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REMARK ON ONE-SIDED A-IDEALS OF SEMIGROUPS

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The autors of papers [2], [3] and [5] are dealing with properties of A-ideals of semigroups. Here the possibility of generating some one-sided A-ideals by functions is discused. This fact enables us to give a simple proof of existence of minimal one-sided A-ideals of free semigroups generated by an infinite set of generators. This result is a generalization of the result obtained in paper [5] to the case of a semigroup having an uncountable set of generators. Unfortunately the construction cannot be used in the case if the free semigroup has a finite set of generators (then the more complicated construction of paper [5] can be used).

Definition 1 (see [2]). *Let S be a semigroup (grupoid), G a nonempty subset of S and let for every $s \in S$ there exist a $g \in G$ such that $sg \in G$. Then G is called left A-ideal of the semigroup (grupoid) S. Similarly the right A-ideal is defined. The set G is a two-sided A-ideal of S if it is a left and a right A-ideal of S.*

Remark (see [2]). Every left ideal of S is a left A-ideal of S. Every set containing a left A-ideal is a left A-ideal.

Lemma 2. *The following statements are equivalent:*
(i) *G is a left A-ideal of S.*
(ii) *There exists a function f: $S \to G$ such that $\{sf(s) \mid a \in S\} \subseteq G$.*
(iii) *There exists a function f: $S \to S$ such that $f(S) \cup \{sf(s) \mid s \in S\} \subseteq G$.*

Lemma 3. *Let S be a grupoid and f: $S \to S$. Then the set $f(S) \cup \{sf(s) \mid s \in S\}$ is a left A-ideal of S.*

Definition 4. *We shall say that the left A-ideal G of S, $G = f(S) \cup \{sf(s) \mid s \in S\}$ is generated by the function f: $S \to S$.*

Theorem 5. *There exists a semigroup S and a left A-ideal of S which cannot be generated by any function f: $S \to S$.*

Proof. Let S be the free semigroup generated by two elements a and b.

The principal left ideal S^1a is a left A-ideal of S. The set $G = S^1a \cup \{b\}$ is also a left A-ideal. We shall show that G cannot be generated by any function f: $S \to S$.

Clearly $S^1a \cap S^1b = \emptyset$ since S is a free semigroup. Let $G = S^1a \cup \{b\} =$

$f(S) \cup \{sf(s) \mid s \in S\}$ hold for a function $f\colon S \to S$. Then we have either $b \in f(S)$ or $b \in \{sf(s) \mid s \in S\}$. In the second case there exists an element $s_0 \in S$ such that $b = s_0 f(s_0)$. But this is impossible, because S is a free semigroup and b is its generator.

In the first case there exists an $s_0 \in S$ such that $b = s_0$. Then we have $s_0 b = s_0 f(s_0) \in G = S^1 a \cup \{b\}$. Hence $s_0 b \in S^1 a$ and $s_0 b \in S^1 b$ hold. But this is impossible.

In both cases we have obtained a contradiction. Therefore G cannot be generated by a function $f\colon S \to S$.

Theorem 6. *Every left A-ideal of a grupoid S is either generated by a function $f\colon S \to S$ or it contains a left A-ideal generated by such a function.*

Proof. Theorem 6 is a consequence of Lemma 2 and Theorem 5.

Theorem 7. *Let S be a grupoid every element of which is idempotent. Then every left A-ideal of S is generated by a function $f\colon S \to S$.*

Proof. Let G be a left A-ideal of the grupoid S. Then there exists such a function $f\colon S \to G$, that $\{sf(s) \mid s \in S\} \subseteq G$. We can form a new function $f^*\colon S \to S$, $f^*(s) = s$ if $s \in G$ and $f^*(s) = f(s)$ if $s \notin G$. Then $f^*(S) = G$ and $\{sf^*(s) \mid s \in S\} = G$, hence $f^*(S) \cup \{sf^*(s) \mid s \in S\} = G$.

Theorem 8. *Let L be a left ideal of a grupoid S. Then there exists a function $f\colon S \to S$ generating L.*

Proof. It is sufficient to take an arbitrary function $f\colon S \to S$ satisfying $f(S) = L$. Since $\{sf(s) \mid s \in S\} \subseteq L$, we have $f(S) \cup \{sf(s) \mid s \in S\} = L$.

Theorem 9. *Let S be a semigroup and $S^1 a$ the principal left ideal generated by the element $a \in S$. Then $S^1 a$ is generated by the function $f\colon S \to S$, $f(s) = a$.*

Proof. Clearly $S^1 a = \{a\} \cup Sa = f(S) \cup \{sa \mid s \in S\} = f(S) \cup \{sf(s) \mid s \in S\}$ holds.

Remark. Theorem 9 need not be true for a grupoid. Let S be the grupoid given by the multiplicative table:

·	a	b	c
a	b	c	a
b	b	c	a
c	b	c	a

Let $(x)_L$ denote the left principal ideal, generated by the element x. Then $(a)_L = (b)_L = (c)_L = S$. But for the function $f\colon S \to S$, $f(s) = a$ we have $f(S) \cup \{sf(s) \mid s \in S\} = \{a, b\} \neq S$. For the function $g\colon S \to S$, $g(s) = b$ we have

$g(S) \cup \{sg(s) \mid s \in S\} = \{b, c\} \neq S$ and for the function $h\colon S \to S$, $h(s) = c$ we have $h(S) \cup \{sh(s) \mid s \in S\} = \{a, c\} \neq S$.

Let $|X|$ denote the cardinality of the set X.

Theorem 10. *Let S be a free semigroup with an infinite set X of generators. Let n be a fixed positive integer and $Y \subset S$ be a set such that $|X| = |Y|$ and every word of Y has length n. Then every bijection $f\colon S \to Y$ generates a minimal left A-ideal of S (i. e. $G = f(S) \cup \{sf(s) \mid s \in S\}$ is a minimal left A-ideal of S).*

First we prove

Lemma 11. *Let S, X and Y satisfy the hypotheses of Theorem* 10 *and let $f\colon S \to Y$ be a bijection. If $s_1 f(s_2) = s_3 f(s_3)$, then $s_1 = s_2 = s_3$.*

Proof. The equality $s_1 f(s_2) = s_3 f(s_3)$ of words $s_1 f(s_2)$ and $s_3 f(s_3)$ implies the equality $f(s_2) = f(s_3)$ of their end segments $f(s_2)$ and $f(s_3)$ and the equality $s_1 = s_3$ of initial segments s_1 and s_3 of these words. The bijectivity implies $s_2 = s_3$.

Proof of Theorem 10. We shall show that if we omit an arbitrary element of the left A-ideal $G = f(S) \cup \{sf(s) \mid s \in S\} = Y \cup \{sf(s) \mid s \in S\}$, we do not get a left A-ideal. From this it follows, that G is a minimal left A-ideal.

I) First we prove that $G' = G \setminus \{f(s_0)\}$ is not a left A-ideal. Suppose that G' is a left A-ideal of S. Then for the element $s_0 \in S$ there exists a $g' \in G'$ such that $s_0 g' \in G' \subset G = f(S) \cup \{sf(s) \mid s \in S\}$. Since $s_0 g' \notin f(S) = Y$, we have $s_0 g' \in \{sf(s) \mid s \in S\}$.

If $g' = f(s_1)$ for some $s_1 \in S$, then $s_0 g' = s_0 f(s_1) = s_2 f(s_2)$. Hence by Lemma 11 we get $s_0 = s_1 = s_2$. This implies that $f(s_0) = f(s_1) = g' \in G' = G \setminus \{f(s_0)\}$ which is a contradiction.

If $g' = s_2 f(s_2)$, then $s_0 g' = s_0 s_2 f(s_2) = s_3 f(s_3)$. From this we get by Lemma 11 that $s_0 s_2 = s_2 = s_3$. But this is again a contradiction since in a free semigroup $s_0 s_2 = s_2$ does not hold.

II) Now we shall prove that $F' = G \setminus \{s_0 f(s_0)\}$ is not a left A-ideal of S. Suppose it is a left A-ideal. Then for the element $s_0 \in S$ there exists a such $f' \in F'$ that $s_0 f' \in F' \subset G = f(S) \cup \{sf(s) \mid s \in S\}$. Since again $s_0 f' \notin f(S) = Y$ we have $s_0 f' \in \{sf(s) \mid s \in S\}$.

If $f' = f(s_1)$ for an element $s_1 \in S$ then we have $s_0 f' = s_0 f(s_1) = s_2 f(s_2)$. From this by Lemma 11 we get that $s_0 = s_1 = s_2$. Hence $s_0 f(s_0) = s_0 f' \in F' = G \setminus \{s_0 f(s_0)\}$, which is a contradiction.

If $f' = s_2 f(s_2)$ then $s_0 f' = s_0 s_2 f(s_2) = s_3 f(s_3)$. By Lemma 11 this implies $s_0 s_2 = s_2 = s_3$. But this is again a contradiction, because in a free semigroup $s_0 s_2 = s_2$ does not hold.

The proof is completed.

Theorem 12. *Every left A-ideal of a free grupoid or of a free semigroup is an infinite set.*

Proof. Suppose that a left A-ideal G of a free grupoid or of a free semigroup S is a finite set. Let $s \in S$ be such a word the length of which is greater than the lengths of all words of G. Then in G there exists no element g such that $sg \in G$ since for every $g \in G$ the length of the word sg is greater than the length of an arbitrary word of G.

In the following we shall present some results about (lower) semilattices without zero.

Lemma 13. *Let S be a semilattice without zero and G an A-ideal of S. Then for every element $s \in S$ there exists such an element $g \in G$ that $g < s$.*

Proof. Since S is a (lower) semilattice, there exists $s' \in S$ such that $s' < s$. Because G is an A-ideal of S, for the element $s' \in S$ there exists a $g' \in G$ such that $g = s'g' \in G$. Clearly $g = s'g' \leqslant s' < s$ holds. Hence $g \in G$ and $g < s$.

Corollary 14. *Let S be a semilattice without zero. Then a nonempty subset G of S is an A-ideal of S iff for every element $s \in S$ there exists an element $g \in G$ such that $g < s$.*

Corollary 15. *Let S be a semilattice without zero. Then the following statements hold.*

(i) *If G is an A-ideal of S, then for every element $s \in S$ there exists an infinite chain of distinct elements of G that are less than s.*

(ii) *If G is an A-ideal of S, then G is an infinite set.*

(iii) *A nonempty subset $G \subseteq S$ is an A-ideal of S iff for every $s \in S$ there exists an infinite chain of distinct elements of G that are less then s.*

(iv) *Let G be an A-ideal of S and $a \in G$. Then $G' = G \setminus \{a\}$ is also an A-ideal of S.*

Proof. The validity of (i), (ii) and (iii) is evident. (iv) follows from (i). It is sufficient for every $s \in S$ to take an element $g \neq a$ from the infinite chain of elements of G that are less than s. Then $g \in G'$ and $sg = g \in G'$. Hence for every element $s \in S$ there exists such a $g \in G'$ that $sg \in G'$ i. e. G' is an A-ideal of S too. (By the way the function f: $S \to S$, $f(s) = g$ generates the A-ideal $F = f(S) \cup \{sf(s) \mid s \in S\}$ for which $F \subset G$, $F \neq G$ and $a \notin F$ hold.)

(iv) of Corollary 15 implies

Theorem 16. *A semilattice without zero has no minimal A-ideal.*

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ЗАМЕТКА ОБ ОДНОСТОРОННЫХ А-ИДЕЛАХ ПОЛУГРУПП

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Резюме

При помощи порождающих функций дана конструкция некоторых односторонных минимальных А-идеалов свободной полугруппы над бесконечным множеством. Доказано, что полуструктура без нулевого элемента минимальные А-идеалы — не содержит.