg. *192* (2003), 1061–1105.
[3] *M. Dobrowolski*: Finite Elemente. University textbook of Würzburg, 1998.
[4] *N. Kopteva, M. Stynes*: Approximation of derivatives in a convection-diffusion two-point boundary value problem. Appl. Numer. Math. *39* (2001), 47–60.
[5] *H.-G. Roos, T. Linss*: Sufficient conditions for uniform convergence on layer-adapted grids. Computing *63* (1999), 27–45.
[6] *H.-G. Roos, M. Stynes, L. Tobiska*: Numerical Methods for Singularly Perturbed Differential Equations. Convection-Diffusion and Flow Problems. Springer-Verlag, Berlin, 1996.
[7] *W. W. Shaidurov, B. M. Bagaev, and E. D. Karepova*: Numerical Methods for Problems with Boundary Layers II. Nauka, Novosibirsk, 2002. (In Russian.)
[8] *M. Stynes, E. O'Riordan*: A uniformly convergent Galerkin method on a Shishkin mesh for a convection-diffusion problem. J. Math. Anal. Appl. *214* (1997), 36–54.

Author's address: *H. G. Roos*, Institute of Numerical Mathematics, Dept. of Mathematics, Technical University of Dresden, Zellerscher Weg 12–14, 01069 Dresden, Germany, e-mail: `roos@math.tu-dresden.de`.