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SYSTEMS WITH ASSOCIATIVE DYNAMICS[1]

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This paper introduces a class of nonlinear discrete-time dynamic models that generalize familiar linear model structures; our motivation is to explore the extent to which known results for the linear case do or do not extend to this nonlinear class. The results presented here are based on a complete characterization of the solution of the associative functional equation $F[F(x,y),z] = F[x,F(y,z)]$ due to J. Aczel, leading to a class of invertible binary operators that includes addition, multiplication, and infinitely many others. We present some illustrative examples of these dynamic models, give a simple explicit representation for their inverses, and present sufficient conditions for bounded-input, bounded-output stability. Finally, we propose a generalization of this model class and we demonstrate that these models have classical state-space realizations, unlike arbitrarily structured NARMA models.

1. INTRODUCTION

The class of finite-dimensional, time-invariant, discrete-time linear dynamic models provides the basis for many important practical results in control theory [10], system identification [11], statistical time-series analysis [3], and digital signal processing [12]. These models may be represented in various ways, including the ARMA(p,q) form:

$$y_k = \sum_{i=1}^{p} a_i y_{k-i} + \sum_{i=0}^{q} b_i u_{k-i}. \tag{1}$$

In many applications (e.g., computer-based control of strongly nonlinear systems or the design of nonlinear digital filters), this linear dynamic model structure is inadequate but the development and characterization of nonlinear alternatives is a difficult problem. As a specific example, a recent survey of industrial applications of nonlinear model predictive control (NMPC) concludes that one of the principal reasons that NMPC has had so much less impact on industrial practice than its linear counterpart is the general difficulty of developing adequate nonlinear dynamic

[1]The results of Sections 1–7 are in Proceedings of 1st IFAC Symposium on System Structure and Control, Prague, Czech Republic, August 29–31, 2001, Paper No 077 and the material of Section 8 is in Proceedings of the Third International Conference on Control Theory & Applications, Pretoria, Republic of South Africa, 2001, Paper No. WP01-4.

models [17]. One of the reasons for this difficulty is that the term "nonlinear discrete-time dynamic models" does not define a single, well-defined model class like the ARMA(p,q) models: instead, many very different model structures fall under the general umbrella of nonlinearity, and the qualitative behavior of these different model structures can vary over an enormous range [13]. To see this point, consider the following two nonlinear model structures:

- the Wiener model $\mathcal{W}$:

$$y_k = g\left(\sum_{i=1}^{p} a_i g^{-1}(y_{k-i}) + \sum_{i=0}^{q} b_i u_{k-i}\right)$$

- the projection-pursuit model $\mathcal{P}$:

$$y_k = g\left(\sum_{i=1}^{p} a_i y_{k-i} + \sum_{i=0}^{q} b_i u_{k-i}\right).$$

The difference between these two models lies in the autoregressive terms in the first sum, but the differences in qualitative behavior between these two models is enormous. For example, if $g(\cdot)$ is continuous, the Wiener model is BIBO stable if the ARMA(p,q) model defined by Eq. (1) is stable, but the projection-pursuit model can exhibit amplitude-dependent stability [14, Fig. 1].

This paper explores an extension of linear systems based on the idea of replacing the addition operations in Eq. (1) with other binary operators that share certain important properties and for which a simple, complete characterization exists.

2. ASSOCIATIVE BINARY OPERATORS

The binary operators $\circ$ considered here may be viewed as a mapping from some domain $D = I \times I$ into I, where I is an interval of real numbers that may be finite or infinite but must be open on at least one side. Further, $\circ$ is *associative* if it satisfies

$$(x \circ y) \circ z = x \circ (y \circ z), \tag{2}$$

for all x, y, and z in I. Equivalently, this binary operation may be written as $x \circ y = F(x,y)$, reducing Eq. (2) to the *associativity equation* [1, Ch. 7]:

$$F[F(x,y),z] = F[x,F(y,z)],$$

for all x, $y, z \in I$. Further, $\circ$ is *continuous* if the map $F : I \times I \to I$ is continuous, and *cancellative* if either of the following conditions implies $t_1 = t_2$: $t_1 \circ z = t_2 \circ z$ or $z \circ t_1 = z \circ t_2$. It has been shown [1, Thm. 1, Ch. 7] that the binary operator $\circ$ is continuous, associative, and cancellative on I if and only if

$$x \circ y = \phi^{-1}[\phi(x) + \phi(y)], \tag{3}$$

where $\phi(\cdot)$ is strictly monotonic and continuous on I. The most common examples are addition, corresponding to $\phi(x) = x$, and multiplication, corresponding to $\phi(x) = \ln x$; the other examples are the *parallel combination* $x||y$, defined as:

$$x||y = \frac{xy}{x+y},$$

arising from the parallel combination of resistances in electrical networks and defined by the function $\phi(x) = 1/x$, and the projective addition operation $\oplus$ defined as

$$\alpha \oplus \beta = \frac{2\alpha\beta - (\alpha + \beta)}{\alpha\beta - 1}$$

in [20] which corresponds to the function $\phi(x) = \frac{x}{x-1}$.

For convenience, the class of all associative, continuous and cancellative binary operators $\circ$ will be denoted $\mathcal{A}$. It follows from Eq. (3) that any binary operator $\circ$ in $\mathcal{A}$ is also *commutative*: $x \circ y = y \circ x$, and, as a consequence, the combination:

$$\bigoplus_{i=1}^{n} x_i \quad = \quad x_1 \circ x_2 \circ \cdots \circ x_n = \phi^{-1}\left[\sum_{i=1}^{n} \phi(x_i)\right]$$

is invariant under arbitrary permutations of the n terms $\{x_i\}$.

Another extremely useful consequence of the representation (3) is that the binary operation $\circ$ is *invertible*, with an inverse operation $\diamond$ given explicitly by:

$$x \diamond y = \phi^{-1}[\phi(x) - \phi(y)]. \tag{4}$$

It follows directly from Eqs. (3) and (4) that $(x \circ y) \diamond y = x$. When $\circ$ denotes addition or multiplication, the inverse operations of subtraction and division are well-known. As less obvious examples, note that the inverses of the parallel combination $x||y$ and the projective addition operation $x \oplus y$ are given by:

$$x \perp y = \frac{xy}{y-x}, \qquad x \odot y = \frac{x-y}{xy - 2y + 1}.$$

3. THE CLASS OF ASSOCIATIVE SYSTEMS

The class of *associative systems* is defined by replacing all additions in Eq. (1) with arbitrary binary compositions $\circ$ from $\mathcal{A}$:

$$y_k = \bigoplus_{i=1}^{p} a_i y_{k-i} \circ \bigoplus_{i=0}^{q} b_i u_{k-i} = \phi^{-1}\left(\sum_{i=1}^{p} \phi(a_i y_{k-i}) + \sum_{i=0}^{q} \phi(b_i u_{k-i})\right). \tag{5}$$

As a specific example, Figure 1 shows four step responses for the model with $p = 1$ and $q = 0$ obtained by taking $a_1 = -0.8$, $b_0 = 0.2$ and $\phi(x) = e^x - 1$. For small amplitude inputs, the behavior is quite similar to the linearized model obtained from the approximation $e^x - 1 \simeq x$, a point seen clearly for step amplitudes $A = \pm 0.2$. In contrast, the behavior changes dramatically with increasing input amplitude: positive step responses become progressively less oscillatory and negative step responses become more oscillatory. In addition, this model exhibits input-dependent stability: negative step inputs larger than approximately 1.24 in amplitude result in unstable responses, as do positive steps larger than approximately 3500, a result that further illustrates the dramatic asymmetry of this model's responses.

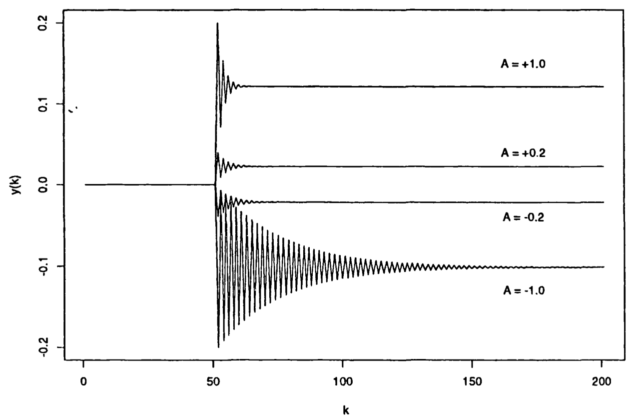

Fig. 1. Amplitude dependent step responses.

4. INVERSE MODELS

One noteworthy feature of the linear ARMA(p,q) model class is the existence of a simple, explicit form for the inverse model relating the output sequence $\{y_k\}$ to the input sequence $\{u_k\}$, a result with practical significance in control [5], spectral estimation and system identification (through the use of prewhitening filters [18]), and a variety of other applications: this inverse is simply the ARMA(q,p) model obtained by interchanging the poles and zeros of the original model. An analogous result may be developed for the general class of associative models using the inverse binary operator $\diamond$ defined in Section 2. Specifically, Eq. (5) may be rearranged to:

$$\bigoplus_{i=0}^{q} b_i u_{k-i} = y_k \diamond \bigoplus_{i=1}^{p} a_i y_{k-i} \Rightarrow b_0 u_k = y_k \diamond \bigoplus_{i=1}^{p} a_i y_{k-i} \diamond \bigoplus_{i=1}^{q} b_i u_{k-i}. \tag{6}$$

Writing this inverse model in terms of the function $\phi(\cdot)$ yields the more explicit representation:

$$u_k = b_0^{-1}\phi^{-1}\left(\phi(y_k) - \sum_{i=1}^{p}\phi(a_i y_{k-i}) - \sum_{i=1}^{q}\phi(b_i u_{k-i})\right). \tag{7}$$

In contrast to the linear case, the general associative class is not closed under model inversion, although there are two important exceptions. First, if $b_0 = 1$ and $\phi(-x) = -\phi(x)$, Eq. (7) may be rearranged into the associative model:

$$u_k = \phi^{-1}\left(\sum_{i=1}^{p} \phi(\beta_i u_{k-i}) + \sum_{i=0}^{q} \phi(\alpha_i y_{k-i})\right) = \bigoplus_{i=1}^{q} \beta_i u_{k-i} \circ \bigoplus_{i=0}^{p} \alpha_i y_{k-i}, \quad (8)$$

where $\alpha_0 = 1$, $\alpha_i = -a_i$ for $i = 1, 2, \ldots, p$, and $\beta_i = -b_i$ for $i = 1, \ldots, q$. The second case where the inverse model remains associative is that of associative homomorphic systems.

5. HOMOMORPHIC SYSTEMS

If $\phi(a_i y_{k-i})$ is replaced with $\alpha_i \phi(y_{k-i})$ and $\phi(b_i u_{k-i})$ is replaced with $\beta_i \phi(u_{k-i})$ in the second line of Eq. (5), we obtain the class of *homomorphic systems* [12, Ch. 10]. A block diagram of these systems is shown in Figure 2 and important examples include the nonlinear mean filters [16], obtained by restricting the linear block in Figure 2 to be a weighted average, i. e.: $y_k = \phi^{-1}\left(\sum_{i=0}^{q} \beta_i \phi(u_{k-i})\right), \sum_{i=0}^{q} \beta_i = 1$.

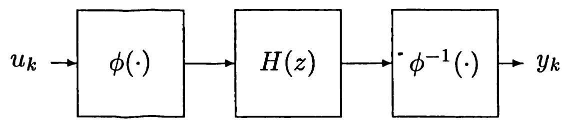

Fig. 2. Representation of a homomorphic system.

As an important specific case, note that restricting consideration to $u_k > 0$, taking $\phi(x) = \ln x$ and $\beta_i = 1/(q+1)$ yields a moving-window geometric mean filter. Similarly, taking $\phi(x) = 1/x$ and $\beta_i = 1/(q+1)$ yields the harmonic mean filter.

It is extremely interesting to ask what is contained in the intersection of these two classes–associative systems and homomorphic systems. First, note that this intersection contains the class of linear ARMA(p, q) systems, obtained by taking $\phi(x) = x$. More generally, membership in both classes requires the following equation to be satisfied:

$$\phi(ax) = \psi(a)\phi(x), \quad (9)$$

for some function $\psi(\cdot)$ and all x. In fact, we can obtain a simple explicit solution to this equation by first noting that, setting $x = 1$ implies $\psi(a) = \phi(a)/\phi(1)$. Dividing Eq. (9) through by $\phi(1)$ then yields:

$$\frac{\phi(ax)}{\phi(1)} = \psi(a)\left[\frac{\phi(x)}{\phi(1)}\right] \Rightarrow \psi(ax) = \psi(a)\psi(x).$$

This equation is *Cauchy's power equation* [2, p. 29], which has only three solutions that are continuous at any point: $\psi(x) = 0, |x|^{\nu}, |x|^{\nu} \operatorname{sign} x$. Of these solutions, only the last is invertible and then only if $\nu \neq 0$; further, note that these invertible functions are continuous at zero if and only if $\nu > 0$.

Hence, the class of associative homomorphic systems corresponds to subset of the associative systems obtained by restricting consideration to the functions $\phi(x) = \alpha|x|^\nu \operatorname{sign} x$ for $\nu > 0$. A particularly interesting characteristic of these systems is that they are *homogeneous*, a result that follows most easily from Cauchy's power equation and the block diagram representation for the homomorphic systems (Figure 2). In particular, if u_k is scaled by λ, it follows that $\phi(u_k) \to \phi(\lambda u_k) = \phi(\lambda)\phi(u_k)$. The effect of this modification of the input of the linear block $H(z)$ in Figure 2 is to simply scale the output by $\phi(\lambda)$. Defining z_k as the output of this linear block in response to the unscaled input u_k, it follows that the output of the homomorphic system is:

$$y_k = \phi^{-1}(\phi(\lambda)z_k) = \lambda\phi^{-1}(z_k),$$

a result that follows from the fact that, if $\phi(\cdot)$ is invertible and satisfies the Cauchy power equation, then so does $\phi^{-1}(\cdot)$. The class of homogeneous systems is discussed further in [13, Ch. 3]. Finally, note that the inverse of any homomorphic system is simply the homomorphic system based on the same nonlinear function $\phi(\cdot)$ and the inverse linear model $H^{-1}(z)$, a result that follows immediately from the block diagram representation. Hence, if $\phi(x) = \alpha|x|^\nu \operatorname{sign} x$ for some $\nu > 0$, the resulting homomorphic system is an associative system with an associative inverse.

6. STABILITY CONDITIONS

It is possible to establish some useful sufficient conditions for the stability of associative systems. In particular, we present conditions under which these systems exhibit the following behavior:

BIBO stability:

> A system $\mathcal{S}\{u_k\} \to \{y_k\}$ is *bounded-input, bounded-output stable* or more simply, *BIBO stable* if, for any $0 < M < \infty$, $|u_k| \le M$ for all k implies the existence of $0 < N < \infty$ such that $|y_k| \le N$ for all k.

The first of these stability conditions is:

Condition A:

> The functions $\phi(\cdot)$ and $\phi^{-1}(\cdot)$ map compact sets into compact sets.

This condition is satisfied by invertible functions that are continuous on R, but it is also satisfied by discontinuous functions like:

$$\phi(x) = \begin{cases} -x & |x| \le 1 \\ x & |x| > 1, \end{cases} \tag{10}$$

which is its own inverse: $\phi^{-1}(y) = \phi(y)$. Conversely, this condition is not satisfied by singular functions like $\phi(x) = 1/x$.

For the special case $p = 0$, it follows immediately that the associative model defined by Eq. (5) is BIBO stable if $\phi(\cdot)$ satisfies Condition A. Similarly, Condition A is also sufficient to guarantee the BIBO stability of associative homomorphic systems, a result that follows directly from the block diagram: if $\{u_k\}$ is uniformly bounded, so is $\{\phi(u_k)\}$; if $H(z)$ is stable, the output sequence $\{z_k\}$ is also uniformly bounded, ultimately implying the output sequence $\{y_k = \phi^{-1}(z_k)\}$ is bounded. Conversely, for the general case, an additional condition is required that extends the usual restriction on the coefficients $\{a_i\}$ in stable linear models [6, Ch. 4]. Here, we introduce the following modified Lipschitz condition:

Condition B:

> The function $\phi(\cdot)$ satisfies $\phi(0) = 0$ and there exists a function $\psi : R \to R^+$ such that $|\phi(ax)| \leq \psi(a)|\phi(x)|$ for all $x \in R$ and all $a \in \mathcal{S}$ where $\mathcal{S}$ is a specified subset of R.

Theorem 1. Suppose $\phi(\cdot)$ satisfies Conditions A and B on some set $\mathcal{S}$. The associative system defined by Eq. (5) is BIBO stable if $a_i \in \mathcal{S}$ for $i = 1, 2, \ldots, p$ and the following linear system $\mathcal{L}$ is stable:

$$x_k = \sum_{i=1}^{p} \psi(a_i) x_{k-i} + u_k.$$

Proof. Applying $\phi(\cdot)$ to Eq. (5) yields: $\phi(y_k) = \sum_{i=1}^{p} \phi(a_i y_{k-i}) + \sum_{i=0}^{q} \phi(b_i u_{k-i})$ $\Rightarrow |\phi(y_k)| \leq \sum_{i=1}^{p} |\phi(a_i y_{k-i})| + \sum_{i=0}^{q} |\phi(b_i u_{k-i})|$. If $|u_k| \leq M$ for all k, it follows from Condition A that there exists a finite upper bound N for the second of these sums; further, by Condition B:

$$|\phi(y_k)| \quad \leq \quad \sum_{i=1}^{p} \psi(a_i) |\phi(y_{k-i})| + N.$$

If $|\phi(y_j)| \leq x_j$ for all $j < k$, it follows by induction that $|\phi(y_k)| \leq x_k$. Hence, if the system $\mathcal{L}$ is stable, its response to a unit step of amplitude N defines a finite overbound on $|\phi(y_k)|$ for all k, establishing the BIBO stability of the associative system. □

Note that, since the linear system $\mathcal{L}$ in this theorem is causal, finite-dimensional, and time-invariant, it follows that BIBO stability is equivalent to ℓ_1 stability (i.e., absolute summability of the impulse response coefficients $\{h_k\}$) [6, p. 339], which implies asymptotic stability. Further, asymptotic stability is equivalent to exponential stability for causal, time-invariant, finite-dimensional linear systems [14, p. 166], and it is easy to show that exponential stability implies ℓ_1 stability:

$$|h_k| \leq C\alpha^k, 0 < \alpha < 1 \Rightarrow \sum_{k=0}^{\infty} |h_k| \leq \frac{C}{1-\alpha} < \infty.$$

Hence, the stability of the system $\mathcal{L}$ required in this theorem may be taken as any of the equivalent forms: BIBO, ℓ_1, asymptotic, or exponential.

This stability result is quite similar to that for the class of *structurally additive* models [13, p. 178]: $y_k = \sum_{i=1}^{p} f_i(y_{k-i}) + \sum_{i=0}^{q} g_i(u_{k-i})$. There, a sufficient condition for stability of the nonlinear model is the stability of a related linear model, derived from the Lipschitz constants of the functions $f_i(\cdot)$. An interesting feature of both of these results is that they relate stability of the original nonlinear system to that of a linear *positive* system, for which very strong stability results are available [7, Ch. 5].

Finally, note that one class of functions satisfying Condition B with $\mathcal{S} = R$ are those defined by Eq. (9), which satisfy the defining condition with equality. As a consequence, the stability of associative homomorphic systems follows, but this result may be obtained more easily from the block diagram argument presented earlier. Hence, it is of particular interest to explore non-homomorphic cases where Condition B is satisfied, as in the following example.

7. A DISCONTINUOUS EXAMPLE

The solution of the associativity equation described here leads to the explicit representation for $\circ$ given by (3) where the function $\phi(\cdot)$ is continuous. In this final example, we relax this condition, considering the discontinuous function defined in Eq. (10). Taking the same first-order linear dynamic model as in the previous example with $a_1 = 0.8$ and $b_0 = 0.2$ gives a model whose responses to various amplitude steps is shown in Figure 1; these plots show the strongly amplitude-dependent dynamic character of this model. Here, however, the stability conditions presented in Secion 6 apply since $\phi(x)$ satisfies: $|\phi(x)| = |x| \Rightarrow |\phi(ax)| = |ax| = |a| \cdot |\phi(x)|$. Hence, so long as the first-order linear model on which this system is based is stable, so too is the overall nonlinear system.

Another interesting feature of this example is that because the function $\phi(x)$ is piecewise linear, the associative model may be expressed as an affine multimodel [13, Ch. 6]. The basis for this result is the observation that

$$\phi(x) + \phi(y) = \begin{cases} -x - y & |x|,\ |y| \leq 1 \\ -x + y & |x| \leq 1,\ |y| > 1 \\ x - y & |x| > 1,\ |y| \leq 1 \\ x + y & |x|,\ |y| > 1. \end{cases}$$

This result leads ultimately to an affine multimodel representation involving the four local models $\pm a_1 y_{k-1} \pm b_0 u_k$ and rather complicated selection conditions. For example, the local linear model $y_k = a_1 y_{k-1} + b_0 u_k$ is selected if either of the following two conditions are satisfied:

1: $|a_1 y_{k-1}| \leq 1,\ |b_0 u_k| \leq 1,\ |a_1 y_{k-1} + b_0 u_k| \leq 1$

2: $|a_1 y_{k-1}| > 1,\ |b_0 u_k| > 1,\ |a_1 y_{k-1} + b_0 u_k| > 1.$

Similar selection criteria hold for the other three local linear models.

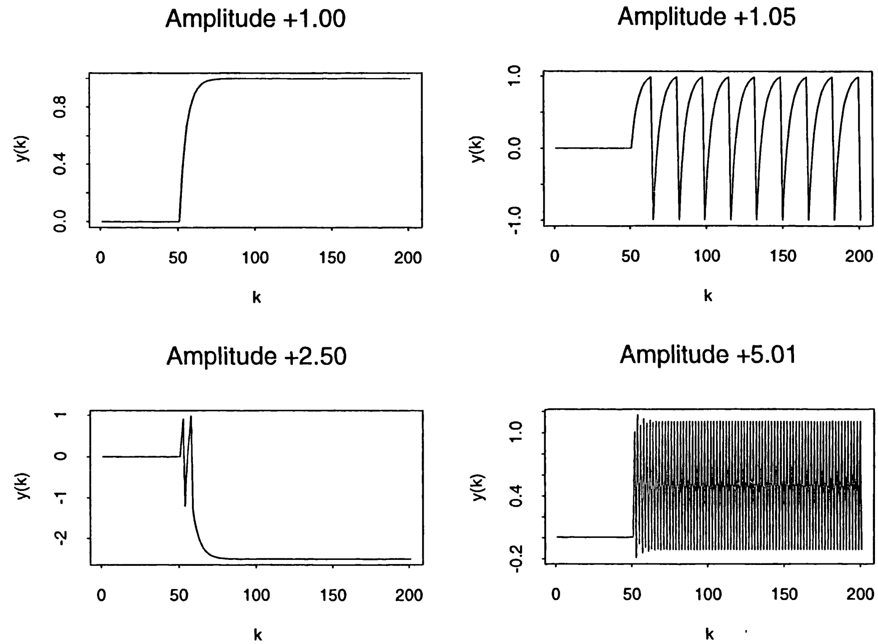

Fig. 3. Step responses, discontinuous model.

8. GENERALIZED ASSOCIATIVE MODELS AND STATE–SPACE REALIZATIONS

The class of empirical NARMA models is defined by the nonlinear input/output relation

$$y_{t+n} = f(y_t, \ldots, y_{t+n-1}, u_t, \ldots, u_{t+n-1}) \tag{11}$$

These models are quite popular, both because they are easier to develop than fundamental first-principles models and because they are better suited to applications like model-based control. Conversely, many control system design and analysis procedures assume the existence of a state-space realization, but not all nonlinear input/output models of the form (11) exhibit such realizations [16,17]. Consequently, it is advantageous to consider a subset of the NARMA class defined by Eq. (11) for which state-space realizations can be shown to exist.

The fundamental reason that general NARMA models do not necessarily exhibit classical state-space realizations is that the nonlinear function $f(\cdot)$ in Eq. (11) admits arbitrary combinations of the variables y_{i-i} and u_{t-j} for all time lags i and j. Alternatively, by restricting the coupling permitted between these variables, it is possible to guarantee the existence of a state-space realization. For example, a subclass of NARMA models is considered in [16] that is guaranteed to have an observable state-space realization. This subclass is specified by requiring the function $f(\cdot)$ in Eq. (11) to be a sum of component functions $f_i(\cdot)$, each depending on a specified subset of the arguments appearing in Eq. (11). The simplest special case of this realizable model is the *additive NARMA (ANARMA)* class, in which the arguments are pairwise decoupled:

$$y_{t+n} = f_1(y_t, u_t) + \ldots + f_n(y_{t+n-1}, u_{t+n-1}). \tag{12}$$

In general, f can be a *sum* of fewer than n component functions, each being a function of more than two arguments

$$\begin{aligned} y_{t+n} &= f_1(y_t, \ldots, y_{t+k}, u_t) + f_2(y_{t+1}, \ldots, y_{t+k+1}, u_{t+1}) + \ldots \\ &\quad + f_{n-k}(y_{t+n-k-1}, \ldots, y_{t+n-1}, u_{t+n-k-1}) \end{aligned} \tag{13}$$

for any $k = 0, 1, \ldots, n-1$.

The purpose of this section is to extend the result of [19] by applying associative binary operations other than addition to decouple the different time lags. The class of generalized associative models is obtained by replacing the addition operations in ANARMA model (12) with arbitrary binary operators $\circ$ from $\mathcal{A}$ defined in Section 2:

$$y_{t+n} = [[\ldots[f_1(y_t, u_t) \circ_1 f_2(y_{t+1}, u_{t+1})] \circ_2 \ldots] \circ_{n-1} f_n(y_{t+n-1}, u_{t+n-1})] \tag{14}$$

where $x \circ_i y = \phi_i^{-1}[\phi_i(x) + \phi_i(y)]$. Note that although all operations $\circ_i$ are individually associative and commutative, these conditions do not hold for their combinations. For example, $(f_1 \circ_1 f_2) \circ_2 f_3 \neq f_1 \circ (f_2 \circ_2 f_3)$ generalizes the familiar situation where $\circ_1$ is defined to be addition and $\circ_2$ multiplication. For that reason (14) is understood to mean that we first apply operator $\circ_1$, then $\circ_2$ and so forth so the order of associative operators is not allowed to change.

Of course, the general structure (14) accommodates the case were all operations $\circ_i$ are identical

$$y_{t+n} = \phi^{-1}\left(\sum_{i=1}^{n} \phi(f_i(y_{t+i-1}, u_{t+i-1})\right). \tag{15}$$

In the special case where $f_i(y_{t+i-1}, u_{t+i-1}) = a_i y_{t+i-1} \circ b_i u_{t+i-1} = \phi^{-1}[\phi(a_i y_{t+i-1}) + \phi(b_i u_{t+i-1})]$ in (15) yields the associative model class studied in Section 3. When $\phi(x) = x$, $f_i(y_{t+i-1}, u_{t+i-1}) = a_i y_{t+i-1} + b_i u_{t+i-1} + c_i y_{t+i-1} u_{t+i-1}$, we obtain the diagonal bilinear model [22].

The class of generalized associative models is shown to have a classical state space realization. Once the associative structure of the model is recognized, the state model construction is direct, allowing a simple translation from input-output model to state-space model. However, it is not always easy to recognize the generalized associative model structure in (11) since it depends on the *existence* of certain function ϕ, not specified in advance. The problem of determining this function is only briefly considered here; a complete solution remains a subject for future research.

An algorithm is now given to check if (11) can be written in the form (14). This algorithm permits computation of the required functions f_i, $i = 1, 2, \ldots, n-1$ step by step, whenever they exist. The algorithm is constructive up to integrating some one - forms which is very common in the nonlinear setting. As noted, additional study is necessary to find ϕ_i's, $i = 1, 2, \ldots, n-1$ and to check if ϕ_i's are strictly monotonic and continous.

Algorithm. Calculate for $i = 1, 2, \ldots, n$

$$\omega_{t+n-i} = \frac{\partial f(\cdot)}{\partial y_{t+n-i}} dy_{t+n-i} + \frac{\partial f(\cdot)}{\partial u_{t+n-i}} du_{t+n-i} \tag{16}$$

Check:

$$d\omega_{t+n-i} \wedge \omega_{t+n-i} = 0 \tag{17}$$

If not, stop; otherwise

$$\omega_{t+n-i} = \lambda_{n-i-1}(y_t, \ldots, y_{t+n-1}, u_t, \ldots, u_{t+n-1})\, df_{n-i+1}(y_{t+n-i}, u_{t+n-i}). \tag{18}$$

As in [19], we assume that the function $f(\cdot)$ defining the NARMA model is meromorphic, since these functions and their derivatives can only vanish at isolated points.

Because we are often interested in various system theoretic properties that can be characterised by the non- vanishing of specific functions defined by the system equations, this restriction allows us to characterise *generic* system properties that hold on an open and dense subset of some suitable domain of definition. The distinction between such generic characterisations and *global* characterisations is that the latter are required to hold everywhere, without exception. In connection with the problem of integrating one forms, by focusing on generic properties, we require that the one-forms be integrable everywhere except possibly at a set of isolated singular points. Note that the class of meromorphic functions is closed with respect to division and the four examples of binary operation given in Section 2 also belong to the class of meromorphic functions, but this is not the case for piecewise linear function ϕ given in Section 7. Therefore, the algorithm only applies to the meromorphic class of systems and at the moment we do not have the procedure to check whether non-meromorphic functions can be rewritten in the form (14). But of course, the realization procedure in the paper is more general and can be applied to (14) independently of which class the functions $f_1, \ldots, f_n$ and $\phi_1, \ldots, \phi_{n-1}$ belong.

The realization problem is to construct the state equations

$$\begin{aligned} x^+ &= f(x, u) \\ y &= h(x) \end{aligned} \tag{19}$$

for the input-output difference equation (14). Note that the superscript + notes the one step forward time shift, i.e. $x^+(t) = x(t+1)$. The sequences $\{u_t, y_t, t \geq 0\}$ generated by (19) (for different initial states) have to be equal to the sequences $\{u_t, y_t, t \geq 0\}$ satisfying equation (14). Then (19) will be called a realization of (14). A system is said to be realizable in the classical state space form if there exists a realization of the form (19).

The main goal of this section is to show that (14) admits a classical state space realization, and to obtain the corresponding state equations. Our analysis is based on the choice of the state coordinates [19] for the ANARMA model (12):

$$\begin{aligned} x_1 &= y_t \\ x_2 &= y_{t+1} - f_n(y_t, u_t) \\ x_3 &= y_{t+2} - f_n(y_{t+1}, u_{t+1}) - f_{n-1}(y_t, u_t) \\ &\vdots \\ x_n &= y_{t+n-1} - f_n(y_{t+n-2}, u_{t+n-2}) - \ldots - f_2(y_t, u_t) \end{aligned} \tag{20}$$

that will yield the state equations

$$\begin{aligned} x_1^+ &= x_2 + f_n(x_1, u) \\ x_2^+ &= x_3 + f_{n-1}(x_1, u) \\ &\vdots \\ x_{n-1}^+ &= x_n + f_2(x_1, u) \\ x_n^+ &= f_1(x_1, u). \end{aligned} \tag{21}$$

The only difference is that the addition and subtraction operations are replaced by the operators $\circ_i$ and $\diamond_i$ respectively, and taking care to preserve the correct order of these operators. Hence, we choose the state coordinates as

$$\begin{aligned} x_1 &= y_t \\ x_2 &= y_{t+1} \diamond_{n-1} f_n(y_t, u_t) \\ x_3 &= [y_{t+2} \diamond_{n-1} f_n(y_{t+1}, u_{t+1})] \diamond_{n-2} f_{n-1}(y_t, u_t) \\ &\vdots \\ x_n &= [\ldots[[y_{t+n-1} \diamond_{n-1} f_n(y_{t+n-2}, u_{t+n-2})] \diamond_{n-2} f_{n-1}(y_{t+n-3}, u_{t+n-3})] \diamond_{n-3} \ldots] \\ &\quad \diamond_1 f_2(y_t, u_t). \end{aligned} \tag{22}$$

This will yield the state equations

$$\begin{aligned} x_1^+ &= x_2 \circ_{n-1} f_n(x_1, u) \\ x_2^+ &= x_3 \circ_{n-2} f_{n-1}(x_1, u) \\ &\vdots \\ x_{n-1}^+ &= x_n \circ_1 f_2(x_1, u) \\ x_n^+ &= f_1(x_1, u) \\ y &= x_1. \end{aligned} \tag{23}$$

If all $\circ_i$'s, $i = 1, \ldots, n-1$ are taken to be the addition operations, and all $\diamond$'s the subtraction, equation (23) reduces to the well-known [19] result for the ANARMA case.

9. EXAMPLE: NONLINEAR ENGINE MODEL

Elsewhere, we will report the development of an empirical model for the dynamics of an internal combustion engine. The structure chosen for this model is

$$\begin{aligned} y_{t+4} &= p_1 u_t y_{t+2} y_{t+3} + p_2 u_t u_{t+2} y_{t+3} + p_3 u_t y_{t+2} u_{t+2} y_{t+3} \\ &\quad + p_4 u_{t+1} y_{t+2} y_{t+3} + p_5 u_{t+1} u_{t+2} y_{t+3} + p_6 u_{t+1} u_{t+2} y_{t+2} y_{t+3} \end{aligned} \tag{24}$$

where the input u_k is the idle-speed air-bleed valve position and the output y_k is the engine speed in RPM. This model structure was chosen both on the basis of certain knowledge about the dynamic behavior of the engine (e. g., the inherent time delay in the response to input changes) and because it belongs to the class of generalized associative models described here. In particular, one can easily verify that condition (17) holds for $i = 1, 2, 3$, and 4; further, the one-forms are easily integrated in this case, from which it follows that the input/output model (24) has the structure (14) with $f_1(y_t, u_t) = \delta u_t$, $f_2(y_{t+1}, u_{t+1}) = \zeta u_{t+1}$, $f_3(y_{t+2}, u_{t+2}) = \alpha y_{t+2} + \beta u_{t+2} + \gamma y_{t+2} u_{t+2}$, $f_4(y_{t+3}, u_{t+3}) = k y_{t+3}$ where $\circ_1$ is addition and $\circ_2$ and $\circ_3$ are multiplications. The coefficients δ, ζ, α, β, γ and k are related to the identified parameters $p_1, \ldots, p_6$ via the following equations $p_1 = \delta\alpha$, $p_2 = \delta\beta k$, $p_3 = \delta\gamma k$, $p_4 = \zeta\alpha$, $p_5 = \zeta\beta$ and $p_6 = \zeta\gamma k$. The general result (22) then leads to the following choice of state coordinates:

$$\begin{aligned} x_1 &= y_t \\ x_2 &= \frac{y_{t+1}}{f_4(y_t, u_t)} = \frac{y_{t+1}}{k y_t} \\ x_3 &= \frac{y_{t+2}}{f_4(y_{t+1}, u_{t+1}) f_3(y_t, u_t)} = \frac{y_{t+2}}{k y_{t+1}(\alpha y_t + \beta u_t + \gamma y_t u_t)} \\ x_4 &= \frac{y_{t+3}}{f_4(y_{t+2}, u_{t+2}) f_3(y_{t+1}, u_{t+1})} - f_2(y_t, u_t) \\ &= \frac{y_{t+3}}{k y_{t+2}(\alpha y_{t+1} + \beta u_{t+1} + \gamma y_{t+1} u_{t+1})} - \zeta u_t \end{aligned} \tag{25}$$

which will yield the state equations

$$\begin{aligned} x_1^+ &= k x_1 x_2 \\ x_2^+ &= \alpha x_1 x_3 + \beta x_3 u + \gamma x_1 x_3 u \\ x_3^+ &= x_4 + \zeta u \\ x_4^+ &= \delta u \\ y &= x_1 . \end{aligned} \tag{26}$$

10. CONCLUSIONS

This paper has introduced a new class of discrete-time dynamic models, obtained by replacing the addition operation with a more general binary operation $\circ$, required only to be associative, continuous, and cancellative. These requirements then lead to a useful, simple representation for the operation $\circ$ in terms of a continuous, strictly monotonic function $\phi(\cdot)$. The spirit of this replacement is similar to that of systems based on max-plus algebras [4], where the addition and multiplication operations on which the standard algebra is based are replaced by the maximum and addition operators, respectively. Useful features of the associative system representation described here are the existence of a simple, explicit inverse, analytically interesting connections with the class of homomorphic systems originally proposed for deconvolution problems and other related applications, and the possibility of developing sufficient conditions for stability. In fact, the associative homomorphic systems represent a limiting case of the fundamental inequality on which this stability result is based (Condition B), raising the question of what other associative systems satisfy these conditions. One such system was described here, based on a discontinuous function $\phi(\cdot)$ that still results in a system that is associative and analytically invertible. Further, this system was also shown to belong to the class of affine multimodels, a class of significant interest in process modeling and one closely related to hybrid systems [13, p. 292].

Finally, in the last section we introduced a class of generalized associative models by replacing the addition operations in additive NARMA model with the associative binary operators which share some important properties with the addition operation; for our purpose the invertibility property (with an inverse operation given explicitly)is most useful. This property allows us to construct the state equations directly from the generalized associative input-output model, generalizing the subclass of realizable NARMA models. A simple algorithm is given to check whether the class of generalized associative models accommodates a higher order input-output difference equation used to construct the state-space realization. Although it is easy to check this property, it is not always easy to convert the original NARMA equation into the form (14), since our algorithm is constructive only up to integrating some one-forms. Moreover, additional study is necessary to find the functions ϕ_i that define the binary operations.

The underlying "linear" structure of (14) should lead to simple characterizations of observability (note that (21) is the well known observer form), accessibility and dynamic feedback linearizability of (14), topics we plan to explore further.

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