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# NONLINEAR BOUNDED CONTROL FOR TIME-DELAY SYSTEMS

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A method to derive a nonlinear bounded state feedback controller for a linear continuous-time system with time-delay in the state is proposed. The controllers are based on an $e$-parameterized family of algebraic Riccati equations or on an $e$-parameterized family of LMI optimization problems. Hence, nested ellipsoidal neighborhoods of the origin are determined. Thus, from the Lyapunov–Krasovskii theorem, the uniform asymptotic stability of the closed-loop system is guaranteed and a certain performance level is attained through a quadratic cost function.

## 1. INTRODUCTION

When dealing with the control design problem, several constraints have to be taken into account in order to obtain a control which operates in practice. Among them, the limitations of actuators are particularly important because they have a direct incidence on the closed-loop system stability and it is not surprising that this problem concentrated the attention of many researchers. See for example [1, 14] and bibliography therein.

The presence of delays in the system is also a source of closed-loop system instability. Some recent results on the control of linear systems with delayed state and bounded inputs have been obtained, see [3, 6, 7, 10] (for independent delay size) or [4, 15] (for dependent delay size). To derive these results, matrix measures, complex Lyapunov equations or Razumikhin-type theorems were used. To have an overview of the more recent results, see [4, 9, 11, 12] or the papers published in this field in the last international conferences as Conference on Decision and Control 2000 or American Control Conference 2000.

This paper presents a method to derive a nonlinear bounded state feedback controller for a linear continuous-time system with time-delay in the state. The idea is based on the existence, under some conditions, of an $e$-parameterized family of bounded linear state feedbacks which asymptotically stabilize the closed-loop system. These controllers are designed from the solutions to an $e$-parameterized family of algebraic Riccati equations or an $e$-parameterized family of LMI optimization problems. From these solutions, it is possible to define invariant ellipsoidal neighborhoods of the origin such that inside them, the control does not saturate. For a Riccati equation approach, when $e = 1$, the ellipsoid corresponds to a region for which the system exhibits a satisfactory behavior. When $e \to +\infty$, the ellipsoid tends to a subset of $\Re^n$. Then it is possible to build a state dependent function $e(x)$, which is used to derive the controller. For the LMI approach, when $e = 1$ we obtain an ellipsoid in which the system has a good behavior, and when $e = 0$ we obtain a subset of $\Re^n$ in which the size of the control gain is low.

It is important to note that this approach is based on a Lyapunov–Krasovskii theorem for analyzing the uniform asymptotic stability of solutions to functional differential equations. A certain performance level for the closed-loop system is taken into account through a quadratic cost function. Modifying slightly the obtained results, it is also possible to deal with model uncertainties. The paper is organized as follows. In the next section, the problem is stated. Section 3 introduces some preliminaries used in Section 4 which addresses the case of the Riccati equation approach. Section 5 presents the Linear Matrix Inequalities method and Section 6 proposes an illustrative example. Finally, a conclusion ends the paper.

**Notations.** $\Re$ denotes the set of real numbers, $\Re^+$ is the set of non-negative real numbers, $\Re^n$ denotes the $n$ dimensional Euclidean space, and $\Re^{n\times m}$ denotes the set of all $n \times m$ real matrices. The notation $X \geq Y$ (respectively, $X > Y$), where $X$ and $Y$ are symmetric matrices, means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). For any real matrix $A$, $A'$ and $A_{(i)}$ denote the transpose and the $i$th row of matrix $A$, respectively. $I$ denotes the identity matrix of appropriate dimensions. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote respectively the maximal and minimal eigenvalue of matrix $P$. $\mathcal{C}_\tau = \mathcal{C}([-\tau, 0], \Re^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\Re^n$ with the topology of uniform convergence. The following norms will be used: $\|\cdot\|$ refers to either the Euclidean vector norm or the induced matrix 2-norm. $\|\phi\|_c = \sup_{-\tau\leq t\leq 0} \|\phi(t)\|$ stands for the norm of a function $\phi \in \mathcal{C}_\tau$. When the delay is finite then "sup" can be replaced by "max". Moreover, we denote by $\mathcal{C}_\tau^v$ the set defined by $\mathcal{C}_\tau^v = \{\phi \in \mathcal{C}_\tau \; ; \; \|\phi\|_c \leq v\}$, where $v$ is a positive real number.

## 2. PROBLEM STATEMENT

Consider the time-delay linear system described by:

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + Bu(t) \tag{1}$$

with the initial condition

$$\begin{gathered} x(t_0 + \theta) = \phi(\theta), \quad \forall\, \theta \in [-\tau, 0], \quad (t_0, \phi) \in \Re^+ \times \mathcal{C}_\tau^v \\ x(t_0) = x_0 \end{gathered} \tag{2}$$

where $x(t) \in \Re^n$ is the state, $u(t) \in \Re^m$ is the control input, $\tau$ is the time-delay of the system. $A$, $A_d$ and $B$ are constant matrices of appropriate dimensions and pair

$(A, B)$ is supposed to be stabilizable. We assume now that the control takes values in a compact set:

$$\mathcal{U} = \left\{u \in \Re^m; -u_{0(i)} \le u_{(i)} \le u_{0(i)},\ u_{0(i)} > 0,\ i = 1, \dots, m\right\}. \tag{3}$$

Associated with system (1)-(2), let us define the following quadratic cost function which defines a performance criterion:

$$\begin{aligned} J &= \int_0^{+\infty} (x(t)'Qx(t) + u(t)'Ru(t))\,\mathrm{d}t \\ Q &= Q' > 0, \quad R = R' > 0. \end{aligned} \tag{4}$$

The problem addressed in this paper is to find a control $u(x)$ such that for all $t$, $u(x) \in \mathcal{U}$ and such that system (1) is asymptotically stable. Moreover, among all possible controls satisfying these properties, we want to select a control which minimizes $J$. In order to solve this problem, some preliminaries are introduced in the following section.

## 3. PRELIMINARY RESULTS

In [17], the problem of designing a linear state feedback which stabilizes system (1) is addressed. An important result stated in this paper is presented in the following lemma.

**Lemma 1.** Given symmetric and positive definite matrices $Q$ and $R$, if there exist two symmetric and positive definite matrices $P$ and $S$ solutions to

$$A'P + PA + PA_dS^{-1}A_d'P - PBR^{-1}B'P + S + Q = 0 \tag{5}$$

then system (1) closed by the state feedback

$$u = Kx = -R^{-1}B'Px \tag{6}$$

is asymptotically stable for all initial conditions $\phi \in \mathcal{B}(\sigma)$ where $\mathcal{B}(\sigma)$ is defined by:

$$\begin{aligned} &\mathcal{B}(\sigma) = \{\phi \in \mathcal{C}_\tau^v; \|\phi\|_c^2 \le \sigma\} \\ &\text{with } \sigma = \frac{\gamma}{\lambda_{\max}(P) + \tau\lambda_{\max}(S)} \end{aligned} \tag{7}$$

$\gamma > 0$ corresponds to the largest ellipsoid

$$\mathcal{D}(P, \gamma) = \{x \in \Re^n; x'Px \le \gamma\} \tag{8}$$

contained in $\mathcal{U}$.

The proof is obtained by showing that

$$\begin{aligned} V(x_t) &= x(t)'Px(t) + \int_{t-\tau}^{t} x(\theta)'Sx(\theta)\,\mathrm{d}\theta \\ P &= P' > 0\,, \quad S = S' > 0 \end{aligned} \tag{9}$$

where $x_t$, $\forall\, t \geq t_0$, denotes the restriction of $x$ to the interval $[t-\tau\,,\,t]$ translated to $[-\tau\,,\,0]$, that is,

$$x_t(\theta) = x(t+\theta),\ \forall\,\theta \in [-\tau\,,\,0],$$

is a Lyapunov functional for the closed-loop system.

If the system satisfies Lemma 1, it is asymptotically stable and as described above, $V(x_t)$ defined by (9) is a Lyapunov functional for the closed-loop system. We can write:

$$\begin{aligned}
\frac{\mathrm{d}V(x_t)}{\mathrm{d}t} &= x(t)'[(A+BK)'P + P(A+BK)]x(t) + 2x(t)'PA_dx(t-\tau) \\
&\quad + x(t)'Sx(t) - x(t-\tau)'Sx(t-\tau) \\
&\leq x(t)'[(A+BK)'P + P(A+BK)]x(t) + x(t)'Sx(t) \\
&\quad + x(t)'PA_dS^{-1}A_d'Px(t) \\
&\leq -x(t)'[Q+K'RK]x(t) \quad \text{by Lemma 1.}
\end{aligned}$$

Then:

$$\begin{aligned}
J &= \int_0^{+\infty} x(t)'[Q+K'RK]x(t)\,\mathrm{d}t \\
&\leq -\int_0^{+\infty} \mathrm{d}V(x_t) = V(x(0)) \quad \text{because the system is stable.}
\end{aligned}$$

We have:

$$J \leq x_0'Px_0 + \int_{-\tau}^{0} x(\theta)'Sx(\theta)\,\mathrm{d}\theta.$$

This inequality suggests the following optimization problem in order to minimize $J$.

$$(\mathrm{P1})\ \begin{cases} \min\left\{\mathrm{trace}(Px_0x_0') + \mathrm{trace}\left(S\displaystyle\int_{-\tau}^{0} x(\theta)\,x(\theta)'\mathrm{d}\theta\right)\right\} \\ \text{under } P = P' > 0,\ S = S' > 0, \text{ and (5).} \end{cases}$$

We can note that the criterion is linear with respect to $P$ and $S$. But $P$ and $S$ appears nonlinearly in (5).

A possibility to solve (5) by standard algorithms consists in fixing $S$ or by using LMI formulation (see Section 5). In fact, a compromise has to be found between the value of $J$ and the size of the initial condition domains (7) and (8). □

As pointed out in the introduction, we can consider model uncertainties. For simplicity, consider uncertainty on matrix $A$ such that

$$\Delta A = A_0 + DFE \tag{10}$$

where $A_0$, $D$ and $E$ are constant matrices of appropriate dimensions and $F'F \leq I$. In this case the Riccati equation (5) is replaced by [13]:

$$A_0'P + PA_0 + PA_dS^{-1}A_d'P + \epsilon PDD'P - PBR^{-1}B'BP + S + \epsilon^{-1}E'E + Q = 0 \quad (11)$$

where the unknowns are $P$, $S$ and $\epsilon > 0$.

## 4. RICCATI EQUATION APPROACH

### 4.1. Single input case

Recall that the control is constrained to belong to $\mathcal{U}$. For the sake of simplicity, consider in a first time, the single-input case, i. e., $u \in [-u_0, u_0]$ and $B \in \Re^n$.

Define:

$$\mathcal{E} = \left\{ x \in \Re^n; x'Px \leq \frac{u_0^2}{R^{-1}B'PBR^{-1}} = c \right\}. \quad (12)$$

Thus, we have the following result.

**Lemma 2.** $\mathcal{E}$ is the maximal ellipsoid defined by the quadratic form $x'Px$ where the feedback $u = -R^{-1}B'Px$ is bounded by $u_0$.

Proof. See [5]. □

The idea in this paper is to fully use the capabilities of actuators without allowing the control saturation. If that is possible, it is hoped that the performance of the system in terms of a speed response will be better. For that, suppose the optimization problem (P1) has been solved obtaining a good compromise between performances and size of initial conditions domains. Then there exist symmetric positive definite matrices $P$ and $S$ solutions to Riccati equation (5):

$$A'P + PA + PA_dS^{-1}A_d'P - PBR^{-1}B'P + S + Q = 0 \quad (13)$$

where $R$ and $Q$ are symmetric positive definite matrices. Now the idea is to parameterize Riccati equation (13) in the following way:

$$A'P(e) + P(e)A + P(e)A_dS^{-1}A_d'P(e) - P(e)BR^{-1}B'P(e) + S + \frac{Q}{e} = 0 \quad (14)$$

with $1 \leq e < \infty$. It is to be noted that $S$ is maintained as constant, only $P$ varies with $e$. For $e = 1$, we recover Riccati equation (13). The first step is to verify that if (14) has a positive definite solution for $e = 1$, it has also a solution for $e > 1$.

**Lemma 3.** Suppose that $P(1) = P(1)' > 0$ and $S = S' > 0$ satisfy Riccati equation (13). Then for all $e > 1$, Riccati equation (14) has a positive definite symmetric solution $P(e) > 0$ and $P(1) \geq P(e)$.

Proof. See [5]. □

When $e \to +\infty$, Riccati equation (14) reads

$$A'P(\infty) + P(\infty)A + P(\infty)A_dS^{-1}A_d'P(\infty) - P(\infty)BR^{-1}B'P(\infty) + S = 0.$$

Recall that $S$ is fixed. Then it follows that [5]:

$$P(1) \geq P(\infty) > 0.$$

Define:

$$\begin{aligned} \mathcal{E}(e) &= \{x \in \Re^n; x'P(e)\,x \leq c(e)\} \\ \text{with } c(e) &= \frac{u_0^2}{R^{-1}B'P(e)BR^{-1}}. \end{aligned} \tag{15}$$

**Lemma 4.** $\mathcal{E}(e)$ is the maximal ellipsoid defined by the quadratic function $x'P(e)\,x$ where the feedback $u = -R^{-1}B'P(e)\,x$ is bounded by $u_0$.

In the sequel, we normalize the ellipsoids (15) defining the positive definite matrix $\mathcal{X}(e) = \frac{P(e)}{c(e)}$, $1 \leq e < \infty$,

$$\Xi(e) = \{x \in \Re^n; x'\mathcal{X}(e)\,x \leq 1\}. \tag{16}$$

**Lemma 5.** $\frac{\mathrm{d}\mathcal{X}(e)}{\mathrm{d}e}$ is negative definite and $\frac{\mathrm{d}c(e)}{\mathrm{d}e} \geq 0$.

Proof. See [5]. □

Following the same lines as in [16], we can prove the following lemmas.

**Lemma 6.** $\Xi(e_1) \subset \mathrm{Int}(\Xi(e_2))$ whenever $e_1 < e_2$, where $\mathrm{Int}(\Xi(e_2))$ denotes the interior of the set $\Xi(e_2)$.

**Lemma 7.** The $e$-parameterized family of ellipsoids $\Xi(e)$ is a nested family set, that is,

$$\Xi(e_1) \subset \mathrm{Int}(\Xi(e_2)) \text{ whenever } e_1 < e_2$$

with a maximal element $\Xi = \bigcup_e \Xi(e)$.

Define the following set:

$$\begin{aligned} \mathcal{B}(e) &= \{\phi \in \mathcal{C}_\tau^v; \|\phi\|_c^2 \leq \tilde{c}(e)\} \\ \text{with } \tilde{c}(e) &= \frac{c(e)}{\lambda_{\max}(P(e)) + \tau\lambda_{\max}(S)}. \end{aligned} \tag{17}$$

It is clear that we have the following result.

**Lemma 8.** The $e$-parameterized family of sets $\mathcal{B}(e)$ has the following property:

$$\mathcal{B}(e_1) \subseteq \mathcal{B}(e_2) \text{ whenever } e_1 < e_2.$$

Proof. The proof follows from the fact that $c(e)$ is an increasing function and that $P(e)$ is a decreasing function of $e$. □

In order to derive the controller, we introduce the function $e(x)$ which is the positive solution to the equation given by

$$x(t)'P(e)\,x(t) + \int_{t-\tau}^{t} x(\theta)'Sx(\theta)\,\mathrm{d}\theta - c(e) = 0 \tag{18}$$

where $P(e)$ is the positive definite symmetric solution to Riccati equation (14). First observe that for a fixed $e$, $\mathcal{E}(e)$ is the largest ellipsoid where the control is bounded by $u_0$, and for all $\phi \in \mathcal{B}(e)$, $x(t) \in \mathcal{E}(e)$.

On the other hand if we define:

$$f(x,e) = x(t)'P(e)\,x(t) + \int_{t-\tau}^{t} x(\theta)'Sx(\theta)\,\mathrm{d}\theta - c(e),$$

one gets $\frac{\partial f}{\partial e} = x(t)'\frac{\mathrm{d}P(e)}{\mathrm{d}e}x(t) - \frac{\mathrm{d}c(e)}{\mathrm{d}e} \neq 0$ and this is a sufficient condition for the differentiability of $e(x)$. The idea behind the definition of $e(x)$ is to have a state dependent function $e(x)$ which takes large values when the system trajectory is far from the origin and small values when the system trajectory is close to the origin. $e(x)$ continuously changes for all $x \in \Re^n\backslash\{0\}$. $e(x)$ can be interpreted as a distance from the origin. The condition $e(x) \leq \mu$, $\mu > 0$ defines a set such that $x$ satisfies

$$x(t)'P(\mu)\,x(t) + \int_{t-\tau}^{t} x(\theta)'Sx(\theta)\,\mathrm{d}\theta \leq c(\mu)$$

if $\phi \in \mathcal{B}(\mu)$. Also, if $\phi \in \mathcal{B}(\mu)$ we can conclude that $x(t) \in \Xi(\mu)$.

We are now in position to introduce the controller. It is defined by:

$$u(x) = \begin{cases} -R^{-1}B'P(e(x))\,x(t) & \text{if} \quad x(t) \in \Xi\backslash\Xi(1) \\ -R^{-1}B'P(1)\,x(t) & \text{if} \quad x(t) \in \Xi(1). \end{cases} \tag{19}$$

**Theorem 1.** Suppose that Riccati equation (13) has positive symmetric solutions $P = P' > 0$ and $S = S' > 0$. Let $P(e)$ the positive definite symmetric solution to Riccati equation (14) and $\Xi(e)$ the ellipsoid defined by (16). Then the controller defined in (19) satisfies the constraint $-u_0 \leq u(x) \leq u_0$ and stabilizes asymptotically the system (1) for all initial conditions $\phi \in \mathcal{B}(1)$.

Proof. To prove that system (1) is asymptotically stable, we have to show that $\Xi(1)$ is a finite attractor in $\Xi$ for the closed-loop system. For this, it suffices to show that for all $x$ in the closure of $\Xi\backslash\Xi(1)$, the time-derivative $\dot{e}$ is negative along the closed-loop vector fields. We have from (18)

$$2x(t)'P(e)\dot{x}(t)+x(t)'\frac{\mathrm{d}P(e)}{\mathrm{d}e}x(t)\dot{e}(t)-\frac{\mathrm{d}c(e)}{\mathrm{d}e}\dot{e}(t)+x(t)'Sx(t)-x(t-\tau)'Sx(t-\tau)=0$$

and therefore

$$\dot{e}(t)=-\frac{2x(t)'P(e)\dot{x}(t)+x(t)'Sx(t)-x(t-\tau)'Sx(t-\tau)}{x(t)'\frac{\mathrm{d}P(e)}{\mathrm{d}e}x(t)-\frac{\mathrm{d}c(e)}{\mathrm{d}e}}.$$

But one gets $2x(t)'P(e)\dot{x}(t)+x(t)'Sx(t)-x(t-\tau)'Sx(t-\tau)=-x(t)'Qx(t)-x(t)'PBR^{-1}B'Px(t)-[x(t-\tau)-S^{-1}A_d'Px(t)]'S[x(t-\tau)-S^{-1}A_d'Px(t)]<0$. Noting that $\frac{\mathrm{d}P(e)}{\mathrm{d}e}<0$ and $\frac{\mathrm{d}c(e)}{\mathrm{d}e}>0$, it follows that $\dot{e}(t)<0$. Now if $e(x)\leq\mu$, we have

$$x(t)'P(\mu)\,x(t)+\int_{t-\tau}^{t}x(\theta)'Sx(\theta)\,\mathrm{d}\theta\leq c(\mu)$$

provided that the initial condition $\phi\in\mathcal{B}(\mu)$ and $x(t)\in\Xi(\mu)$. But from Lemma 8, one gets:

$$\mathcal{B}(1)\subseteq\mathcal{B}(\mu),\ \forall\,\mu>1.$$

And then this fact with $\dot{e}(t)<0$ complete the proof of the theorem. □

### 4.2. The multi-inputs case

In this section, we move to the multi-inputs case. Matrix $B\in\Re^{n\times m}$ is written as $B=\begin{bmatrix}B_1 & \dots & B_m\end{bmatrix}$ with $B_i\in\Re^n$, $i=1,\dots,m$, and we take for simplicity $R=\rho^{-1}I>0$.

Define also

$$c_i(e)=\frac{u_{0(i)}^2}{\rho^2B_i'P(e)B_i},\forall\,i=1,\dots,m$$

and

$$\mathcal{C}(e)=\min_i c_i(e).$$

Note from the previous section that $c_i(e)$, $i=1,\dots,m$, are increasing functions of $e$. Hence $\mathcal{C}(e)$ is also an increasing function of $e$. Nevertheless, $\mathcal{C}(e)$ is not necessarily differentiable for any $e>0$, but its right-hand side derivative is well-defined as

$$D\mathcal{C}(e)=\lim_{\xi\to0^+}\frac{\mathcal{C}(e+\xi)-\mathcal{C}(e)}{\xi}.$$

Using this definition and the previous notations, it is possible to extend the results of the previous section as follows.

**Theorem 2.** Let $P(e)$ and $S$ be the positive definite symmetric solutions to Riccati equation (14). Define the function $e(x)$ in the following way:

— For $x(t) \in \Xi(1)$ and $\phi \in \mathcal{B}(1)$ as the positive solution $e(x) = 1$.

— For $x(t) \in \Xi \backslash \Xi(1)$ and $\phi \in \mathcal{B}(1)$ as the positive solution to the equation

$$x(t)' P(e)\, x(t) + \int_{t-\tau}^{t} x(\theta)' S x(\theta)\, \mathrm{d}\theta - \mathcal{C}(e) = 0.$$

Then the control $u(x)$ defined by

$$u(x) = \begin{cases} -R^{-1} B' P(e(x))\, x(t) & \text{if} \quad x(t) \in \Xi \backslash \Xi(1) \\ -R^{-1} B' P(1)\, x(t) & \text{if} \quad x(t) \in \Xi(1) \end{cases} \tag{20}$$

satisfies the constraints $-u_{0(i)} \leq u_{(i)}(x) \leq u_{0(i)}$, $i = 1, \ldots, m$, and drives any point of $\Xi$ to the origin.

Suppose now that $e \to +\infty$, Riccati equation (14) becomes:

$$A' P(\infty) + P(\infty) A + P(\infty) A_d S^{-1} A_d' P(\infty) - P(\infty) B R^{-1} B' P(\infty) + S = 0.$$

If the pair $(S^{\frac{1}{2}}, A)$ is observable and if $(A, B)$ is stabilizable, then $P(\infty)$ is positive definite. Hence the set $\Xi$ is characterized by the following theorem.

**Theorem 3.** Suppose that pair $(A, B)$ is stabilizable. Then

$$\Xi = \left\{ x \in \Re^n ; \max_i \left[ \frac{\rho^2 B_i' P(\infty) B_i}{u_{0(i)}^2} \right] x' P(\infty)\, x < 1 \right\}, \quad i = 1, \ldots, m.$$

Proof. The proof is a direct consequence of assumptions and elementary results on the behavior of the solutions to a Riccati equation. □

From a practical point of view, it is not possible to solve equation (18). In practice, to implement the control it is possible to use the following algorithm.

— *Step 0.* Choose N values of $e$ such that $e_0 = 1 < e_1 < e_2 < \ldots < e_N < \infty$. For $e = e_N$, solve Riccati equation (5). We obtain the corresponding ellipsoid $\Xi(e_N)$, the set $\mathcal{B}(e_N)$ and the control $K(e_N)$. Set

$$\Xi = \{\Xi(e_N)\}, \ \mathcal{B} = \{\mathcal{B}(e_N)\}, \ K = \{K(e_N)\}.$$

— *Step i.* Take $e = e_{N-i}$. Solve Riccati equation (5) for $e = e_{N-i}$. We obtain $\Xi(e_{N-i})$, $\mathcal{B}(e_{N-i})$ and $K(e_{N-i})$. Set

$$\Xi = \{\Xi, \Xi(e_{N-i})\}, \ \mathcal{B} = \{\mathcal{B}, \mathcal{B}(e_{N-i})\}, \ K = \{K, K(e_{N-i})\}.$$

Go to step i+1.

At the end of the algorithm, we obtain a nested family of ellipsoids $\Xi$, sets $\mathcal{B}$ and corresponding control gains. To apply the control, we measure $x(t)$ and identify the outer ellipsoid in $\Xi$, which contains $x(t)$, and the corresponding control is applied. With this method, a piecewise control is obtained.

## 5. LMI APPROACH

Another parameterization of the control gain matrix is possible using an LMI formulation. From (5), it is easy to see that condition of Lemma 1 can be expressed as follows:

Find matrices $K$ and $P = P' > 0$, $S = S' > 0$ such that

$$(A+BK)'P + P(A+BK) + PA_dS^{-1}A_d'P + S < 0.$$

Multiplying on the left and the right by $P^{-1} = W$ and denoting $Y = KW$, we obtain rearranging some terms

$$AW + WA' + BY + Y'B' + \begin{bmatrix} A_dS^{-1} & W \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} S^{-1}A_d' \\ W \end{bmatrix} < 0.$$

Introducing $U = S^{-1}$ we arrive at

$$\begin{bmatrix} AW + WA' + BY + Y'B' & A_dU & W \\ UA_d' & -U & 0 \\ W & 0 & -U \end{bmatrix} < 0$$

We can deduce the following lemma which is similar to Lemma 1 in the context of the LMI formulation.

**Lemma 9.** If there exist a solution $W = W' > 0$, $U = U' > 0$ and $Y$ matrices of appropriate dimensions such that

$$\begin{bmatrix} AW + WA' + BY + Y'B' & A_dU & W \\ UA_d' & -U & 0 \\ W & 0 & -U \end{bmatrix} < 0 \tag{21}$$

$$\begin{bmatrix} W & Y_{(i)}' \\ Y_{(i)} & u_{0(i)}^2 \end{bmatrix} \geq 0, \; i = 1, \ldots, m \tag{22}$$

then system (1) closed by the state feedback

$$u = Kx = YW^{-1} \tag{23}$$

is asymptotically stable for all initial conditions $\phi \in \mathcal{B}(\sigma)$ where $\mathcal{B}(\sigma)$ is defined by:

$$\mathcal{B}(\sigma) = \{\phi \in \mathcal{C}_\tau^v; \|\phi\|_c^2 \leq \sigma\}$$
$$\text{with } \sigma = \frac{1}{\lambda_{\max}(W^{-1}) + \tau\lambda_{\max}(U^{-1})} \tag{24}$$

and

$$\mathcal{D}(W^{-1}, 1) = \{x \in \Re^n; x'W^{-1}x \leq 1\} \tag{25}$$

is contained in $\mathcal{U}$.

The proof follows from the previous manipulations and from Lemma 1. $\mathcal{D}(W^{-1}, 1)$ is contained in $\mathcal{U}$ from inequalities (22) [18]. □

If the model is affected by norm bounded uncertainties, supposing that only matrix $A$ is affected, we can replace (21) by:

$$\begin{bmatrix} AW + WA' + BY + Y'B' + \epsilon DD' & A_d U & W & WE' \\ UA'_d & -U & 0 & 0 \\ W & 0 & -U & 0 \\ EW & 0 & 0 & -\epsilon I \end{bmatrix} < 0.$$

Now to deal with a quadratic cost as defined in (4), a similar development as previously leads to inequality

$$\begin{bmatrix} AW + WA' + BY + Y'B' & A_d U & W & Y' & W \\ UA'_d & -U & 0 & 0 & 0 \\ W & 0 & -U & 0 & 0 \\ Y & 0 & 0 & -R^{-1} & 0 \\ W & 0 & 0 & 0 & -Q^{-1} \end{bmatrix} < 0 \qquad (26)$$

with $K = YW^{-1}$ and

$$J \le \mathrm{trace}(W^{-1} x_0 x'_0) + \mathrm{trace}\left(U^{-1} \int_{-\tau}^{0} x(\theta)x(\theta)' \mathrm{d}\theta\right). \qquad (27)$$

While (26) is linear with respect to the unknowns, it is not easy to minimize $J$ because it is nonlinear in the unknowns. A way to obtain a linear problem consists in minimizing the following problem:

$$(\mathrm{P2}) \begin{cases} \min\limits_{W,U,Y,\gamma,\delta} H(\gamma,\delta) = \gamma\, \mathrm{trace}(x_0 x'_0) + \delta\, \mathrm{trace}\left(\int_{-\tau}^{0} x(\theta)x(\theta)' \mathrm{d}\theta\right) \\ \text{under relations (26), (22)} \\ \begin{bmatrix} \gamma I & I \\ I & W \end{bmatrix} \ge 0, \ \begin{bmatrix} \delta I & I \\ I & U \end{bmatrix} \ge 0. \end{cases}$$

The main advantage is now that the problem is linear. Conditions $\begin{bmatrix} \gamma I & I \\ I & W \end{bmatrix} \ge 0$ and $\begin{bmatrix} \delta I & I \\ I & U \end{bmatrix} \ge 0$ ensure that $\mathrm{trace}(W^{-1}) \le n\gamma$ and $\mathrm{trace}(U^{-1}) \le n\delta$, respectively. Problem (P2) is solvable by an LMI solver when a solution exists. We have the following lemma.

**Lemma 10.** If Problem (P2) is solvable. Then for all initial condition belonging to $\mathcal{B}(\sigma)$ defined in (24), the system is asymptotically stable by the control $u = YW^{-1}x$, which belongs to $\mathcal{U}$.

The idea is to parameterize problem (P2) in order to obtain nested family of ellipsoids and sets $\mathcal{B}(\sigma)$ as in the case of Riccati equation approach. For that, introduce the following optimization problem:

$$
(\mathrm{P3})\quad \begin{cases}
\min\limits_{W,U,Y,\gamma} \{eH(\gamma,\delta) + (1-e)\log(\det(W^{-1}))\} \\[2mm]
\text{under } \begin{bmatrix} W & Y'_{(i)} \\ Y_{(i)} & u^2_{0(i)} \end{bmatrix} \geq 0,\ i = 1,\ldots,m \\[2mm]
\begin{bmatrix} AW + WA' + BY + Y'B' & A_dU & W & eY' & eW \\ UA'_d & -U & 0 & 0 & 0 \\ W & 0 & -U & 0 & 0 \\ eY & 0 & 0 & -R^{-1} & 0 \\ eW & 0 & 0 & 0 & -Q^{-1} \end{bmatrix} < 0 \\[2mm]
0 \leq e \leq 1 \\[2mm]
\begin{bmatrix} \gamma I & I \\ I & W \end{bmatrix} \geq 0,\ \begin{bmatrix} \delta I & I \\ I & U \end{bmatrix} \geq 0.
\end{cases}
$$

When $e = 0$, Problem (P3) reduces to the following problem:

$$
\min_{W,U,Y} \{\log(\det(W^{-1}))\}
$$

$$
\text{under } \begin{bmatrix} W & Y'_{(i)} \\ Y_{(i)} & u^2_{0(i)} \end{bmatrix} \geq 0,\ i = 1,\ldots,m
$$

$$
\begin{bmatrix} AW + WA' + BY + Y'B' & A_dU & W \\ UA'_d & -U & 0 \\ W & 0 & -U \end{bmatrix} < 0.
$$

This problem, when a solution exists, solves the stabilization problem by a control belonging to $\mathcal{U}$ and maximizes the size of $\mathcal{D}(W^{-1}, 1)$ .

When $e = 1$, we obtain Problem (P2) in which performances are taken into account by minimizing $H(\gamma, \delta)$, the size of $\mathcal{D}(W^{-1}, 1)$ being not a priori maximized.

When $0 < e < 1$, we obtain a problem which gives a compromise between the size of $\mathcal{D}(W^{-1}, 1)$ and performances taken into account through $H(\gamma, \delta)$. The control gain depends on the parameter $e$ and is written

$$
K(e) = Y(e)W(e)^{-1} \tag{28}
$$

where $Y(e)$ and $W(e)$ are the solutions obtained by solving (P3).

Now the idea is to let $e$ vary from 0 to 1. When $e_1 < e_2$, since constraints (22) are satisfied, the size of $K(e_1)$ is lower than the one of $K(e_2)$. To be sure that the domains $\mathcal{D}(W^{-1}, 1)$ and $\mathcal{B}(\sigma)$ are nested, for two values of $e$, $e_1$ and $e_2$ such that

$$0 \leq e_1 < e_2 \leq 1$$

we have to impose that

$$\begin{aligned} W(e_1) &< W(e_2) \\ U(e_1) &\leq U(e_2). \end{aligned} \tag{29}$$

As in the Riccati equation approach, since $W$, $U$ and $Y$ are $e$-depending, we denote in the sequel the sets $\mathcal{D}(W^{-1}, 1)$ and $\mathcal{B}(\sigma)$ by $\mathcal{D}(e)$ and $\mathcal{B}(e)$, respectively.

All these remarks suggest the following algorithm to build a piecewise linear control law.

— *Step 0.* Choose $N$ values of $e$ such that $e_N = 0 < e_{N-1} < e_{N-2} < \ldots < e_0 = 1$. Solve LMI problem (P3) for $e = e_0$. We obtain the corresponding ellipsoid $\mathcal{D}(e_0)$, the set $\mathcal{B}(e_0)$ and the control $K(e_0) = Y(e_0)W(e_0)^{-1}$. Set

$$\mathcal{D} = \{\mathcal{D}(e_0)\},\ \mathcal{B} = \{\mathcal{B}(e_0)\},\ K = \{K(e_0)\}$$

— *Step i.* Take $e = e_i$. Solve LMI problem (P3) for $e = e_i$ by adding the constraints:

$$\begin{aligned} W(e_i) &< W(e_{i-1}) \\ U(e_i) &\leq U(e_{i-1}) \end{aligned}$$

We obtain $\Xi(e_i)$, $\mathcal{B}(e_i)$ and $K(e_i)$. Set

$$\mathcal{D} = \{\mathcal{D}, \mathcal{D}(e_i)\},\ \mathcal{B} = \{\mathcal{B}, \mathcal{B}(e_i)\},\ K = \{K, K(e_i)\}$$

Go to step i+1.

At the end of the algorithm we obtain a nested family of ellipsoids $\mathcal{D}$, sets $\mathcal{B}$, with the corresponding control gains. To implement this control, we proceed as for the Riccati equation approach.

### 5.1. Decentralized control

The interest of a solution based on a LMI formulation lies on the possibility of adding some structural constraints provided they do not destroy the linearity of the optimization problem. Among the problems which is possible to investigate, we present in what follows the decentralized state feedback design problem [2]. If the system is formed from geographically separated subsystems, the control is composed of $q$ channels i. e. $u_i \in \Re^{m_i}$, $i = 1, \ldots, q$ and the decentralised state feedback design consists in finding matrices $K_i \in \Re^{m_i \times n_i}$ where:

$$\sum_{i=1}^{q} m_i = m, \quad \sum_{i=1}^{q} n_i = n.$$

A solution to solve this problem is to add to problem (P3) the constraints

$$W = \mathrm{diag}(W_1, \dots, W_q), \quad W_i \in \Re^{n_i \times n_i}, \quad Y = \mathrm{diag}(Y_1, \dots, Y_q), \quad S_i \in \Re^{m_i \times n_i}$$

which are convex and do not destroy the linearity. The control gain can be written:

$$K = YW^{-1} = \mathrm{diag}(Y_1 W_1^{-1}, \dots, Y_q W_q^{-1})$$

and has a diagonal structure.

## 6. ILLUSTRATIVE EXAMPLE

Let us consider system (1) described by the following data:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; A_d = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with $\tau = 0.5s$ and $u_0 = 2$.

By selecting

$$S = I_2; R = 1; Q = I_2$$

the solution to Riccati equation (5) writes:

$$P(e) = \begin{bmatrix} \frac{1}{2}\sqrt{4+\frac{2}{e}}\sqrt{4+\frac{2}{e}+4\sqrt{4+\frac{2}{e}}} & \sqrt{4+\frac{2}{e}} \\ \sqrt{4+\frac{2}{e}} & \sqrt{4+\frac{2}{e}+4\sqrt{4+\frac{2}{e}}} \end{bmatrix}$$

and

$$c(e) = \frac{4}{\sqrt{4+\frac{2}{e}+4\sqrt{4+\frac{2}{e}}}}.$$

When $e \to \infty$ one gets:

$$\begin{aligned} \lim P(e) &= \begin{bmatrix} 2\sqrt{3} & 2 \\ 2 & 2\sqrt{3} \end{bmatrix} \\ \lim c(e) &= \frac{2}{\sqrt{3}}. \end{aligned}$$

The set of initial conditions is defined as

$$\mathcal{B}(1) = \left\{ \phi \in \mathcal{C}_{\frac{1}{2}}; \|\phi\|_c^2 \le \frac{c(1)}{\lambda_{\max}(P(1)) + 0.5\lambda_{\max}(S)} = 0.136 \right\}$$

which implies that

$$\|\phi\|_c^2 = \left[ \sup_{0.5 \le \theta \le 0} \|\phi(\theta)\| \right]^2 \le 0.136 \Rightarrow \sup_{\substack{0.5 \le \theta \le 0 \\ \phi \in C_{\frac{1}{2}}}} \|\phi(\theta)\| \le 0.369.$$

## 7. CONCLUSION

In this paper, a nonlinear bounded state feedback controller design for a linear continuous-time system with state delayed is proposed. The controller is designed from the solutions to an $\epsilon$-parameterized family of algebraic Riccati equations or linear matrix inequalities which allow to define invariant ellipsoidal neighborhoods of the origin.

From the Lyapunov–Krasovskii theorem, it is possible to show that uniform asymptotic stability is ensured and a certain performance level is attained using a quadratic cost function. In this paper, feedback control is addressed. For practical reasons, the output feedback design problem have to be considered in a near future.

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