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ON THEOREMS OF NIVEN AND DRESSLER

ŠTEFAN PORUBSKÝ

Niven [2] and Dressler [1] have proved that the (asymptotic) density of sets $\{n: (n, \varphi(n)) \leqq k\}$, $\{n: (n, \sigma_j(n)) \leqq k\}$ and $\{n: (\varphi(n), \sigma_j(n)) \leqq k\}$ is zero, where, as usual, $\varphi(n)$ is the Euler totient function and $\sigma_j(n) = \sum_{d \mid n} d^j$ $(j = 1, 2, \ldots)$. In this note we intend to extend these results to sets of the type $\{n: (f_1(n), f_2(n)) \leqq g(n)\}$, where $f_1(n)$ and $f_2(n)$ are integer-valued multiplicative functions. The ideas behind our proofs are borrowed from those of Niven and Dressler.

Every positive integer n can be written in the form $n = n_1 \cdot n_2$, where n_1 is square-free and n_2 is square-full with $(n_1, n_2) = 1$. We shall call n_1 the square-free part of n and denote it by $s(n)$.

Theorem 1. *Let $g(n)$ be an integer-valued arithmetical function and $f_1(n)$, $f_2(n)$ multiplicative ones. Assume, moreover, that there exists a set of primes $P = \{p_i\}$ which satisfies the following conditions:*

(i) $\sum_{p_i \in P} p_i^{-1}$ diverges,

(ii) $f_1(p_i) \nmid g(n)$ for every p_i from P and positive integer n,

(iii) If $P_i = \{q_{i,j}\}$ denotes the set of primes such that $f_1(p_i) \mid f_2(q_{i,j})$, then $\sum_{q_{i,j} \in P_i} q_{i,j}^{-1}$ diverges for every i.

Then the set $T = \{n: (f_1(n), f_2(n)) = g(n)\}$ has the density $d(T) = 0$.

Proof. Given a sequence A of positive integers and a prime q, A_q will denote the set of those elements n of A such that $q \mid s(n)$.

In the proof we shall use the following result due to I. Niven [2] or [3], p. 254:

(1) *If there is a set of primes $\{q_i\}$ such that $\sum q_i^{-1}$ diverges and $d(A_{q_i}) = 0$ for $i = 1, 2, \ldots$, then $d(A) = 0$.*

Of particular interest will be the case of A_{q_i} being void for every i:

(2) *If there is a set of primes $\{q_i\}$ such that $\sum q_i^{-1}$ diverges and no member of A has its square-free part divisible by some q_i, then $d(A) = 0$.*

Now, according to (1) it is sufficient to show that $d(T_{p_i}) = 0$ for every p_i from P. This will immediately follow if we show that the set $M^{(i)} = \{m: m \,.\, p_i \in T_{p_i}, (m, p_i) = 1\}$ has density zero. To do this we shall use (2) with $q_{i,j}$ in place of q_j. It can be easily verified that the hypotheses of (2) are really satisfied in this case. In fact, the required divergence follows from (*iii*) and on the other hand, (*ii*) tohether with the fact that f_1 and f_2 are multiplicative implies that $q_{i,j} \neq p_i$ and that $M^{(i)}_{q_{i,j}}$ is void for every i.This proves the theorem.

Corollary 1. *Let $f(n)$ be an integer-valued multiplicative function. Suppose that there exists a set of primes $\{p_i\}$ such that:*

(j) $\sum p_i^{-1}$ diverges,

(jj) for every i the series $\sum_{p_i \mid f(q)} q^{-1}$ diverges, where the summation is extended over all the primes q satisfying the indicated property.

Then for every positive k *the density of the set of positive integers n for which* $(n, f(n)) \leqq k$ is zero.

Corollary 2. *Let $f(n)$ be an integer-valued multiplicative function. Suppose that there exists a non-constant polynomial $z(n)$, $z(0) \neq 0$ (with integer coefficients) such that $f(p) = z(p)$ for every prime p. Then for every positive k the density of the set of positive integers n for which $(n, f(n)) \leqq k$ is zero.*

Proof. By the preceding corollary it is sufficient to find a set $P = \{p_i\}$ of primes satisfying (*j*) and (*jj*). Define P' as the set of those primes for which the congruence

$$(Z) \qquad z(n) \equiv 0 \pmod{p_i}$$

has a solution. Then Schinzel's theorem on p. 403 of [4] asserts that

$$\sum_{\substack{p \leqq x \\ p \in P'}} p^{-1} = \lambda \operatorname{loglog} x + D + O((\log x)^{-1})$$

with suitable contants λ and D. Now, if P denotes the set of those primes from P' for which there exists a solution n of (Z) with $(n, p_i) = 1$, then P differs from P' only in a finite number of terms. Thus Schinzel's result implies that (*j*) is fulfilled for this P.

Finally, for $p_i \in P$ the relation $p_i \mid f(q)$ is equivalent to $q \equiv n_0 \pmod{p_i}$, where n_0 denotes a solution of (Z) such that $(n_0, p_i) = 1$. At this stage, Dirichlet's theorem on primes in arithmetical progression yields that also (*jj*) is satisfied; and the proof is finished.

Thus for instance, for $f(n) = \varphi(n)$ or $f(n) = \sigma_j(n)$ we obtain the above mentioned results of Niven and Dressler. Moreover, the first results can be extended in turn to functions $\varphi_j(n) = n^j \prod_{p \mid n} (1 - p^{-j})$ denoting the number of ordered sets of j (equal or not) positive integers, none of which exceeds n and whose g.c.d. is prime to n ($j = 1, 2, 3, \ldots$).

In [1] Dressler proved using a completely different idea that the set $\{n: (\varphi(n),$

$\sigma_j(n))\leqq k\}$ has density zero. In the case $j=1$ this result follows also from our theorem. However, Dressler's idea of the proof of this statement will be employed in the proof of the next theorem. In what follows, $\omega(n)$ and $\Omega(n)$ will denote the number of distinct prime divisors and the total number of prime divisors of n, resp.

Lemma. *Let $h(n)$ be an arithmetical function such that*

$$\limsup_{n\to\infty}\frac{h(n)}{\log\log n}=a<1\,. \tag{H}$$

Then the set $T_h=\{n\colon \omega(s(n))\geqq h(n)\}$ has density one.

Proof. Let α be such that

$$1>\alpha>a>\max\{0, 3\alpha-2\}\,.$$

Such an α always exists, e.g., if $a<2/3$ put $\alpha=2/3$, and if $a>2/3$ take α from the interval $(a, (a+2)/3)$. Let $\delta=\delta(\alpha)$ be determined by the equality $a=\delta\,.\,\alpha$. Obviously

$$1>\delta>\max\left\{0, 3-\frac{2}{\alpha}\right\}.$$

Further, put $g(n)=\delta^{-1}\,.\,h(n)$. Then

$$\limsup_{n\to\infty}\frac{g(n)}{\log\log n}=\alpha<1\,. \tag{G}$$

Finally put $\varepsilon=(1-\delta)/(3-\delta)$. Then $0<\varepsilon<1-\alpha$ and $\varepsilon<1/3$.

It follows from (G) that $g(n)<(1-\varepsilon)\log\log n$ holds for all sufficiently large n. On the other hand, $(1-\varepsilon)\log\log n<\omega(n)$ is satisfied for almost all n, because $\log\log n$ is the normal order of $\omega(n)$. Thus the density of the set $M=\{n\colon g(n)<\omega(n)\}$ is one. Therefore to finish the proof it is sufficient to show that $d(M-T_h)=0$, that is, that the density of the set of integers m for which $g(m)<\omega(m)$ and simultaneously $h(m)>\omega(m)$ is zero. But $h(m)=\delta g(m)=(1-3\varepsilon)/(1-\varepsilon)\,.\,g(m)$, and therefore

$$\Omega(m)\geqq\omega(s(m))+2(\omega(m)-\omega(s(m)))>2\omega(m)-\delta g(m)>$$

$$>\frac{1+\varepsilon}{1-\varepsilon}\,\omega(m)\,.$$

On the other hand, $\Omega(m)$ and $\omega(m)$ have the same normal order $\log\log n$. This yields that the inequality $\Omega(m)/\omega(m)\leqq(1+\varepsilon)/(1-\varepsilon)$ holds for almost all n, which proves the lemma.

Remark. By the way, it follows from our lemma that the set of integers n with $\omega(s(m))\leqq$const. has density zero.

Theorem 2. *Let $g(n)$ and $h(n)$ be arithmetical functions such that h(n) satisfies*

(H). Moreover, let $f_1(n)$ and $f_2(n)$ be multiplicative integer-valued functions. Finally assume, that for every n from $T_h = \{n: \omega(s(n)) \geqq h(n)\}$ we have

$$\prod_{p|s(n)} (f_1(p), f_2(p)) > g(n) .$$

Then the set of positive integers n for which $(f_1(n), f_2(n)) \leqq g(n)$ has density zero.

The theorem is an immediate consequence of our Lemma, because the investigated set is a subset of a set of density zero, more precisely

$$\{n: (f_1(n), f_2(n)) \leqq g(n)\} \subseteqq \text{compl}\,(T_h) .$$

Corollary. *If $g(n) = o$ (loglog n), then the set of positive integers such that $(\varphi(n), \sigma_j(n)) \leqq g(n)$ is zero for every $j = 1, 2, \dots$.*

Proof. If p is an odd prime, the $2 | (\varphi(p), \sigma_j(p))$. Therefore, if $n \in T_{2g}$, then

$$\prod_{p|n} (\varphi(p), \sigma_j(p)) \geq 2^{\omega(s(n))-1} \geqq \frac{1}{2}\, \omega(s(n)) \geqq g(n)$$

and the conclusion follows.

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О ТЕОРЕМАХ НИВЕНА И ДРЕСЛЕРА

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резюме

Пусть $g(n)$, $f_1(n)$, $f_2(n)$ — целозначные арифметические функции, причем $f_1(n)$, $f_2(n)$ — мультипликативны. В работе обобщаются некоторые результаты Нивена и Дреслера. Приводится несколько достаточных условий таких, чтобы множества типа $\{n: (f_1(n), f_2(n)) \leqq g(n)\}$ имели нулевую асимптотическую плостность.