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ASYMPTOTIC DENSITIES OF SETS OF POSITIVE INTEGERS

TIBOR ŠALÁT—ROBERT TIJDEMAN

1. Introduction.

Let N denote the set of all positive integers and let 2^N be the set of all subsets of N. For $A \subset N$ we define the counting function $A(n) = \sum_{a \in A:\, a \leqq n} 1$, the asymptotic lower density $\underline{d}(A) = \liminf_{n \to \infty} n^{-1} A(n)$ and the asymptotic upper density $\bar{d}(A) = \limsup_{n \to \infty} n^{-1} A(n)$. If $\underline{d}(A) = \bar{d}(A)$, then we call this value the asymptotic density, $d(A)$, of A.

A set function $v: 2^N \to R$ is said to have the the property **P** if for every pair $A, B \subset N$ with the $\lim_{n \to \infty} A(n)/B(n) = t$ one has $v(A)/v(B) = t$. It is obvious that both $\underline{d}$ and $\bar{d}$ and d posses the property **P**. Here we shall show that every density measure has the property **P**.

Using the well-known theorem of Hahn and Banach, the set function d which is defined on the sets having asymptotic density can be extended to a finitely additive measure μ defined on all subsets of N (cf. [1], p. 231). We call such a measure μ a density measure on 2^N. (Thus if $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$). Density measures have been studied by several authors, see, e. g. R. C. Buck [2], [3], R. B. Deal [4] and D. Maharam [7], [8]. The last mentioned author ([7], p. 174) proved that for every $A \subset N$ and $t \in R$ with $0 \leqq t \leqq 1$, there exists a subset B of A with $\mu(B) = t\mu(A)$. Lemma 1 states the simple fact that for given $A \subset N$ and t with $0 \leqq t \leqq 1$ there exists a subset B of A with $\lim_{n \to \infty} B(n)/A(n) = t$. The following can therefore be considered as a generalization of the result of Maharam.

Theorem 1. *Let μ be any density measure on 2^N. Let A and B be subsets of N with $\lim_{n \to \infty} A(n)/B(n) = t$. Then $\mu(A) = t\mu(B)$.*

Thus every density measure on 2^N has the property **P**.

If a set function has the property **P**, then almost all subsets of a given set have a measure which is equal to one half of the measure of the original set. To be more precise, let $A=\{a_1, a_2, \dots\}$ be a given infinite set of positive integers with $a_1<a_2\dots$. Let $S(A)$ be the set of all infinite subsets of A. To the subset $B=\{a_{k_1}, a_{k_2}, \dots\}$ with $k_1<k_2<\dots$ we adjoin the number $\varrho(B)=\sum_{j=1}^{\infty} 2^{-k_j}$. Hence, ϱ is a one-to-one mapping of $S(A)$ onto $(0, 1\rangle$. For $0<x\leqq 1$ we denote the uniquely determined subset $\varrho^{-1}(x)$ of A by A_x. The mapping ϱ^{-1} transfers the Lebesgue measure on $(0, 1\rangle$ to a measute on $S(A)$. The above mentioned assertion is expressed more precisely by the following result.

Theorem 2. *Let v be a set function of 2^N having the property* **P**. *Let A an infinite set of positive integers. Then $v(A_x)=\frac{1}{2}v(A)$ for almost all $x\in(0, 1\rangle$.*

For v one can take $\underline{\mathrm{d}}$, $\bar{\mathrm{d}}$ or any density measure μ. It follows that the measure of the set of x with $v(A_x)=tv(A)$ and $t\neq\frac{1}{2}$ is 0. The concept of the Hausdorff dimension is useful for measuring such sets. For simplicity we restrict ourselves to sets having asymptotic density. Let dim C denote the Hausdorff dimension of C and put $h(0=h(1)=0$ and

$$h(t)=-\frac{t\log t+(1-t)\log(1-t)}{\log 2}, \qquad 0<t<1.$$

Theorem 3. *Let A be a set of positive integers with $\mathrm{d}(A)=d>0$. Let $0\leqq t\leqq 1$. Let $W(t)$ be the set of all $x\in(0, 1\rangle$ for which $\mathrm{d}(A_x)$ exists and equals td. Then* $\dim W(t)=h(t)$.

Closely related is the following result.

Theorem 4. *Let A be a set of positive integers with $\mathrm{d}(A)=d>0$. Let $0\leqq t\leqq 1$. Let $Z(t)$ be the set of all $x\in(0, 1\rangle$ for which dt is a limit point of the sequence $\{n^{-1}A_x(n)\}_{n=1}^{\infty}$. Then $\dim Z(t)=h(t)$. Further, if $0<\xi\leqq\frac{1}{2}$, then* $\dim \bigcup_{0\leqq t\leqq\xi} Z(t)=h(t)$.

Our final result has more topological character and is closely related to the main result of T. Šalát [11]. We denote the set of all limit points of sequence $\{a_n\}_{n=1}^{\infty}$ by $\{a_n\}_n'$.

Theorem 5. *Let $\bar{\mathrm{d}}(A)=\alpha>0$. Then the set of all $x\in(0, 1\rangle$ for which*

$$\left\{\frac{A_x(n)}{n}\right\}_n'=\langle 0, \alpha\rangle \tag{1}$$

is a residual set in $(0, 1\rangle$.

Corollary. *If* $\bar{d}(A)>0$, *then the set* H *of all* $x\in(0, 1\rangle$ *for which* A_x *has an asymptotic density, is a set of the first Baire category in* $(0, 1\rangle$.

We refer to G. Grekos [5] for a precise and elegant characterization of the possibilities of the asymptotic density of a subset of a given set A.

Before giving proofs of the above mentioned assertions we define two other properties of a set function v on 2^N.

a) If $A(n)\geqq B(n)$ for all n, then $v(A)\geqq v(B)$.

b) If $A, B\subset N$ and $\lim_{n\to\infty} A(n)/B(tn)=1$, then $v(A)=t\cdot v(B)$.

It is obvious that both $\underline{d}$ and $\bar{d}$ have these properties. It is an open problem whether every density measure possesses them, too. Our second author conjectures that there exists a density measure having **a)** and **b)**, but that **a)** and **b)** are not a common property of all density measures.

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2. Some elementary properties of density measures

We derive some auxiliary results for the proof of Theorem 1. Throughout this section we assume that μ is a density measure on 2^N and that the elements of sets of positive integers are arranged in increasing order.

Lemma 1. *Let* $C\subset N$ *be an infinite set and* $0\leqq t\leqq 1$. *There exists a subset* D *of* C *such that* $\lim_{n\to\infty} D(n)/C(n)=t$.

Proof. We construct D by induction. Suppose $D(n-1)\leqq tC(n-1)$. If $n\notin C$, then $n\notin D$. If $n\in C$ and $D(n-1)+1 > tC(n)$, then $n\notin D$, otherwise $n\in D$. It is obvious that $D(n)\leqq tC(n)$ for every n. On the other hand, let $n\in N$ and let m be the largest number with $m\leqq n$, $m\in C$, $m\notin D$. Then

$$C(n)-D(n)=C(m)-D(m-1)<(1-t)C(m)+1\leqq(1-t)C(n)+1.$$

Hence, $D(n)>tC(n)-1$. Thus

$$\lim_{n\to\infty}\frac{D(n)}{C(n)}=t.$$

Lemma 2. *Let* A, B, C, D *be subsets of* N *with* $D\subset A\cap B$, $A\cap C=\emptyset$ *and* $B\cap C=\emptyset$. *Then*

$$\mu(A)-\mu(B)=\mu((A\cup C)\setminus D)-\mu((B\cup C)\setminus D).$$

Proof. Since μ is finitely additive, we have

$$\mu(A\setminus D)+\mu(D)=\mu(A),\quad \mu(B\setminus D)+\mu(D)=\mu(B).$$

Hence, $\mu(A)-\mu(B)=\mu(A\setminus D)-\mu(B\setminus D)$. Similarly,

$$\mu(A\setminus D)-\mu(B\setminus D)=\mu((A\setminus D)\cup C)-\mu((B\setminus D)\cup C).$$

Since $C\cap D=\emptyset$, this gives the assertion.

Lemma 3. *If $A=\{a_1, a_2, \dots\}$ and $B=\{b_1, b_2, \dots\}$ are disjoint subsets of N such that $\lim_{n\to\infty} A(n)/B(n)=1$, then $\mu(A)=\mu(B)$.*

Proof. Let $\varepsilon>0$. Then

$$(1-\varepsilon)A(n)<B(n)<(1+\varepsilon)A(n)$$

for n larger than some p. Put $D=A\cup B$. Then

$$-\varepsilon n\leqq-\varepsilon A(n)<2B(n)-D(n)<\varepsilon A(n)\leqq\varepsilon n$$

for $n>p$. Put $E=\{e_1, e_2, \dots\}=N\setminus D$ and $C=\{e_1, e_3, \dots, e_{2k-1}, \dots\}$. Then $0\leqq 2C(n)-E(n)\leqq 1$ for every n in N. Hence,

$$-\frac{\varepsilon n}{2}\leqq B(n)+C(n)-\frac{n}{2}\leqq\frac{\varepsilon n+1}{2}$$

for $n>p$. Since $2B(n)-D(n)=D(n)-2A(n)$ for every n, we further have

$$-\frac{\varepsilon n}{2}\leqq A(n)+C(n)-\frac{n}{2}\leqq\frac{\varepsilon n+1}{2}$$

for $n>p$. Hence $\mathrm{d}(A\cup C)=\mathrm{d}(B\cup C)=\frac{1}{2}$. Since μ extends d, we obtain $\mu(A\cup C)=\mu(B\cup C)=\frac{1}{2}$. By Lemma 2 we find

$$\mu(A)-\mu(B)=\mu(A\cup C)-\mu(B\cup C)=0,$$

because $A\cap C=B\cap C=\emptyset$. This proves the assertion.

Lemma 4. *If $A=\{a_1, a_2, \dots\}$, $B=\{b_1, b_2, \dots\}$ are subsets of N with $\lim_{n\to\infty} A(n)/B(n)=1$, then $\mu(A)=\mu(B)$.*

Proof. We may assume that B is an infinite set. Let us put $D=A\cap B$ with $D=\{d_1, d_2, \dots\}$. Put $D_1=\{d_1, d_3, \dots, d_{2k-1}, \dots\}$, $D_2=\{d_2, d_4, \dots, d_{2k}, \dots\}$, $A^*=A\setminus D_1$, $B^*=B\setminus D_2$. Hence $A^*\cap B^*=A^*\cap D_1=B^*\cap D_2=D_1\cap D_2=\emptyset$. Furthermore $D_1(n)\leqq A^*(n)+1$ and $D_2(n)\leqq B^*(n)$ for every n. Observe that

$$0\leqq D_1(n)-D_2(n)\leqq 1 \tag{2}$$

for every n.

If $\{D_1(n)\}_{n=1}^{\infty}$ is bounded, then $\mu(D_1)=\mu(D_2)=0$. If $\{D_1(n)\}_{n=1}^{\infty}$ is unbounded, then, by (2) $\lim_{n\to\infty} D_1(n)/D_2(n)=1$ and hence, by Lemma 3, $\mu(D_1)=\mu(D_2)$. Thus in both cases

(3) $$\mu(D_1)=\mu(D_2).$$

It is given that for every $\varepsilon>0$ there exists an m such that $(1-\varepsilon)B(n)<A(n)<(1+\varepsilon)B(n)$ for $n>m$. Hence, by (2),

$$-1+(1-\varepsilon)B(n)-D_2(n)<A(n)-D_1(n)<(1+\varepsilon)B(n)-D_2(n).$$

It follows that

$$(1-\varepsilon)B^*(n)-\varepsilon D_2(n)-1<A^*(n)<(1+\varepsilon)B^*(n)+\varepsilon D_2(n)$$

for $n>m$. Since $B^*(n)\geqq D_2(n)$, $B^*(n)>\varepsilon^{-1}$ for n larger than some constant m', we obtain

$$1-3\varepsilon<1-\varepsilon-\frac{\varepsilon D_2(n)}{B^*(n)}-\frac{1}{B^*(n)}<\frac{A^*(n)}{B^*(n)}<1+\varepsilon+\frac{\varepsilon D_2(n)}{B^*(n)}\leqq 1+2\varepsilon$$

for $n>\max\{m, m'\}$. Thus $\lim_{n\to\infty} A^*(n)/B^*(n)=1$. By Lemma 3 we have $\mu(A^*)=\mu(B^*)$. Recall that $A^*\cap D_1=B^*\cap D_2=\emptyset$. We conclude in view of (3)

$$\mu(A)=\mu(A^*\cup D_1)=\mu(A^*)+\mu(D_1)=\mu(B^*)+\mu(D_2)=\mu(B).$$

Lemma 5. *Let $p, q\in N$. Let $A=\{a_1, a_2, \ldots\}$ and $B=\{b_1, b_2, \ldots\}$ be subsets of N such that $\lim_{n\to\infty} A(n)/B(n)=p/q$. Then $q\mu(A)=p\mu(B)$.*

Proof. It follows from Lemma 4 that each pair of subsets

$$\{a_{p+j}, a_{2p+j}, a_{3p+j}, \ldots\} \text{ with } j\in\{1, 2, \ldots, p\}$$

has the same density measure. Hence, by the finite additivity of μ each subset has the density measure $\mu(A)/p$. In particular $\mu(A^*)=\mu(A)/p$, where $A^*=\{a_p, a_{2p}, a_{3p}, \ldots\}$ and, similarly, $\mu(B^*)=\mu(B)/q$, where $B^*=\{b_q, b_{2q}, b_{3q}, \ldots\}$. Note that $A^*(n)\sim A(n)/p$ and $B^*(n)\sim B(n)/q$ as $n\to\infty$. Hence,

$$\lim_{n\to\infty}\frac{A^*(n)}{B^*(n)}=\frac{q}{p}\lim_{n\to\infty}\frac{A(n)}{B(n)}=1.$$

A second application of Lemma 4 implies that $\mu(A^*)=\mu(B^*)$. Thus $\mu(A)/p=\mu(B)/q$.

3. Proof of Theorem 1

Let μ be some density measure on 2^N and let A and B be subsets of N with $\lim_{n\to\infty} A(n)/B(n)=t$. If A is a finite set and $t\neq 0$, then B is finite, too. Therefore we can assume that A is an infinite set.

If $t=0$, then $A(n)=o(B(n))=o(B(n))=o(n)$ and hence $\mu(A)=\mathrm{d}(A)=0$.

Let $t\neq 0$. Let p/q be any rational number with $t>p/q$. Then we can construct a subset A^* of A with $\lim\limits_{n\to\infty} A^*(n)/B(n)=p/q$ by applying Lemma 1. It follows from Lemma 5 that $q\mu(A^*)=p\mu(B)$. Since

$$\mu(A)=\mu(A^*)+\mu(A\setminus A^*)\geqq(A^*),$$

we obtain

$$\mu(A)\geqq\mu(A^*)=\frac{p}{q}\,\mu(B).$$

This result holds for every rational number p/q less than t. Hence $\mu(A)\geqq t\mu(B)$. On the other hand, $\lim\limits_{n\to\infty} B(n)/A(n)\geqq t^{-1}$. By the same argument

$$\mu(B)\geqq t^{-1}\mu(A). \text{ Thus } \mu(A)=t\mu(B).$$

4. Proofs of Theorems 2, 3 and 4

To understand the proofs it is important to note that $\{n^{-1}\cdot A_x(a_n)\}_{n=1}^{\infty}$ measures the relative density of A_x with respect to A and it is therefore independent of the structure of A itself.

Denote by E the set of all dyadically normal numbers from $(0, 1\rangle$. A well-known theorem of Borel states that the Lebesgue measure of E is 1 (cf. [9], Ch. 8). If $x\in E$, then

$$\lim_{n\to\infty}\frac{A_x(a_n)}{A(a_n)}=\lim_{n\to\infty}\frac{A_x(a_n)}{n}=\frac{1}{2}.$$

Since ν has the property **P**, we have $\nu(A_x)/\nu(A)=1/2$ for $x\in E$. Thus $\nu(A_x)=\nu(A)/2$ for almost all x. This proves Theorem 2. $W(t)$ in Theorem 3 corresponds to the set of $x\in(0, 1\rangle$ with

$$\lim_{n\to\infty}\frac{A_x(a_n)}{A(a_n)}=\lim_{n\to\infty}\frac{A_x(a_n)}{a_n}\cdot\lim_{n\to\infty}\frac{a_n}{A(a_n)}=dtd^{-1}=t$$

The Hausdorff dimension of this set is $h(t)$ (cf. [10], p. 195). This proves Theorem 3.

In order to prove Theorem 4 we first show that $Z(t)$ is precisely the set of numbers x for which t is a limit point of the sequence $\{n^{-1}A_x(a_n)\}_{n=1}^{\infty}$.

If t is a limit point of $\{n^{-1}A_x(a_n)\}_{n=1}^{\infty}$, then there exists a sequence $\{n_i\}_{i=1}^{\infty}$ such that

$$n_i^{-1}A_x(a_{n_i})\to t\quad(i\to\infty).$$

Hence

$$a_{n_i}^{-1}A_x(a_{n_i})=\frac{n_i}{a_{n_i}}\cdot\frac{A(a_{n_i})}{n_i}\to dt(i\to\infty).$$

This implies that $d\cdot t$ is a limit point of the sequence

$$\{n^{-1}A_x(n)\}_{n=1}^{\infty}.$$

Conversely, suppose $d\cdot t$ is a limit point of

$$\{n^{-1}A_x(n)\}_{n=1}^{\infty}.\text{ Let } A=\{a_1, a_2, \ldots\}$$

with elements in increasing order. There exists a sequence $\{k_i\}_{i=1}^{\infty}$ such that $k_i^{-1}A_x(k_i)\to d\cdot t$, $i\to\infty$. For every k_i there is an n_i such that $a_{n_i}\leqq k_i<a_{n_i+1}$. We have $A_x(a_{n_i})=A_x(k_i)$ and hence

$$\frac{A_x(a_{n_i})}{a_{n_i}}\cdot\frac{a_{n_i}}{a_{n_i+1}}<\frac{A_x(k_i)}{k_i}\leqq\frac{A_x(a_{n_i})}{a_{n_i}}.$$

Since

$$\frac{a_{n_i}}{a_{n_i+1}}=\frac{a_{n_i}}{n_i}\cdot\frac{n_i}{n_i+1}\cdot\frac{n_i+1}{a_{n_i+1}}=d^{-1}\cdot 1\cdot d=1$$

as $i\to\infty$, we obtain

$$\frac{A_x(a_{n_i})}{a_{n_i}}\to d\cdot t \quad\text{as}\quad i\to\infty.$$

It follows that

$$\frac{A_x(a_{n_i})}{n_i}=\frac{A_x(a_{n_i})}{a_{n_i}}\cdot\frac{a_{n_i}}{n_i}\to t \quad\text{as}\quad i\to\infty.$$

We conclude that t is a limit point of $\{n^{-1}A_x(a_n)\}_{n=1}^{\infty}$. It is known that the Hausdorff dimension of the set

$$\{x\in(0,1\rangle\,;\,t\in\{n^{-1}A_x(a_n)\}'\}$$

is $h(t)$ (cf. [10], p. 195). This proves the first part of Theorem 4. The second part follows from the first part of the proof on using Theorem 7 of [10], p. 195.

5. Proof of Theorem 5

Let $D=\{k2^{-n}\,;\,k,n\in N,\,0<k\leqq 2^n\}$. Put $X=(0,1\rangle\setminus D$. For $z\in(0,\alpha)$ denote by $M(z)$ the set of $x\in X$ for which $z\in\{n^{-1}A_x(n)\}_n'$. For $k,n\in N$ put

$$M(z,k,n)=\left\{x\in X;\left|\frac{A_x(n)}{n}-z\right|<\frac{1}{k}\right\}.$$

It follows from the construction of the dyadic expansions of real numbers that $M(z, k, n)$ is a union of some sets of the form

$$\left(\frac{j}{2^n}, \frac{j+1}{2^n}\right) \cap X \quad (0 \leqq j < 2^n).$$

Hence the set $M(z, k, n)$ is open in X. This implies that

$$M(z) = \bigcap_{k=1}^{\infty} \bigcup_{n-1}^{\infty} M(z, k, n)$$

is a G_δ-set in X.

We shall show that $M(z)$ is dense in X.

Let $x_0 = \sum_{k=1}^{\infty} \varepsilon_k(x_0) 2^{-k}$ (non-terminating dyadic expansion of x_0), $x_0 \in X$, $\eta > 0$. Choose a positive integer m such that $2^{-m} < \eta$. It follows from Lemma 1 that A has a subset A^* such that $\lim_{n \to \infty} A^*(n)/A(n) = z/\alpha$. Hence $\bar{d}(A^*) = z$. Let $A^* = \{a_{k_1}, a_{k_2}, \ldots\}$ with $a_{k_1} < a_{k_2} < \ldots$. Put

$$\varepsilon_j = \begin{matrix} \varepsilon_j(x_0) & \text{for } j \leqq m, \\ 0 & \text{for } j \neq k_i \ (i = 1, 2, \ldots), \\ 1 & \text{for } j > m, j = k_i \text{ for some } i. \end{matrix}$$

Put $x = \sum_{j=1}^{\infty} \varepsilon_j 2^{-j} \in (0, 1\rangle$. Since $\bar{d}(A^*) = z$ we have $x \in M(z)$. According to the choice of m we have $|x - x_0| < \eta$. Thus $M(z)$ is dense in X. Since $M(z)$ is a dense G_δ-set in X, it is residual in X (cf. [6], p. 49).

Now let $Z \subset (0, \alpha)$ be a countable dense set in $(0, \alpha)$. Then it follows that $M = \bigcap_{z \in Z} M(z)$ is residual in X. Since $(0, 1\rangle \setminus X$ is countable, the set M is also residual in $(0, 1\rangle$. This finishes the proof, for (1) holds for every $x \in M$.

Proof of the Corollary. The set H is a subset of the set H^* of all $x \in (0, 1\rangle$ for which (1) does not hold. According to Theorem 5 the set H^* is a set of the first category in $(0, 1\rangle$. Thus H is also a set of the first category in $(0, 1\rangle$.

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АСИМПТОТИЧЕСКИЕ ПЛОТНОСТИ МНОЖЕСТВ НАТУРАЛЬНЫХ ЧИСЕЛ

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Резюме

В работе рассматриваются свойства конечно-аддитивных мер на 2^N в связи с асимптотическими плотностями множеств $A \subset N = \{1, 2, \ldots, n, \ldots\}$.