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ON REVERSES OF SOME BINARY OPERATORS

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The notion of reverse of any binary operation on the unit interval is introduced. The properties of reverses of some binary operations are studied and some applications of reverses are indicated.

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1. INTRODUCTION

The notion of reverse of a t-norm was introduced under the name invert by C. Kimberling in [8]. Klement, Mesiar and Pap defined in [9] the t-reverse of t-norms, studied self-reversibility and t-reversibility of ordinal sums of t-norms and they also formulated some open problems. Almost all problems concerning reversibility of t-norms were solved in [5, 9, 18]. In this paper we follow the name "reverse" used by Fodor and Jenei in [5]. The notion of reverse of a copula is known under the name survival copula [10, 17, 20] and its applications can be found, e. g., in [3, 10, 11]. In this paper, the reverse of any binary operation on the unit interval is defined. Next, some interesting properties of the introduced operation, such as involutivity, preserving monotonicity, self-reversibility and continuity are studied. Finally, we discuss the possibility of applications of reverses bringing some new results.

The paper is organized as follows. After recalling the necessary notations and definitions in Section 2, we list the known results related to reversible t-norms in Section 3. The reverse of any binary operation on the unit interval is introduced in Section 4, as well as some basic properties of reverses are derived. Section 5 deals with some known and unknown applications of reverses of binary operations. Section 6 is devoted to the class of self-reversible binary operations.

2. NOTATIONS AND DEFINITIONS

A triangular norm (t-norm) T is a binary operation $T : [0,1]^2 \rightarrow [0,1]$ which is associative, commutative, nondecreasing in both arguments and satisfies the boundary condition $T(x,1) = x$ for all $x \in [0,1]$. Analogously, a triangular conorm (t-conorm)

S is a binary operation $S : [0,1]^2 \to [0,1]$ which is associative, commutative, nondecreasing in both arguments and satisfies the boundary condition $S(x,0) = x$ for all $x \in [0,1]$. Note that t-conorms are dual operations of t-norms, in the sense:

$$S(x,y) = 1 - T(1-x, 1-y) \quad \text{for all} \quad x,y \in [0,1].$$

Let T be a t-norm and S be its dual with respect to the standard negation N given by $N(x) = 1 - x$ for all $x \in [0,1]$. Then the triplet (T,S,N) is called a De Morgan triplet. Recall that the most frequently used t-norms and their dual t-conorms are given by: (for all $x,y \in [0,1]$)

$$\begin{array}{ll} M(x,y) = \min\{x,y\} & M'(x,y) = \max\{x,y\} \\ P(x,y) = xy & P'(x,y) = x + y - xy \\ W(x,y) = \max\{0, x+y-1\} & W'(x,y) = \min\{1, x+y\}. \end{array}$$

Note, that the mentioned t-norms and t-conorms belong to the class of Frank t-norms and Frank t-conorms. Therefore the notation $M = T_0^F$ for the minimum t-norm, $P = T_1^F$ for the product t-norm and $W = T_\infty^F$ for the Lukasiewicz t-norm is sometimes used. An ordinal sum of t-norms is defined as follows. Let $]a_k, b_k[, k \in K$ be a family of pairwise disjoint open subintervals of $[0,1]$ and $T_k, k \in K$ be a family of t-norms. Then the binary operation T defined on $[0,1]^2$ by

$$T(x,y) = \begin{cases} a_k + (b_k - a_k) T_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & \text{if } x,y \in [a_k, b_k], \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$

is a t-norm. It is denoted by $\{\langle a_k, b_k, T_k\rangle\}_{k\in K}$ and called the ordinal sum of the summands $\langle a_k, b_k, T_k\rangle, k \in K$. For more details about t-norms, t-conorms and negations we recommend [12].

The notion of copula was introduced in the study of the relationship between multivariate distribution functions and their marginals. A copula C is a binary operation $C : [0,1]^2 \to [0,1]$ satisfying:

1. $C(x,0) = C(0,x) = 0, C(x,1) = C(1,x) = x$
2. C is 2-increasing, i. e., for all $x_1, x_2, y_1, y_2 \in [0,1]$, with $x_1 \le x_2, y_1 \le y_2$ we have $C(x_1,y_1) + C(x_2,y_2) \ge C(x_1,y_2) + C(x_2,y_1)$.

For an extensive overview over copulas see, e. g., [17].

3. THE REVERSES OF t-NORMS AND t-CONORMS

The well-known formula of classical two-valued logic says that the truth values of any two propositions p, q, their disjunction $p \vee q$ and their conjunction $p \wedge q$ fulfill the equation

$$val(p \vee q) + val(p \wedge q) = val(p) + val(q).$$

In fuzzy set theory, the generalization of the former equation leads to the Frank equation

$$T(x,y) + S(x,y) = x + y \tag{F}$$

where T is a t-norm and S is a t-conorm and $x, y \in [0,1]$. The Frank family of t-norms was introduced as a family of solutions of this functional equation, see [6].

Let a t-conorm S be the dual of the t-norm H, i. e., $S(x,y) = 1 - H(1-x, 1-y)$. Then the equation (F) can be written in the form

$$T(x,y) = x + y - 1 + H(1-x, 1-y)$$

or

$$H(x,y) = x + y - 1 + T(1-x, 1-y). \tag{H}$$

The t-norm H can be considered as a transformation of a t-norm T. Similarly, if a t-norm T is the dual of the t-conorm R, the equation (F) can be written in the form

$$R(x,y) = x + y - 1 + S(1-x, 1-y). \tag{R}$$

It is known [6] that if T and H are Frank t-norms, and moreover $T = H$, then the equation (H) holds. Similarly, if R and S are Frank t-conorms and $S = R$, then the equation (R) holds. It is clear that for some t-norm T the expression $x + y - 1 + T(1-x, 1-y)$ can be negative and for some t-conorm S the expression $x + y - 1 + S(1-x, 1-y)$ can exceed 1. For example, we can choose a t-norm $T < W$ or a t-conorm $S > W'$. Therefore a binary operation T^* (and similarly S^*) was defined [6, 9] as follows

Definition 1. Let T be a t-norm. Then the binary operation $T^* : [0,1]^2 \to [0,1]$ defined by

$$T^*(x,y) = \max\{0, x + y - 1 + T(1-x, 1-y)\}$$

is called *the reverse of t-norm T.*

Definition 2. Let S be a t-conorm. Then the binary operation $S^* : [0,1]^2 \to [0,1]$ defined by

$$S^*(x,y) = \min\{1, x + y - 1 + S(1-x, 1-y)\}$$

is called *the reverse of t-conorm S.*

It can be shown [9], that the monotonicity and associativity may not hold for T^*. For example, put $T = \{\langle 0, \frac{1}{2}, Z\rangle\}$, where Z is the weakest t-norm given by

$$Z(x,y) = \begin{cases} \min\{x,y\} & \text{if } \max\{x,y\} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If T^* is also t-norm, then we say that T is *reversible.* Analogously, if S^* is t-conorm, then we say, that S is *reversible.*

The problem of reversibility of t-norms has already been intensively studied [5, 9, 18]. In summary we briefly list the most relevant results from the mentioned papers.

— Let $T \leq W$. Then $T^* = W$ (W is the Łukasiewicz t-norm) [9].

— Let $T \geq W$. Then T is reversible if and only if T is a Frank t-norm or an ordinal sum of Frank t-norms [18].

— If T is a continuous Archimedean t-norm which is not comparable with W then T is not reversible [5].

— If T is a reversible t-norm then T^* is continuous [18].

— If T is a Frank t-norm then $T = T^*$ [9].

— $T = T^*$ if and only if T is a Frank t-norm or an ordinal sum of Frank t-norms $\{\langle a_k, b_k, T_k\rangle\}_{k\in K}$ such that for any $i \in K$ there is $j \in K$ such that

$$[a_i, b_i] = [1 - b_j, 1 - a_j] \quad \text{and} \quad T_i = T_j.$$

We already know that all Frank t-norms are self-reversible, i. e., $T^* = T$. There is no problem to verify that Frank t-conorms have the same property.

4. THE REVERSES OF BINARY OPERATIONS

Now we will define the reverse of arbitrary binary operation on the unit interval.

Definition 3. Let $B : [0,1]^2 \to [0,1]$ be a binary operation on the unit interval. *The reverse of the binary operation* B is the binary operation $B^* : [0,1]^2 \to [0,1]$ such that

$$B^*(x,y) = /x + y - 1 + B(1-x, 1-y)/$$

where $/a/ = \min\{1, \max\{0, a\}\}$. If $B = B^*$ then the binary operation B is called self-reversible.

It is clear that for the reverse of any binary operation B holds

$$W \leq B^* \leq W'.$$

Geometrically B^* can be obtained by the rotation of the 3-dimensional graph of B by 180^o around the axis $x = y = 1/2$, adding the plane $z = x + y - 1$ and replacing the overlapping parts by 0 or 1.

Example 1. Let B be the t-norm ordinal sum $B = \{\langle 0, 0.5, P\rangle\}$, i. e.,

$$B(x,y) = \begin{cases} 2xy & \text{if } (x,y) \in [0,1/2]^2, \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$

Then $B^* = \{\langle 0.5, 1, P\rangle\}$. Observe that B is an ordinal sum with one summand $P = T_1^F$.

Recall that a t-subnorm is a binary operation on the interval $[0,1]$ which is associative, commutative, nondecreasing in both arguments and not greater than the minimum t-norm [7, 15].

Example 2. Let $B : [0,1]^2 \to [0,1]$ be a t-subnorm [7,15] given by

$$B(x,y) = xy/2.$$

Then $B^*(x,y) = \max\{0, x+y-1+(1-x)(1-y)/2\} = \max\{0,(x+y-1+xy)/2\}$ is the Sugeno–Weber t-norm [12] (and so a t-subnorm).

Example 3. Let $B : [0,1]^2 \to [0,1]$ be the Łukasiewicz implicator given by

$$I(x,y) = \min\{1, 1-x+y\}.$$

$$\text{Then } I^*(x,y) = \begin{cases} /2x/, & \text{if } y > x \\ /x+y/, & \text{otherwise.} \end{cases}$$

The next lemma deals with binary operations which lie between W and W'. We note that the assumption $x+y-1 \le B(x,y) \le x+y$, for all $x \in [0,1]$ is equivalent to

$$W \le B \le W'.$$

Lemma 1. Let B be a binary operation on $[0,1]$. If for all $x, y \in [0,1]$

$$x+y-1 \le B(x,y) \le x+y$$

then

(i) $B^*(x,y) = x+y-1+B(1-x,1-y)$,

(ii) $B^{**} = (B^*)^* = B$.

Proof. Let $x+y-1 \le B(x,y) \le x+y$. Replacing x, y by $1-x, 1-y$ we obtain

$$(1-x)+(1-y)-1 \le B(1-x,1-y) \le (1-x)+(1-y)$$

and adding $x+y-1$ yields

$$0 \le x+y-1+B(1-x,1-y) \le 1$$

which implies (i). We shall prove (ii). Using (i) we obtain $B^{**}(x,y) = x+y-1+B^*(1-x,1-y) = x+y-1+(1-x)+(1-y)-1+B(x,y) = B(x,y)$. □

Due to Lemma 1 we have that $A^* = A^{***}$ for any binary operation A on the unit interval. The following lemmas can be easily proved as they are generalizations of the known properties of t-norms.

Lemma 2. Let A be a binary operation on the unit interval and $A(x,y) \le W(x,y)$ for all $x, y \in [0,1]$. Then $A^* = W$.

Lemma 3. Let A be a binary operation on the unit interval and $A(x,y) \geq W'(x,y)$ for all $x,y \in [0,1]$. Then $A^* = W'$.

Let us denote by $/A/_L$ the restriction of a binary operation A by the Łukasiewicz t-norm and t-conorm, i. e., for all $x,y \in [0,1]$, we define

$$/A/_L(x,y) = \max\{W(x,y), \min\{A(x,y), W'(x,y)\}\}.$$

Lemma 4. Let A be a binary operation on the unit interval. Then $A^* = (/A/_L)^*$ and $A^{**} = /A/_L$.

There is no relation between the monotonicity of a binary operation B and its reverse B^*. E. g., the t-norm $T = \{\langle 0, 0.5, Z\rangle\}$ is nondecreasing in both arguments but T^* is not nondecreasing in both arguments. On the contrary, the implicator I in Example 3 is decreasing in first argument, but I^* is nondecreasing in both arguments. Now, we will try to solve the problem of the monotonicity of reverses.

Recall that a binary operation $A : [0,1]^2 \to [0,1]$ has the 1-Lipschitz property if $|A(x,y) - A(z,t)| \leq |x - z| + |y - t|$ for all $x,y,z,t \in [0,1]$.

Lemma 5. Let $A : [0,1]^2 \to [0,1]$ be nondecreasing in both arguments such that also A^* is nondecreasing in both arguments and let $W \leq A \leq W'$. Then A and A^* are 1-Lipschitz (and therefore continuous).

Proof. Conversely, suppose that A is not 1-Lipschitz. Then there exist $x_1 \leq x_2, y_1 \leq y_2$ such that

$$A(x_2,y_2) - A(x_1,y_1) > (x_2 - x_1) + (y_2 - y_1).$$

This implies that

$$A(x_2,y_2) - A(x_2,y_1) > y_2 - y_1 > 0 \text{ or } A(x_2,y_1) - A(x_1,y_1) > x_2 - x_1 > 0.$$

Let $A(x_2,y_2) - A(x_2,y_1) > y_2 - y_1 > 0$. Then $1 - y_1 > 1 - y_2$ and

$$\begin{aligned} & A^*(1-x_2, 1-y_1) - A^*(1-x_2, 1-y_2) \\ = \; & 1 - x_2 + 1 - y_1 - 1 + A(x_2,y_1) - (1 - x_2 + 1 - y_2 - 1 + A(x_2,y_2)) \\ = \; & y_2 - y_1 + A(x_2,y_1) - A(x_2,y_2) < (y_2 - y_1) + (y_1 - y_2) = 0, \end{aligned}$$

which contradicts the monotonicity of A^*. By Lemma 1 we have $A^{**} = A$ and

$$x + y - 1 \leq A^*(x,y) \leq x + y$$

Now, we can replace A by A^* in the proof above and repeat the procedure. □

Theorem 1. Let $A : [0,1]^2 \to [0,1]$ be a binary operation nondecreasing in both arguments such that $W \leq A \leq W'$. Then A^* is nondecreasing in both arguments if and only if the binary operation A is 1-Lipschitz.

Proof. The necessity follows from Lemma 5. Let A be 1-Lipschitz, $x_1 \leq x_2$, $y_1 \leq y_2$. Then

$$\begin{aligned}
& A^*(x_2, y_2) - A^*(x_1, y_1) \\
= \; & (x_2 - x_1) + (y_2 - y_1) + A(1 - x_2, 1 - y_2) - A(1 - x_1, 1 - y_1) \\
= \; & (x_2 - x_1) + (y_2 - y_1) - (A(1 - x_1, 1 - y_1) - A(1 - x_2, 1 - y_2)) \\
\geq \; & (x_2 - x_1) + (y_2 - y_1) - (x_2 - x_1) + (y_2 - y_1) = 0.
\end{aligned}$$

□

The following theorem is a consequence of Theorem 1 and Lemma 5.

Theorem 2. Let $A : [0,1]^2 \to [0,1]$ be a binary operation which is nondecreasing in both arguments. Then A^* is nondecreasing in both arguments if and only if $/A/_L$ is 1-Lipschitz.

Proof. Let A^* be nondecreasing in both arguments. Using Lemma 4 and Theorem 1 we obtain that $/A/_L = A^{**}$ is 1-Lipschitz. Conversely, if $/A/_L$ is 1-Lipschitz then $(/A/_L)^* = A^*$ is nondecreasing in both arguments. □

Finally, we will discuss the problem of the continuity of reverses.

Lemma 6. Let A be a binary operation on the unit interval such that A and A^* are nondecreasing in both arguments. Then if A is not continuous at the point (x_0, y_0) then one of the following statements holds

(i) $x_0 + y_0 \leq 1$ and $A(x_0, y_0) \geq x_0 + y_0$

(ii) $x_0 + y_0 \geq 1$ and $A(x_0, y_0) \leq x_0 + y_0 - 1$.

The proof is left to the reader. The following theorem is a direct consequence of Lemma 6 and Lemma 4.

Theorem 3. Let A be nondecreasing binary operation on the unit interval such that also A^* is nondecreasing in both arguments. Then A^* is continuous.

5. APPLICATION OF REVERSES OF SOME BINARY OPERATIONS

We have already mentioned the role of reverses in the theory of t-norms and t-conorms. The notion of reverse has also appeared in the theory of copulas [17, 20]. It is also known that every copula C fulfills the assumptions of Lemma 1, $W \leq C \leq \min \leq W'$, and therefore

$$C^*(x,y) = x + y - 1 + C(1-x, 1-y).$$

Moreover, the reverse of a copula is always a copula. Self-reversible copulas play an important role in several applications, e.g. in aggregation based on Choquet integral [11], in fuzzy similarity measurement [3] and in statistics (see Section 6 or [17]). Now we show another application of reverses that leads to some noteworthy results.

A preference structure is a basic concept of preference modelling [1, 2, 4, 21, 22]. In a classical preference structure (PS) a decision-maker makes three decisions for any pair (a, b) from the set $\boldsymbol{A}$ of all alternatives. His decisions define a triplet (P, I, J) of crisp binary relations on $\boldsymbol{A}$:

1) a is preferred to b $\Leftrightarrow (a,b) \in P$ (strict preference).

2) a and b are indifferent $\Leftrightarrow (a,b) \in I$ (indifference).

3) a and b are incomparable $\Leftrightarrow (a,b) \in J$ (incomparability).

Definition 4. A preference structure (PS) on a set $\boldsymbol{A}$ is a triplet (P, I, J) of binary relations on $\boldsymbol{A}$ such that

(ps1) I is reflexive, P and J are irreflexive.

(ps2) P is asymmetric, I and J are symmetric.

(ps3) $P \cap I = P \cap J = I \cap J = \emptyset$.

(ps4) $P \cup I \cup J \cup P^t = A \times A$ where $P^t(x,y) = P(y,x)$.

Using characteristic mappings [1, 2, 21, 22] a minimal definition of (PS) can be formulated as a triplet (P, I, J) of binary relations on $\boldsymbol{A}$ such that

(m1) I is reflexive and symmetric.

(m2) $P(a,b) + P^t(a,b) + I(a,b) + J(a,b) = 1$ for all $(a,b) \in \boldsymbol{A}^2$.

A preference structure can be characterized by the reflexive relation $R = P \cup I$ called the *large preference relation.* The relation R can be interpreted as

$$(a,b) \in R \Leftrightarrow a \text{ is preferred to } b \text{ or } a \text{ and } b \text{ are indifferent.}$$

It can be easily proved that

$$co(R) = P^t \cup J$$

where $coR(a,b) = 1 - R(a,b)$ and

$$P = R \cap co(R^t), I = R \cap R^t, J = co(R) \cap co(R^t).$$

This allows to construct a preference structure (P, I, J) from a reflexive binary relation R only. A natural demand led researchers to the introduction of a fuzzy preference structure (FPS) [1, 2, 4, 19, 21, 22]. Of course, the attempts simply to replace the notions used in the definition of (PS) by their fuzzy equivalents have led to some problems. It was proved [4, 21, 22], that reasonable constructions of a fuzzy preference structure (FPS) strongly recommend to use a nilpotent t-norm only. Because any nilpotent t-norm (t-conorm) is isomorphic to the Łukasiewicz t-norm W (t-conorm W'), for simplicity, we will use the Łukasiewicz triplet $(W, W', 1-x)$. Then we can define (FPS) as the triplet of binary fuzzy relations (P, I, J) on the set of alternatives $\boldsymbol{A}$ satisfying:

(fm1) I is reflexive and symmetric.

(fm2) $\forall (a,b) \in \boldsymbol{A}^2, P(a,b) + P^t(a,b) + I(a,b) + J(a,b) = 1.$

The idea to use some reflexive large preference relation R to construct (PS) can be fuzzified too. Fodor and Roubens [4] (among others) proposed an axiomatic construction of a fuzzy preference structure generated by a reflexive binary relation R $(FPS - R)$. They supposed that (in the frame of the Łukasiewicz De Morgan triplet $(W, W', 1-x)$)

— the values $P(a,b)$, $I(a,b)$, $J(a,b)$ depend only on the values $R(a,b)$, $R(b,a)$, i. e., there exist binary operations $p, i, j : [0,1]^2 \to [0,1]$ such that for any $(a,b) \in \boldsymbol{A}^2$

$$P(a,b) = p(R(a,b), R(a,b)),$$
$$I(a,b) = i(R(a,b), R(b,a)),$$
$$J(a,b) = j(R(a,b)), R(b,a)).$$

— $p(x, N(y)), i(x,y), j(N(x), N(y))$ are nondecreasing in arguments x and y.

— i and j are symmetric

— (P, I, J) is (FPS) for any reflexive binary relation R on a set of alternatives $\boldsymbol{A}$ such that $W(P, I) = R$, $W'(P, J) = 1 - R^t$, where $R^t(a,b) = R(b,a)$.

It was proved [1, 4] that for all $x, y \in [0,1]$ it holds

$$W(x,y) \leq p(x, 1-y), i(x,y), j(1-x, 1-y) \leq M\{x,y\}.$$

The mentioned triple (p, i, j) is called a *monotone generator triplet* (De Baets and Fodor [2]). Summarizing, the monotone generator triplet is a triplet (p, i, j) of $[0,1]^2 \to [0,1]$ mappings such that: for all $x, y \in [0,1]$

(gt1) $p(x, 1-y), i(x,y), j(1-x, 1-y)$ are nondecreasing in arguments x and y.

(gt2) $W(x,y) \leq p(x,1-y), i(x,y), j(1-x,1-y) \leq M\{x,y\}$.

(gt3) $i(x,y) = i(y,x)$.

(gt4) $p(x,y) + p(y,x) + i(x,y) + j(x,y) = 1$.

(gt5) $p(x,y) + i(x,y) = x$. [F1]

It can be easily shown that (gt3) and (gt4) imply the symmetry of j and (gt4) and (gt5) imply

$$p(x,y) + j(x,y) = 1 - y. \qquad \text{[F2]}$$

Now we are looking for a monotone generator triplet defined above. The following theorem gives a necessary condition.

Theorem 4. Let (p,i,j) be a monotone generator triplet. Then for all $x,y \in [0,1]$

$$i^*(x,y) = j(1-x,1-y) \text{ and } j^*(x,y) = i(1-x,1-y).$$

Proof. Let (p,i,j) be a generator triplet. Then [F1], [F2] imply

$$i(x,y) = x + y - 1 + j(x,y).$$

Using (gt2) and Lemma 1 we obtain

$$\begin{aligned} i^*(x,y) &= x + y - 1 + i(1-x,1-y) \\ &= x + y - 1 + (1-x) + (1-y) - 1 + j(1-x,1-y) = j(1-x,1-y). \end{aligned}$$

Using Lemma 1 we obtain $j^*(x,y) = i(1-x,1-y)$. □

It can be shown that the symmetry of the operation $G : [0,1]^2 \to [0,1]$ given by $G(x,y) = p(x,1-y)$ is equivalent to the self-reversibility of indifference generator i. Theorem 1, Theorem 4 and [F1] imply that all members of the monotone generator triplet (p,i,j) must be 1-Lipschitz, compare with [2, 13].

Recall that a binary operation $C : [0,1]^2 \to [0,1]$ that is nondecreasing in both arguments, $C(x,0) = C(0,x) = 0$, $C(x,1) = C(1,x) = x$ for all $x \in [0,1]$, and C is 1-Lipschitz is called a *quasi-copula.* Therefore, if (p,i,j) is a monotone generator triplet then p, i, j are 1-Lipschitz and i is a quasi-copula. Moreover, the 1-Lipschitz property implies continuity. Therefore, if (p,i,j) is a monotone generator triplet then p, i, j are continuous. Thanks to the functional equations [F1], [F2], a generator triplet can be determined by one of its members only. The following theorem gives the hint how to construct a monotone generator triplet from some preference generator, compare with [2].

Theorem 5. Assume that a binary operation $p : [0,1]^2 \to [0,1]$ fulfils

(a) The binary operation $C : [0,1]^2 \to [0,1]$ given by $C(x,y) = p(x, 1-y)$ is a quasi-copula.

(b) $p(x,y) - p(y,x) = x - y$ for all $(x,y) \in [0,1]^2$.

Then the triplet (p,i,j) where

$$i(x,y) = x - p(x,y), \quad j(x,y) = 1 - y - p(x,y) \quad \text{for all } x, y \in [0,1])$$

is a monotone generator triplet.

Example 4. Put $p(x,y) = x(1-y)$ for all $x, y \in [0,1]$. Then

$$p(x, 1-y) = xy, \quad p(x,y) - p(y,x) = x - y$$

$$i(x,y) = xy, \qquad j(x,y) = (1-x)(1-y).$$

Since p fulfils the assumption of Theorem 5 we have obtained a generator triplet (p,i,j).

Example 5. Let $p(x,y) = \max\{0, x-y, x-2xy\}$. Then the binary operation $C : [0,1]^2 \to [0,1]$ given by $C(x,y) = p(x,1-y)$ is a quasi-copula and $p(x,y) - p(y,x) = x - y$. Using Theorem 5 we obtain (for all $(x,y) \in [0,1]^2$)

$$i(x,y) = x - p(x,y) \text{ and } j(1-x, 1-y) = i^*(x,y).$$

It means that

$$i = \{\langle 0, 0.5, P\rangle\}, i^*(x,y) = \{\langle 0.5, 1, P\rangle\}$$

and (p,i,j) is a monotone generator triplet. Recall that $i \neq i^*$, therefore $p(x, 1-y)$ is not symmetric (see Figure). Especially, take notice of the validity of (gt4).

6. SELF-REVERSIBILITY

The characterization of the class S of all self-reversible binary operations on the unit interval is also an interesting problem. As we have already mentioned, self-reversible copulas play important roles in statistics. Recall that copulas [17] are the functions that join a two-dimensional distribution function H to its one-dimensional margins F, G. It means that for the mentioned triplet F, G, H there exists a copula C such that for all $x, y \in R \cup \{-\infty, \infty\}$

$$H(x,y) = C(F(x), G(y))$$

([17], Sklar's theorem). For a pair (X,Y) of continuous random variables with the joint distribution function H, the joint survival function H_s is given by

$$H_s(x,y) = \text{Prob}[X > x, Y > y] = C^*(1 - F(x), 1 - G(y)),$$

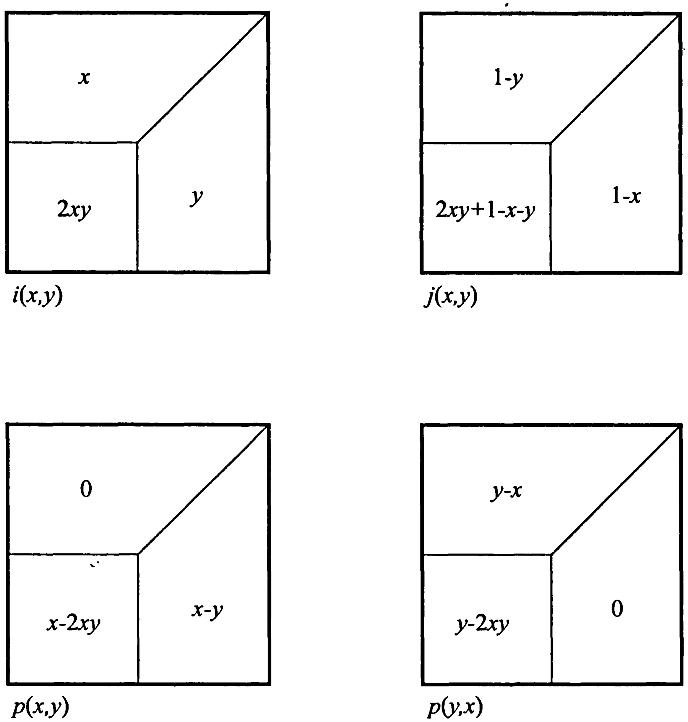

Figure. Description of the functions from Example 5 on the unit square.

where C^* is the reverse of C. Moreover, if X, Y are symmetric about a and b respectively, i. e., for all $x, y \in R \cup \{-\infty, \infty\}$

$$F(a+x) = 1 - F(a-x), \quad G(b+y) = 1 - G(b-y)$$

and C is self-reversible, then H is radially symmetric about (a, b), i. e., the three-dimensional graph of H is symmetric with respect to the point $[a, b, 0.5]$.

Of course, there exist self-reversible binary operations which are neither copulas nor t-norms (observe that each self-reversible t-norm is also a copula due to the fact that each 1-Lipschitz t-norm is a copula [16]). For example, the arithmetic mean or the binary operation $K : [0,1]^2 \to [0,1]$ given by

$$K(x,y) = \frac{x^2 + y^2}{2}, \quad x, y \in [0,1]$$

are also self-reversible. As it can be shown, the class $\boldsymbol{S}$ is convex. It means that for any $A, B \in \boldsymbol{S}$, $k \in [0,1]$, the binary operation given by

$$V(x,y) = kA(x,y) + (1-k)B(x,y), \quad x, y \in [0,1] \tag{M}$$

is self-reversible as well. However, the combination $L(A, B)$ of self-reversible binary operations L, A, B need not be self-reversible. For example, if $L = A = B = P$

(product), then the binary operation U given by

$$U(x,y) = L(A(x,y), B(x,y)) = (xy)^2, \quad x, y \in [0,1]$$

is no longer self-reversible. The following theorem gives a necessary and sufficient condition for L to ensure the self-reversibility of $L(A,B)$ for all A, $B \in \boldsymbol{S}$. We say that a binary operation A on the unit interval is *shift-stable* if

$$A(x+t, y+t) = t + A(x,y)$$

for all $x, y \in [0,1]$ and $t \in [-\min\{x,y\}, 1-\max\{x,y\}]$. Note that shift-stable aggregation operators are sometimes called shift-invariant [14].

Theorem 6. Consider a binary operation $L : [0,1]^2 \to [0,1]$. Then the following statements are equivalent:

(i) $L(A,B) \in \boldsymbol{S}$ for all A, $B \in \boldsymbol{S}$;

(ii) L is shift-stable.

Proof. Observe first that $A \in \boldsymbol{S}$ if and only if for all $x, y \in [0,1]$

$$A(x,y) = x + y - 1 + A(1-x, 1-y).$$

It implies that $W \leq A \leq W'$ whenever $A \in \boldsymbol{S}$.

(i)$\Rightarrow$ (ii) Let $U = L(A,B) \in \boldsymbol{S}$ for all A, $B \in \boldsymbol{S}$. Then for all x, $y \in [0,1]$ and all A, $B \in \boldsymbol{S}$;

$$U(x,y) = L(A(x,y), B(x,y))$$

and

$$\begin{aligned} U^*(x,y) &= x + y - 1 + L(A(1-x,1-y), B(1-x,1-y)) \\ &= x + y - 1 + L(1 - x - y + A(x,y), 1 - x - y + B(x,y)) \\ &= L(A(x,y), B(x,y)). \end{aligned}$$

Putting $u = A(x,y)$, $v = B(x,y)$, $q = 1 - x - y$, we obtain

$$L(u+q, v+q) = q + L(u,v).$$

It remains to prove that the previous equation holds for all $(u,v) \in [0,1]^2$ and all $q \in [-\min\{u,v\}, 1-\max\{u,v\}]$.

From the fact that $W, W^* \in \boldsymbol{S}$ and from the self-reversibility of operator V in (M) it follows that all convex combinations of W and W' are also self-reversible. Hence, for any $(u,v) \in [0,1]^2$ there exist $(x,y) \in [0,1]^2$ and binary operations A, $B \in \boldsymbol{S}$ such that $u = A(x,y)$, $v = B(x,y)$. It is sufficient to take the $y = 1-x$ and put

$$A = uW' + (1-u)W, B = vW' + (1-v)W.$$

Given u, v and any $q \in [-\min\{u,v\}, 1-\max\{u,v\}]$, we are looking for $(x_0, y_0) \in [0,1]^2$ and $A, B \in \boldsymbol{S}$ such that

$$u = A(x_0, y_0), v = B(x_0, y_0) \text{ and } q = 1 - x_0 - y_0.$$

As $W \leq A, B \leq W'$ we have:

$$u = A(x_0, y_0), v = B(x_0, y_0) \text{ for some A, B} \in \boldsymbol{S}$$

if and only if

$$\begin{aligned} & (x_0, y_0) \in \{(x,y) |\max\{u,v\} \leq x + y \leq 1 + \min\{u,v\}\} \\ = \quad & \{(x,y)| -\min(u,v) \leq 1 - x - y \leq 1 - \max\{u,v\}\}. \end{aligned}$$

It is sufficient to take $(x_0, y_0) \in [0,1]^2$ for which $x_0 + y_0 = 1 - q$. Then $1 - x_0 - y_0 = q \in [-\min\{u,v\}, 1 - \max\{u.v\}]$. Hence, there exist $A, B \in \boldsymbol{S}$ such that $u = A(x_0, y_0), v = B(x_0, y_0)$. It proves (ii).

(ii)$\Rightarrow$(i) Let L be shift-stable. It means that

$$L(u + q, v + q) = q + L(u, v)$$

for all $(u, v) \in [0,1]^2$ and all $q \in [-\min\{u,v\}, 1 - \max\{u,v\}]$. Let $A, B \in \boldsymbol{S}$ and $(x, y) \in [0,1]$. Putting $u = A(x,y)$, $v = B(x,y)$, $q = 1 - x - y$ in previous equation we obtain that

$$q \in [-\min\{u,v\}, 1 - \max\{u,v\}]$$

and

$$L(A(x,y) + 1 - x - y, B(x,y) + 1 - x - y) = 1 - x - y + L(A(x,y), B(x,y)).$$

It proves that $U = L(A, B) \in \boldsymbol{S}$. This completes the proof. □

The previous theorem allows to construct self-reversible binary operations from a shift-stable and two self-reversible binary operations. As a shift-stable binary operation need not be monotone we have that the monotonicity is not necessary condition for the self-reversibility of binary operations. For example, the binary operation $U : [0,1]^2 \to [0,1]$ given by

$$U(x,y) = \min\{\max\{x,y\}, 2x - y + 1, 2y - x + 1\}\}$$

is not monotone, but it is shift-stable. Using Theorem 6 we have that the binary operations $U(M, M')$, $U(M, P)$ are self-reversible but they are not monotone.

7. CONCLUSION

We have shown that the reverses of binary operations defined in this paper have many interesting properties. We have tried to answer the common questions concerning the most important properties of monotonicity, self-reversibility and continuity of reverses. We also demonstrated that the applications of reverses in other fields of interest can give some new results. The problem of commutativity and associativity of reverses was solved for t-norms and t-conorms only. For general binary operations, these problems remain still open. Also, the problem of a full characterization of the class of all self-reversible binary operations on the unit interval is still open.

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