Eugen Ružický; Jozef Tvarožek

A conjecture on Lie algebras admitting a regular automorphism of finite order

Mathematica Slovaca, Vol. 35 (1985), No. 1, 77--81

Persistent URL: http://dml.cz/dmlcz/136377

Math. Slovaca 35, 1985, No. 1, 77—81

A CONJECTURE ON LIE ALGEBRAS ADMITTING A REGULAR AUTOMORPHISM OF FINITE ORDER

EUGEN RUŽICKÝ—JOZEF TVAROŽEK

Let $\mathscr{L}$ be a Lie algebra over a field F of characteristic $p \geqslant 0$ admitting a regular automorphism[1]) $A: \mathscr{L} \to \mathscr{L}$ of order n, $n \geqslant 2$. According to V. A. Kreknin, [2], the Lie algebra $\mathscr{L}$ is solvable and the length $l(\{\mathscr{L}^{(i)}\})$ of the derived series $\{\mathscr{L}^{(i)}\}$ of $\mathscr{L}$ is bounded from above by the integer 2^{n-1}. This estimate is rather rough, it seems to be possible to improve it. O. Kowalski in 1981 proposed the following

Conjecture. $l(\{\mathscr{L}^{(i)}\}) \leqslant n-1$.

The purpose of this paper is to prove the Conjecture for $n = 2, \ldots, 7$.

First we recall some basic notions and facts. Without loss of generality the field F can be supposed to be algebraicly closed. Further, if $p > 0$ (a prime number), we can suppose that $(n, p) = 1$, i.e. n, p are relatively prime. In fact, let r be the greatest number such that $p^r \mid n$. Then $A^{p^r}: \mathscr{L} \to \mathscr{L}$ is a regular automorphism of order $n' = n/p^r$ and $(n', p) = 1$ (see [2]). Since $(n, p) = 1$, all roots of the minimal polynomial of the automorphism A are different.

Choose some primitive[2]) nth root of $1 \in F$ and denote it by α. Let $\mathscr{L}_i$ be the characteristic subspace corresponding to the root $\alpha_i = \alpha^i$ of the minimal polynomial of the automorphism A, $i = 1, \ldots, n-1$. Then $\mathscr{L} = \sum_{i=1}^{n-1} \mathscr{L}_i$ and $A(x_i) = \alpha_i x_i$ for all $x_i \in \mathscr{L}_i$, $i = 1, \ldots, n-1$. Since A is an automorphism of the Lie algebra $\mathscr{L}$, we have

$$[\mathscr{L}_i, \mathscr{L}_j] \subset \mathscr{L}_{i+j} \tag{1}$$

for all $i, j \in \{1, \ldots, n-1\}$, see [1]. As usually, the index $i+j$ is taken modulo n and $\mathscr{L}_0 = \mathbf{0}$ in the formula (1).

Let $\mathscr{L} = \mathscr{L}^{(0)} \supset \mathscr{L}^{(1)} \supset \ldots \supset \mathscr{L}^{(k)} \supset \ldots$ be the derived series of the Lie algebra $\mathscr{L}$. Every $\mathscr{L}^{(k)}$ is a vector subspace of the vector space $\mathscr{L}$. Let $\mathscr{L}_i^{(k)} = \mathscr{L}^{(k)} \cap \mathscr{L}_i$, $i = 0, 1,$

[1]) Automorphism without non zero fixed vectors.

[2]) An element α of the field F is a primitive nth root of $1 \in F$ if $\alpha^n = 1$ and $\alpha^k \neq 1$ for all k, $0 < k < n$.

..., $n-1$, $k \in N$. The subspace $\mathscr{L}_i^{(k)}$ is generated by the set $\{[x, y];\ x \in \mathscr{L}_p^{(k-1)}, y \in \mathscr{L}_q^{(k-1)},\ i = p+q\}$, shortly

$$\mathscr{L}_i^{(k)} = \sum_{i=p+q} [\mathscr{L}_p^{(k-1)}, \mathscr{L}_q^{(k-1)}], \tag{2}$$

where the indices i, p and q are taken modulo n.

Further, for all h, $k \in N$, $h \leqslant k$, and for every $i \in \{0, 1, \ldots, n-1\}$ we have

$$\mathscr{L}_i^{(k)} \subset \mathscr{L}_i^{(h)}. \tag{3}$$

Let $r \in \{1, \ldots, n-1\}$ be a given number for which $(r, n) = 1$. Denote G_n the multiplicative group of nth roots of $1 \in F$, i.e. $G_n = \{\alpha^i;\ i = 0, 1, \ldots, n-1\}$. The map f_r: $G_n \to G_n$, $f_r(\alpha_i) = \alpha_i^r$ is a group isomorphism. The isomorphism f_r represented on the aditive group Z_n of cosets modulo n (under the identification $\alpha^i \equiv i$) will be denoted by F_r.

Let the symbol $\mathscr{L}'_i$ denotes some subspace $\mathscr{L}_i^{(k)}$ in the case when it is not necessary to specify k, $i = 0, 1, \ldots, n-1$. Since α is a primitive nth root of 1 and since f_r is a isomorphism of G_n, $f_r(\alpha)$ is a primitive nth root of 1 too. Making use of this fact and (1), we get the following

Proposition 1. Let Ω be any inclusion or equality, derived from (2) using (1), (3) and Jacobi's identity, containing sums of vector subspaces $\mathscr{L}'_i$, $[\mathscr{L}'_j, \mathscr{L}'_k]$ for some $i, j, k \in \{0, \ldots, n-1\}$. Then Ω is preserved if all terms $\mathscr{L}'_1, \ldots, \mathscr{L}'_{n-1}$ contained in Ω are replaced by the terms $\mathscr{L}'_{F_r(1)}, \ldots, \mathscr{L}'_{F_r(n-1)}$.

Corollary. Let $\mathscr{L}_i^{(k)} = \mathbf{0}$ for some $i \in (1, \ldots, n-1\}$. Then $\mathscr{L}_j^{(k)} = \mathbf{0}$ for all $j \in \{1, \ldots, n-1\}$ such that $(i, n) = (j, n)$.

Proof. Since $(i, n) = (j, n)$, there is an integer $r \in \{1, \ldots, n-1\}$ such that $(r, n) = 1$ and $f_r(\alpha_i) = \alpha_j$. Applying Proposition 1 we get that $\mathscr{L}_{F_r(i)}^{(k)} = \mathbf{0}$, i.e. $\mathscr{L}_j^{(k)} = \mathbf{0}$.

The next proposition is useful for the practical computation.

Proposition 2. Let i, j, $k \in \{1, \ldots, n-1\}$. Then

a) $i + j = n \Rightarrow [[\mathscr{L}_i, \mathscr{L}_i], \mathscr{L}_j] = \mathbf{0}$

b) $i + j = n \Rightarrow [[\mathscr{L}_i, \mathscr{L}_i], [\mathscr{L}_j, \mathscr{L}_j]] = \mathbf{0}$

c) $i + k = n$, $j + 2k \equiv 0 (\text{mod } n) \Rightarrow [[\mathscr{L}_i, \mathscr{L}_j], [\mathscr{L}_k, \mathscr{L}_k]] = \mathbf{0}$.

Proof. We prove only part a) because the rest of the proof is similar. Taking use of Jacobi's identity and (1) we get: $[[\mathscr{L}_i, \mathscr{L}_i], \mathscr{L}_j] \subset [[\mathscr{L}_i, \mathscr{L}_j], \mathscr{L}_i] \subset [\mathscr{L}_0, \mathscr{L}_i] = \mathbf{0}$.

Proof of the Conjecture for $n = 2, \ldots, 7$.

The case $n=2$ is trivial because $\mathscr{L}=\mathscr{L}_1$ and $\mathscr{L}_1^{(1)}=[\mathscr{L}_1, \mathscr{L}_1]=\mathbf{0}$.
In order to simplify our next formulae we shal introduce the following notation:

$$\mathbf{i}=\mathscr{L}_i$$
$$\mathbf{i}^p=\mathscr{L}_i^{(p)}$$
$$\mathbf{ij}=[\mathscr{L}_i, \mathscr{L}_j]$$
$$\mathbf{i}^p\mathbf{j}^q=[\mathscr{L}_i^{(p)}, \mathscr{L}_j^{(q)}],$$

where $i, j \in \{1, \ldots, n-1\}$, $p, q \in N$, $p>0$, $q>0$.

$n=3$. The Lie algebra $\mathscr{L}$ decomposes in a direct sum of the subspaces $\mathbf{1}$ and $\mathbf{2}$. Using (2) we get $\mathbf{1}^1=\mathbf{22}$ and $\mathbf{2}^1=\mathbf{11}$. Then $\mathbf{1}^2=\mathbf{2}^1\mathbf{2}^1=\mathbf{2}^1(\mathbf{11})=\mathbf{0}$ according to Proposition 2. Applying Corollary of Proposition 1 we get $\mathbf{2}^2=\mathbf{0}$. Hence $l(\{\mathscr{L}^{(i)}\})\leqslant 2$.

$n=4$. As in the case $n=3$ we get $\mathbf{1}^1=\mathbf{23}$, $\mathbf{2}^1=\mathbf{11}+\mathbf{33}$, $\mathbf{3}^1=\mathbf{12}$. Then $\mathbf{1}^2=\mathbf{2}^1\mathbf{3}^1=$ $=(\mathbf{11}+\mathbf{33})(\mathbf{12}\subset(\mathbf{11})\mathbf{3} + (\mathbf{33})(\mathbf{12})=\mathbf{0}$, $\mathbf{2}^2=\mathbf{1}^1\mathbf{1}^1+\mathbf{3}^1\mathbf{3}^1$ and $\mathbf{3}^2=\mathbf{0}$ by Corollary of Proposition 1. Further $\mathbf{2}^3=\mathbf{1}^2\mathbf{1}^2+\mathbf{3}^2\mathbf{3}^2=\mathbf{0}$ and $l(\{\mathscr{L}^{(i)}\})\leqslant 3$.

$n=5$. By the direct computation using (2), Jacobi's identity and Proposition 1 it can be shown that

$$\mathbf{1}^1\mathbf{1}^1\subset\mathbf{34},\ \mathbf{1}^1\mathbf{2}^1\subset\mathbf{44},\ \mathbf{1}^1\mathbf{3}^1\subset\mathbf{22}. \tag{4}$$

From (2) (4) and Proposition 1 we get $\mathbf{2}^2\subset\mathbf{11}$ and $\mathbf{3}^2\subset\mathbf{44}$, $\mathbf{12}$, i.e. $\mathbf{3}^2\subset\mathbf{44}$ and $\mathbf{3}^2\subset\mathbf{12}$. Then $\mathbf{1}^3 = \mathbf{2}^2\mathbf{4}^2 + \mathbf{3}^2\mathbf{3}^2 \subset (\mathbf{11})\mathbf{4}^2 + (\mathbf{44})(\mathbf{12})=\mathbf{0}$. Hence $\mathbf{2}^3=\mathbf{3}^3=\mathbf{4}^3=\mathbf{0}$ by Corollary of Proposition 1, thus $l(\{\mathscr{L}^{(i)}\})\leqslant 3$. We see that in this case the Conjecture holds in the stronger form $l(\{\mathscr{L}^{(i)}\})\leqslant n-2$.

$n=6$. After some computation we get from (2) that $\mathbf{1}^2\subset\mathbf{25}$, $\mathbf{2}^2\subset\mathbf{11}+\mathbf{35}$, $\mathbf{11}+\mathbf{44}$, $\mathbf{35}+\mathbf{44}$, $\mathbf{3}^2\subset\mathbf{12}, \mathbf{45}$. Proposition 1 for $r=5$ implies that $\mathbf{4}^2\subset\mathbf{55}+\mathbf{13}$, $\mathbf{55}+\mathbf{22}$, $\mathbf{13}+\mathbf{22}$, $\mathbf{5}^2\subset\mathbf{14}$. Then $\mathbf{3}^2\mathbf{5}^2 \subset (\mathbf{12})(\mathbf{14}) \subset \mathbf{11}$ and $\mathbf{4}^2\mathbf{4}^2 \subset (\mathbf{13}+\mathbf{22})(\mathbf{22}+\mathbf{55}) \subset$ $\subset (\mathbf{13})(\mathbf{22}) + (\mathbf{22})(\mathbf{22}) + (\mathbf{13})(\mathbf{55}) + (\mathbf{22})(\mathbf{55})\subset\mathbf{11}$.

We have just proved that

$$\mathbf{2}^3\subset\mathbf{11} \tag{5}$$

and by Proposition 1 also

$$\mathbf{4}^3\subset\mathbf{55}. \tag{6}$$

Using (5) and (6) we get $\mathbf{1}^3\mathbf{4}^3\subset\mathbf{1}^3(\mathbf{55})=\mathbf{0}$ and $\mathbf{2}^3\mathbf{3}^3\subset(\mathbf{11})(\mathbf{45})=\mathbf{0}$. Then $\mathbf{2}^3\mathbf{5}^3=$ $=\mathbf{3}^3\mathbf{4}^3=\mathbf{0}$ and

$$\mathbf{1}^4=\mathbf{5}^4=\mathbf{0}. \tag{7}$$

From (2) and (7) we have

$$\mathbf{1}^5=\mathbf{3}^5=\mathbf{5}^5=\mathbf{0},\ \mathbf{2}^5=\mathbf{4}^4\mathbf{4}^4,\ \mathbf{4}^5=\mathbf{2}^4\mathbf{2}^4. \tag{8}$$

Making use of $\mathbf{2}^3(\mathbf{44})=\mathbf{2}^3(\mathbf{3}^2\mathbf{5})=\mathbf{0}$ we prove

$$\mathbf{2}^4\mathbf{2}^4=\mathbf{0}. \tag{9}$$

In fact, $\mathbf{2}^4\mathbf{2}^4 = (\mathbf{1}^3\mathbf{1}^3)(\mathbf{1}^3\mathbf{1}^3) \subset ((\mathbf{2}^2\mathbf{5}^2 + \mathbf{3}^2\mathbf{4}^2)\mathbf{1})(\mathbf{11}) \subset (((\mathbf{11}+\mathbf{44})\mathbf{5}^2)\mathbf{1})(\mathbf{11}) +$ $+ (((\mathbf{45})\mathbf{4})\mathbf{1})(\mathbf{11}) \subset ((\mathbf{45})\mathbf{5}^2)(\mathbf{11}) + ((\mathbf{55})4(\mathbf{11}) + ((\mathbf{45})\mathbf{5})(\mathbf{11})=\mathbf{0}$ using Jacobi's identity and Proposition 2.

Applying Proposition 1 for $r=5$ we get from (9) that

$$\mathbf{4}^4\mathbf{4}^4=\mathbf{0}. \tag{10}$$

Results (8) – (10) imply $l(\{\mathscr{L}^{(i)}\})\leqslant 5$.

$n=7$. By the standard computation using (2), Proposition 1 and Proposition 2 one can obtain the following inclusions:

$$\mathbf{2}^2\mathbf{6}^2\subset\mathbf{35},\ \mathbf{3}^2\mathbf{5}^2\subset\mathbf{26},\ \mathbf{4}^2\mathbf{4}^2\subset\mathbf{26},\ \mathbf{35}. \tag{11}$$

Then

$$\mathbf{1}^3\subset\mathbf{26},\ \mathbf{35}. \tag{12}$$

Computing $\mathbf{3}^3\mathbf{6}\subset\mathbf{11}$ and $\mathbf{4}^2\mathbf{5}^3\subset\mathbf{11}$ we get

$$\mathbf{2}^4\subset\mathbf{11}. \tag{13}$$

Taking use of the equalities $\mathbf{1}^3(\mathbf{2}^3\mathbf{6}^3)=(\mathbf{3}^3\mathbf{5}^3)(\mathbf{4}^3\mathbf{4}^3)=(\mathbf{3}^3\mathbf{5}^3)(\mathbf{3}^3\mathbf{5}^3)=(\mathbf{44})(\mathbf{4}^3\mathbf{4}^3)=\mathbf{0}$ we prove that

$$\mathbf{1}^4\mathbf{1}^4=\mathbf{0}. \tag{14}$$

In fact, we have $\mathbf{1}^4\mathbf{1}^4 = (\mathbf{2}^3\mathbf{6}^3 + \mathbf{3}^3\mathbf{5}^3 + \mathbf{4}^3\mathbf{4}^3)(\mathbf{2}^3\mathbf{6}^3+\mathbf{3}^3\mathbf{5}^3+\mathbf{4}^3\mathbf{4}^3) \subset \mathbf{1}^3(\mathbf{2}^3\mathbf{6}^3) +$ $+(\mathbf{3}^3\mathbf{5}^3)(\mathbf{3}^3\mathbf{5}^3) + (\mathbf{44})(\mathbf{4}^3\mathbf{4}^3) + (\mathbf{3}^3\mathbf{5}^3)(\mathbf{4}^3\mathbf{4}^3)=\mathbf{0}$. Further, from (12) and (13) using Proposition 1 and Proposition 2 one can get

$$\mathbf{1}^4\mathbf{2}^4\subset(\mathbf{44})(\mathbf{36})=\mathbf{0},\ \mathbf{1}^4\mathbf{3}^4\subset(\mathbf{44})\mathbf{3}=\mathbf{0}. \tag{15}$$

From (14), (15) and Proposition 1 it follows that $\mathbf{a}^4\mathbf{b}^4=\mathbf{0}$ for every $\mathbf{a},\mathbf{b}\in\{\mathbf{1},\dots,\mathbf{6}\}$. Thus $l(\{\mathscr{L}^{(i)}\})\leqslant 5$. As in the case $n=5$ the Conjecture holds in the stronger form $l(\{\mathscr{L}^{(i)}\})\leqslant n-2$.

REFERENCES

[1] JACOBSON, N.: Lie algebras. Interscience Publishers, New York—London, 1962.
[2] KREKNIN, V. A.: O razrešimosti algebr Li s reguľarnym avtomorfizmom konečnogo poriadka. DAN SSSR, 150, 3 1963, 467—469.

Received September 3, 1982

Katedra geometrie
Matematicko-fyzikálnej fakulty
Univerzity Komenského
Mlynská dolina
842 15 Bratislava

Katedra matematiky
Elektrotechnickej fakulty
Slovenskej vysokej školy technickej
Gottwaldovo nám. 19
812 19 Bratislava

ГИПОТЕЗА О АЛГЕБРАХ ЛИ, ДОПУСКАЮЩИХ РЕГУЛЯРНЫЙ АВТОМОРФИЗМ КОНЕЧНОГО ПЕРИОДА

Eugen Ružický—Jozef Tvarožek

Резюме

Пусть $\mathscr{L}$-алгебра Ли над полем характеристики $p \geqq 0$, допускающая регулярный автоморфизм $A: \mathscr{L} \to \mathscr{L}$ конечного периода n, $n \geqslant 2$. В. А. Крекнин доказал, что длина $l(\{\mathscr{L}^{(i)}\})$ производного ряда $\{\mathscr{L}^{(i)}\}$ алгебры $\mathscr{L}$ не превосходит 2^{n-1}. В настоящей заметке гипотеза $l(\{\mathscr{L}^{(i)}\}) \leqslant n-1$ проверена для $n = 2, 3, \ldots, 7$.