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A REMARK ON THE EXISTENCE OF SMALL SOLUTIONS TO A FOURTH ORDER BOUNDARY VALUE PROBLEM WITH LARGE NONLINEARITY

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ABSTRACT. The existence of small solutions to nonlinear boundary value problem for the fourth order is proved. The main technique used is obtaining a priori bounds and applying Leray-Schauder degree arguments.

In this paper we show the existence of at least one small solution to the nonlinear boundary value problem

$$\mathcal{L}y = Ly+\eta y^{2l} = y^{(4)} + (m^2+n^2)y'' + m^2n^2y + \eta y^{2l} = f\,,$$

$0 < m < n$, $l \geq 4$, $l,m,n \in \mathbb{N}$, $\eta = \pm 1$ with periodic boundary conditions $y^{(i)}(0) = y^{(i)}(2\pi)$, $i = 0,1,2,3$, under the assumption that the function f is in $L^1([0,2\pi])$, and that the norm $\|f\|_1$ is sufficiently small. We call the solution of that problem small if it is lying inside a small ball in $BC = \{y \in D(\mathcal{L})\}$, where $D(\mathcal{L}) = \{y(t) \in C^3([0,2\pi])\,,\ y^{(4)} \in L^1([0,2\pi])\colon\ y^{(i)}(0) = y^{(i)}(2\pi)\,,\ i = 0,1,2,3\}$. L. Lefton in [5] has considered the existence of at least one small solution of the second order nonlinear boundary value problem $L_1y+\eta y^3 = y'' + p(x)y' + q(x)y + \eta y^3 = f$ with the boundary conditions $M_1y = \alpha_1 y(a) + \alpha_2 y(b) + \alpha_3 y'(a) + \alpha_4 y'(b) = 0$, $M_2y = \beta_1 y(a) + \beta_2 y(b) + \beta_3 y'(a) + \beta_4 y'(b) = 0$, $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1,2$. He supposed that the operator L_1 has a one-dimensional null space spanned by φ, and that $\varphi^3 \in R(L_1)$ (the range of the operator L_1). In this paper the null space of L is a four-dimensional space generated by the functions $\cos mt$, $\sin mt$, $\cos nt$, $\sin nt$. The special form of these functions enables easy calculations of a priori bounds. The form of the operator $\mathcal{L}$ has been taken from [7]. In this paper J. D. Schuur has considered the boundary value

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problem $x^4 + (m^2+n^2)x'' + m^2n^2 x + h(x) = p(t)$, $x^i(0) = x^i(2\pi)$, $i = 0,1,2,3$, where he has assumed that the function h is $|h(x)| \le c_1 + c_2|x|$, $0 \le c_1$, $0 < c_2$ and hence our result is not a consequence of the Schuur theorem. He has used a modification of the Cesari method. However, we apply a modification of the Mawhin method, proposed by L. Lefton in [5]. The main difference between this paper and [5] consists in considering a four-dimensional null space of L and in the other form of the nonlinearity. This has made difficulties in degree calculations. S. H. Ding and J. Mawhin in [2] considered the more general resonance problem $L_2\big(u(t)\big) + g\big(u(t)\big) = s + e\big(t, u(t)\big)$, where L_2 is a Fredholm operator of index zero. The order of L_2 is $m \ge 3$. They assumed that the null space of L_2 is generated by the constant, $\lim\limits_{|v|\to\infty} g(v) = \infty$, s is the parameter and $e\big(t, u(t)\big)$ is the Caratheodory function.

1. Introduction

Consider the fourth order nonlinear differential operator $\mathcal{L}y = Ly + \eta y^{2l} = y^{(4)} + (m^2+n^2)y'' + m^2n^2y + \eta y^{2l}$, where $0 \le m \le n$, $l \ge 4$, $l, m, n \in \mathbb{N}$. The linear part of $\mathcal{L}$ is $Ly = y^{(4)} + (m^2+n^2)y'' + m^2n^2y$. The operator $\mathcal{L}$ as well as L is defined on the domain

$$D(\mathcal{L}) = \big\{y(t) \in C^3\big([0,2\pi]\big),\ y^{(4)} \in L^1\big([0,2\pi]\big):\ y^{(i)}(0) = y^{(i)}(2\pi),\ i = 0,1,2,3\big\}.$$

Hence $\mathcal{L}\colon D(\mathcal{L}) \to L^1\big([0,2\pi]\big)$. We will study the existence of solutions of

$$\mathcal{L}y = f \tag{1.1}$$

with periodic boundary conditions

$$y^{(i)}(0) = y^{(i)}(2\pi), \qquad i = 0,1,2,3, \tag{1.2}$$

and $f \in L^1\big([0,2\pi]\big)$. Define

$$BC = \big(D(\mathcal{L}), \|\cdot\|_\infty\big),$$

where $\|y\|_\infty = \sup\limits_{t\in[0,2\pi]} |y(t)|$ for all $y \in D(\mathcal{L})$.

Note the null space of $L\colon BC \to L^1\big([0,2\pi]\big)$ as $NS(L)$. $NS(L)$ is four-dimensional and consists of the functions

$$NS(L) = \big\{y \in BC:\ y(t) = c_1 \cos mt + c_2 \sin mt + c_3 \cos nt + c_4 \sin nt,\ c_i \in \mathbb{R},\ i = 1,2,3,4\big\}.$$

Let the range of the operator be denoted as $R(L)$ and I be the identity operator in BC. First we study the operator L.

LEMMA 1.1. *Let the operator L be defined on BC. Then $NS(L) \cap R(L) = \{0\}$.*

Proof. The problem $L(x) = 0$, $x^{(i)}(0) = x^{(i)}(2\pi)$, $i = 0, 1, 2, 3$ is self-adjoint and therefore the assertion of the lemma is true.

The functions $\varphi_1(t) = \cos mt$, $\varphi_2(t) = \sin mt$, $\varphi_3(t) = \cos nt$, $\varphi_4(t) = \sin nt$ form a fundamental system of solutions of the equation $Ly = 0$ and satisfy the boundary conditions (1.2). It is obvious that zero is the eigenvalue of the operator L. In this case the Green function does not exist. In the next lemma we show that the operator $L + K \cdot I$ has not the eigenvalue 0 for some $K \in \mathbb{R}$.

LEMMA 1.2. *Let $K > \frac{1}{4}(n^2 - m^2)^2$. Then 0 is not the eigenvalue of the operator $L + K \cdot I$.*

Proof. λ is the eigenvalue of the problem $Ly = \lambda y$ if and only if there exists such a $k \in \mathbb{Z}$ that ik is the root of the characteristic equation

$$r^4 + (m^2 + n^2) r^2 + m^2 n^2 - \lambda = 0 .$$

This happens if and only if k satisfies the equation

$$k^4 - (m^2 + n^2) k^2 + m^2 n^2 - \lambda = 0 .$$

Denote by $g \colon \mathbb{R} \to \mathbb{R}$ the function

$$g(k) = k^4 - (m^2 + n^2) k + m^2 n^2 .$$

The eigenvalues of the problem $Ly = \lambda y$ are the values of the function g at $k \in \mathbb{Z}$. The function g is an even function and $\min g(k) = -\frac{1}{4}(n^2 - m^2)^2$, $k \in \mathbb{R}$, and hence all eigenvalues $\lambda_j \geq -\frac{1}{4}(n^2 - m^2)^2$. By the form of the function g it follows that all its eigenvalues form a sequence $\{\lambda_j\}$ which approaches to infinity as $j \to \infty$. If we add a constant $K > \frac{1}{4}(n^2 - m^2)^2$ to the function g, then $g + K$ will be positive for all k. The corresponding characteristic equation will be

$$r^4 + (m^2 + n^2) r^2 + m^2 n^2 + K = 0 ,$$

and the corresponding differential operator will be $Ly + Ky = 0$, where $K > \frac{1}{4}(n^2 - m^2)^2$.

From this lemma it follows that the equation $(L + K \cdot I) y = 0$ has only the trivial solution for $K > \frac{1}{4}(n^2 - m^2)^2$. By [3, Lemma 4.3, p. 145] it follows that the operator $L + K \cdot I$ is one-to-one and maps BC onto $L^1([0, 2\pi])$. Therefore the operator $(L + K \cdot I)^{-1}$ is completely continuous ([3, Lemma 4.4, p. 145]).

COROLLARY 1.1. *The operator* $L\colon BC \subset L^1([0,2\pi])$ *into* $L^1([0,2\pi])$ *is*

(i) *a Fredholm operator of index zero,*
(ii) *a closed operator.*

Moreover,

(iii) $L^1([0,2\pi]) = NS(L) \oplus R(L)$, *where* $\oplus$ *is a topological direct sum.*

Proof. The conditions of Theorem 1 [7, p. 555] are satisfied.

2. Construction of the operator K_P

Define a projection P_0 by

$$P_0\, y(t) = \frac{1}{\pi} \sum_{i=1}^{4} \int_0^{2\pi} y(t)\varphi_i(t)\ \mathrm{d}t \cdot \varphi_i(t) \qquad \text{for} \quad y \in BC\,.$$

Note that P_0 maps BC onto $NS(L)$ and that $L^1([0,2\pi]) = NS(L) \oplus NS(P_0)$ holds, where $NS(P_0)$ is the null space of P_0. The operator L is one-to-one on BC but its restriction to $BC_{P_0} = BC \cap NS(P_0)$ is one-to-one and onto $R(L)$. Therefore there exists the inverse operator $K_P\colon R(L) \to BC \cap NS(P_0)$ to the operator $L|_{BC \,\cap\, NS(P_0)}$. Now we construct the operator K_P. The Cauchy function for the equation $L(x) = 0$ is

$$K_1(t,s) = \left[mn(n^2-m^2)\right]^{-1} \cdot \left[n \sin m(t+s) - m \sin(t+s)\right], \qquad \text{for} \quad 0 \le s < t \le 2\pi\,.$$

Let $x \in BC_{P_0} \cap NS(P_0)$ be the solution of the equation $Lx = y$, $y \in R(L)$. Then it has the form

$$x(t) = \sum_{n=1}^{4} c_i\varphi_i(t) + \left[mn(n^2-m^2)\right]^{-1} \cdot \int_0^t K_1(t,s)y(s)\ \mathrm{d}s\,, \qquad \text{for} \quad 0 \le t \le 2\pi\,. \tag{2.1}$$

The function $x \in BC_{P_0}$ and therefore it is true that for all $\varphi_i \in NS(L)$ we have

$$\int_0^{2\pi} x(t) \cdot \varphi_i(t)\ \mathrm{d}t = 0\,, \qquad i = 1,2,3,4\,.$$

For the constants c_i, $i = 1, 2, 3, 4$ we obtain

$$0 = \pi \cdot c_i + \int_0^{2\pi}\int_0^t K_1(t,s)\,y(s)\,\varphi_i(t)\ \mathrm{d}s\ \mathrm{d}t\,, \qquad i = 1,2,3,4\,. \tag{2.2}$$

By (2.2) we see that the constants c_i, $i = 1,2,3,4$ are uniquely determined. From periodic conditions it follows that $y \in R(L)$ if and only if

$$\int_0^{2\pi} \frac{\partial^i K_1(2\pi, s)}{\partial t^i} \cdot y(s)\ \mathrm{d}s = 0\,, \qquad i = 1,2,3,4\,, \tag{2.3}$$

is true. Therefore $R(L)$ consists of the functions which fulfil (2.3). By Fubini's theorem in (2.2) as well as by putting the constants c_i, $i = 1,2,3,4$ in (2.3) we get that

$$\begin{aligned} x(t) = &-\frac{1}{\pi}\int_0^{2\pi}\int_s^{2\pi}[\cos mt + \sin mt + \cos nt + \sin nt]\cdot K_1(t,s)\ \mathrm{d}t\ y(s)\ \mathrm{d}s \\ &+ \int_0^t K_1(t,s)\,y(s)\ \mathrm{d}s\,, \qquad 0 \le t \le 2\pi\,. \end{aligned} \tag{2.4}$$

Denote the inner integral by $I(s)$ and compute

$$\begin{aligned} I(s) = &\int_s^{2\pi}[\cos mt + \sin mt + \cos nt + \sin nt]\cdot K_1(t,s)\ \mathrm{d}s \\ = &\big[mn(n^2 - m^2)\big]^{-1}\cdot\Big\{\frac{1}{2}(2\pi - s)\big[n(\sin ms + \cos ms) - m(\sin ns + \cos ns)\big] \\ &- \frac{m}{2n}\sin ns[\sin 2ns + \cos 2ns] - \frac{n}{2m}\sin ms[\sin 2ms + \cos 2ms] \\ &+ \frac{n}{n-m}\sin\frac{(m-n)s}{2}\left[\cos\frac{(3m-n)s}{2} - \sin\frac{(3m-n)s}{2}\right] \\ &- \frac{m}{n-m}\sin\frac{(n-m)s}{2}\left[\cos\frac{(3n-m)s}{2} - \sin\frac{(3n-m)s}{2}\right] \\ &+ \frac{n}{n+m}\sin\frac{(m+n)s}{2}\left[\sin\frac{(3n+m)s}{2} - \cos\frac{(3n+m)s}{2}\right] \\ &- \frac{n}{n+m}\sin\frac{(m+n)s}{2}\left[\sin\frac{(3m+n)s}{2} - \cos\frac{(3m+n)s}{2}\right]\Big\}. \end{aligned}$$

Putting it in (2.4) we obtain

$$x(t) = \left[-\pi\, mn(n^2 - m^2)\right]^{-1} \int\limits_0^{2\pi} I(s)y(t)\ \mathrm{d}s + \int\limits_0^{t} K_1(t,s)y(s)\ \mathrm{d}s\,,$$
$$\text{for} \quad 0 \le t \le 2\pi\,.$$

Define the function $K(t,s)$ in $[0,2\pi] \times [0,2\pi]$

$$K(t,s) = \begin{cases} K_1(t,s)\,, & 0 \le s < t \le 2\pi\,,\\ 0\,, & 0 \le t < s \le 2\pi\,. \end{cases} \tag{2.5}$$

Then we can write the function $x(t)$ in the form

$$\begin{aligned} x(t) &= \left[-\pi mn(n^2 - m^2)\right]^{-1} \int\limits_0^{2\pi} \left\{I(s) - \pi\left[mn(n^2 - m^2)\right] K(t,s)\right\} y(s)\ \mathrm{d}s \\ &= \left[-\pi mn(n^2 - m^2)\right]^{-1} \int\limits_0^{2\pi} \left\{I(s) - \pi\left[n \cdot \sin m(t+s) - m \cdot \sin n(t+s)\right]\right\} y(s)\,\mathrm{d}s\,. \end{aligned}$$

Denote by

$$K^*(t,s) = I(s) - \pi\left[n \cdot \sin m(t+s) - m \cdot \sin n(t+s)\right]. \tag{2.6}$$

THEOREM 2.1. *The form of the inverse operator* $K_P \colon R(L) \to BC \cap NS(P_0)$ *to* $L|_{BC \cap NS(P_0)}$ *is*

$$K_P\, y(t) = \left[-\pi\, mn(n^2 - m^2)\right]^{-1} \int\limits_0^{2\pi} K^*(t,s)\, y(s)\ \mathrm{d}s\,, \qquad 0 \le t \le 2\pi\,, \tag{2.7}$$

$y \in R(L)$ *and* $K^*(t,s)$ *is determined by* (2.6).

Estimate the function $K^*(t,s)$ as

$$|K^*(t,s)| \le 3\pi(n+m) + \frac{n^2 + m^2}{mn} + \frac{8mn}{n^2 - m^2}\,. \tag{2.8}$$

Using the estimate (2.8) we obtain that

$$\int_0^{2\pi}\int_0^{2\pi} |K^*(t,s)|^2 \, ds \, dt < +\infty ,$$

and therefore K_P is the Hilbert-Schmidt operator. We have the following estimate for the norm of the operator K_P in $L^2([0,2\pi]\times[0,2\pi])$

$$\|K_P\| < \left[\int_0^{2\pi}\int_0^{2\pi} \left[\pi^2 m^2 n^2 (n^2-m^2)^2\right]^{-1} \cdot |K^*(t,s)|^2 \, dt \, ds\right]^{\frac{1}{2}}$$
$$\leq \frac{6}{mn(n-m)} + \frac{2(n^2+m^2)}{m^2n^2(n^2-m^2)} + \frac{16}{n^2-m^2} < +\infty .$$

By (2.6) it follows that K_P is a continuous operator on $[0,2\pi]$ and by [3, Lemma 4.4, p. 145] we have that the operator K_P is a completely continuous operator on $R(L)$.

3. A priori bound for $\mathcal{L}y$

The next lemma is true.

LEMMA 3.1. *Let $a,b,c,d \in \mathbb{R}$ be arbitrary constants. Then $(a\cos mt + b\sin mt + c\cos nt + d\sin nt)^{2l} \in R(L)$, i.e., there exists such a $w \in BC$, $w \in NS(L)^{\perp}$ that $Lw = (a\cos mt + b\sin mt + c\cos nt + d\sin nt)^{2l}$.*

Proof. Denote by $y(t) = (a\cos mt + b\sin mt + c\cos nt + d\sin nt)^{2l}$. It follows from the definition of the projection P_0 and Corollary 1.1 (iii) that $y \in R(L)$ if and only if $P_0\, y(t) = 0$. We investigate the following integrals

$$I_1 = \int_0^{2\pi} \cos^i mt \cdot \cos^j kt \, dt , \quad I_2 = \int_0^{2\pi} \sin^i mt \sin^j kt \, dt ,$$
$$i+j = 2l+1 , \quad k = m,n ,$$

$$I_3 = \int_0^{2\pi} \cos^i mt \cdot \cos^j nt \sin kt \, dt , \quad I_4 = \int_0^{2\pi} \sin^i mt \sin^j nt \cos kt \, dt ,$$
$$i+j = 2l , \quad k = m,n ,$$

$$I_5 = \int_0^{2\pi} \sin^i kt \cdot \cos^j pt \sin rt \, \mathrm{d}t\,, \quad I_6 = \int_0^{2\pi} \sin^i kt \sin^j pt \cos rt \, \mathrm{d}t\,,$$

$$i + j = 2l\,, \quad k, p, r = m, n\,.$$

In the integrals I_1, I_2 we have the functions $\cos mt$, $\sin mt$ with an odd exponent for $k = m$ and therefore $I_1 = I_2 = 0$. If $k = n$ we first multiply the trigonometric functions and then we integrate and get that $I_i = 0$, $i = 1, 2, \dots, 6$. Similarly $I_i = 0$, $i = 3, 4, 5, 6$ for $k = m$. Therefore $P_0 y(t) = 0$.

LEMMA 3.2. *Let $\|\mathcal{L}y_k\|_1 = o(\|y_k\|_\infty^{2l})$ for some sequence $\{y_k\} \subset BC$, $y_k \to 0$ uniformly. Then $\|y_k\|_\infty = O\Big(\sum_{j=m,n} (|A_j^k| + |B_j^k|)\Big)$, for $k \to \infty$, where A_j^k, B_j^k are the Fourier coefficients of the function y_k for $j = m, n$.*

P r o o f. We can write that

$$y_k(t) = A_m^k \cos mt + B_m^k \sin mt + A_n^k \cos nt + B_n^k \sin nt + w_k(t)\,,$$

where $w_k(t) \in NS(L)^\perp$. Observe that

$$L w_k = \mathcal{L} y_k - \eta y_k^{2l}\,. \tag{3.1}$$

Hence $\mathcal{L}y_k - \eta y_k^{2l} \in R(L)$. Apply the operator K_P to (3.1) to get $w_k = K_P(\mathcal{L}y_k - \eta y_k^{2l})$. Using the assumption of this lemma and the continuity of K_P we find

$$\|w_k\|_\infty \le C\,(\|\mathcal{L}y_k\|_1 + \|y_k\|_1^{2l}) \le C\,(o(\|y_k\|_\infty^{2l}) + (2\pi)^{2l}\|y_k\|_\infty^{2l}) = O\,(\|y_k\|_\infty^{2l}) \tag{3.2}$$

From the form of the function y_k and (3.2) it follows that

$$\|y_k\|_\infty \le M \sum_{j=m,n} (|A_j^k| + |B_j^k|) = O\Big(\sum_{j=m,n} (|A_j^k| + |B_j^k|)\Big), \qquad \text{for} \quad k \to \infty\,. \tag{3.3}$$

This completes the proof.

R e m a r k 3.1. The case that there exists such a subsequence of the sequence $\sum_{j=m,n} (|A_j^k| + |B_j^k|)$ which is a null sequence cannot happen. If it were true, then there would exist $y_k = w_k$ and (3.2) would contradict to the assumption on y_k. Therefore there exists such a k_0 that for all $k \ge k_0$

$$\sum_{j=m,n} (|A_j^k| + |B_j^k|) > 0\,.$$

Denote

$$\alpha_k = \sqrt{|A_m^k|^2 + |B_m^k|^2 + |A_n^k|^2 + |B_n^k|^2}\,,$$
$$\beta_k = |A_m^k| + |B_m^k| + |A_n^k| + |B_n^k|\,.$$

As the $\alpha_k > 0$ for $k \geq k_0$ we consider the sequences

$$a_m^k = \frac{A_m^k}{\alpha_k}\,, \quad b_m^k = \frac{B_m^k}{\alpha_k}\,, \quad a_n^k = \frac{A_n^k}{\alpha_k}\,, \quad b_n^k = \frac{B_n^k}{\alpha_k}\,. \tag{3.4}$$

It is true that $(a_m^k)^2 + (b_m^k)^2 + (a_n^k)^2 + (b_n^k)^2 = 1$ for $k \geq k_0$. So there exists such a subsequence of indices $\{k_p\}$ that

$$\lim_{p\to\infty} a_m^{k_p} = a_m\,, \quad \lim_{p\to\infty} b_m^{k_p} = b_m\,, \quad \lim_{p\to\infty} a_n^{k_p} = a_n\,, \quad \lim_{p\to\infty} b_n^{k_p} = b_n\,,$$

and

$$a_m^2 + b_m^2 + a_n^2 + b_n^2 = 1\,. \tag{3.5}$$

From this it follows that at least one of the numbers a_m, b_m, a_n, b_n is different from zero.

PROPOSITION 3.1. *If $\|\mathcal{L}y_k\|_1 = o\big(\|y_k\|_\infty^{2l}\big)$ for some sequence $\{y_k\} \subset BC$, $y_k \to 0$ uniformly, then there exists such a subsequence $\{y_{k_p}\}$ that the corresponding sequences of coefficients $\{a_m^{k_p}\}$, $\{b_m^{k_p}\}$, $\{a_n^{k_p}\}$, $\{b_n^{k_p}\}$ have the limits $\lim\limits_{p\to\infty} a_m^{k_p} = a_m$, $\lim\limits_{p\to\infty} b_m^{k_p} = b_m$, $\lim\limits_{p\to\infty} a_n^{k_p} = a_n$, $\lim\limits_{p\to\infty} b_n^{k_p} = b_n$, and these limits satisfy* (3.5).

LEMMA 3.3. *Let the function $w \in BC$ be the solution of the equation $Lw = (a_m \cos mt + b_m \sin mt + a_n \cos nt + b_n \sin nt)^{2l}$, $w \in NS(L)^\perp$. Suppose further that there is no solution $v \in BC$ of $Lv = (a_m \cos mt + b_m \sin mt + a_n \cos nt + b_n \sin nt)^{2l-1} w$. Then there exist $\delta > 0$, $c > 0$, such that $\|\mathcal{L}\|_1 \geq c\,\|y\|_\infty^{4l-1}$ for all $y \in BC$ with $\|y\|_\infty < \delta$.*

Lew Lefton proved the same lemma for $4l - 1 = 5$ in [5, Lemma 1.4, p. 175].

4. Degree calculation

In this section we show the existence of at least one small solution in BC of the equation $\mathcal{L}y = f$ for small enough $f \in L^1([0, 2\pi])$. First we describe the neighbourhood of the origin which will act as the domain of our compact operator. We will use the constants δ and c from Lemma 3.3. Let $0 < \varepsilon < c$ and Δ be a ball, centered at the origin in BC, defined by

$$\Delta = \Big\{ y \in BC: \ \|y\|_\infty < \Big(\frac{1}{c-\varepsilon}\|f\|_1\Big)^{\frac{1}{4l-1}} < \delta \Big\}.$$

Note that the radius of Δ monotonically depended on $\|f\|_1$. Therefore, if we need to consider smaller functions y, we need only reduce $\|f\|_1$.

Consider the operator $A_t \colon \Delta \to BC$ defined as

$$A_t y = P_0 y + P_0(tf - \eta y^{2l}) + K_P \cdot P_1\big(tf - \eta y^{2l}\big), \qquad 0 \le t \le 1, \tag{4.1}$$

where $P_1 \colon L^1\big([0,2\pi]\big) \to R(L)$ is a continuous projection onto $R(L)$.

LEMMA 4.1. *$A_t y = y$ if and only if $\mathcal{L}y = tf$.*

Proof. The proof of this lemma is similar as in [5, Lemma 2.1, p. 176].

We have shown that the solutions of the problem $\mathcal{L}y = f$ are precisely the fixed points of A_1. The next step is to show that the Leray-Schauder degree $d(I - A_1, \Delta, 0) \neq 0$ and hence the equation $\mathcal{L}y = f$ has at least one solution in Δ. We construct the homotopy in two steps. $P_0 \colon L^\infty\big([0,2\pi]\big) \to L^\infty\big([0,2\pi]\big)$ is continues projection into the finite-dimensional space, $K_P \colon L^1\big([0,2\pi]\big) \to BC$ is the completely continuous operator, therefore the operator $A_t \colon L^\infty\big([0,2\pi]\big) \to L^\infty\big([0,2\pi]\big)$ is a completely continuous operator too for all $t \in [0,2\pi]$. Let $K(\Delta)$ be the set of all compact mappings $A \colon \Delta \to BC$ with the norm $\|A\| = \sup\limits_{x\in\Delta} \|Ax\|$. Define $h(t) = A_t$ and note that $h \colon [0,1] \to K(\Delta)$ is continuous. This defines a homotopy of compact transformations.

LEMMA 4.2. *The equation $(I - A_t)y = 0$ has no solution on $\partial\Delta$ for any $t \in [0,2\pi]$.*

Proof. The proof of this lemma is the same as in [5, Lemma 2.2, p. 177] for $2l - 1 = 5$.

By the homotopy invariance of the Leray-Schauder degree we get that $d(I - A_t, \Delta, 0)$ is independent of t. In particular

$$d(I - A_1, \Delta, 0) = d(I - A_0, \Delta, 0).$$

For the second step of the homotopy we define

$$\begin{aligned} \tilde h(\lambda)y &= A^\lambda y \\ &= P_0 y - P_0\big(\eta y^{2l}\big) - (1-\lambda)\, K_P\, P_1\big(\eta y^{2l}\big) - \lambda\, P_w\, K_P\, P_1\big(\eta y^{2l}\big). \end{aligned} \tag{4.2}$$

Here P_w is the projection onto the space generated by the function w, $w \in NS(L)^{\perp}$ and $Lw = (a_m \cos mt + b_m \sin mt + a_n \cos nt + b_n \sin nt)^{2l}$. It follows immediately that $\tilde{h}(t)$ is a homotopy of compact transformations. It is true that $A^0 = A_0$. So the homotopies can be combined appropriately. We need to show that $A^{\lambda} y = y$ has no solution on $\partial \Delta$.

LEMMA 4.3. *Assume that the assumptions of Lemma 3.2 and Lemma 3.3 hold. Then for an arbitrary $\varepsilon > 0$ the equation $A^{\lambda} y = y$ for $\lambda \in [0,1]$ has no solution y with the norm $\|y\|_{\infty} < \varepsilon$ except $y = 0$.*

Proof. Suppose that there exists such a sequence of solutions $\{y_k\}_{k=1}^{\infty}$ that for all $\frac{1}{k} > 0$ $A^{\lambda_k} y_k = y_k$ with the norm $\|y_k\|_{\infty} < \frac{1}{k}$, $y_k \neq 0$. Then we can write y_k in the form

$$y_k(t) = A_m^k \cos mt + B_m^k \sin mt + A_n^k \cos nt + B_n^k \sin nt + z_k(t),$$

where $z_k \in NS(L)$. Applying $I - P_0$ to $y_k = A^{\lambda_k} y_k$, we obtain

$$z_k(t) = -(1-\lambda) K_P P_1 (\eta y_k^{2l}) - \lambda P_w K_P P_1 (\eta y_k^{2l}).$$

By the continuity of the operator K_P and of the projections P_1, P_w we get

$$\|z_k\|_{\infty} = \left\|(1-\lambda) K_P P_1 (\eta y_k^{2l}) - \lambda P_w K_P P_1 (\eta y_k^{2l})\right\|_{\infty} \leq C \cdot \|y_k\|_{\infty}^{2l}. \tag{4.3}$$

Note that the constant C does not depend on λ_k. Similarly as in Lemma 3.2 in (3.2), (3.3) we get $\|y_k\|_{\infty} = O(\beta_k)$. By the continuity of P_0 we obtain that $\sum\limits_{j=m,n} (A_j^k \cos jt + B_j^k \sin jt) = O(\|y_k\|_{\infty})$. Using this fact we come to the equality

$$y_k^{2l}(t) = \Big[\sum_{j=m,n} (A_j^k \cos jt + B_j^k \sin jt)\Big]^{2l} + O(\beta_k^{4l-1}), \tag{4.4}$$

and

$$\begin{aligned} y_k^{2l}(t) = {} & \Big[\sum_{j=m,n} (A_j^k \cos jt + B_j^k \sin jt)^{2l}\Big]^{2l} \\ & + 2l \Big[\sum_{j=m,n} (A_j^k \cos jt + B_j^k \sin jt)^{2l}\Big]^{2l-1} z_k(t) + O(\beta_k^{6l-2}). \end{aligned} \tag{4.5}$$

Apply P_0 to $A^{\lambda_k} y_k = y_k$ to see that $P_0(\eta\, y_k^{2l}) = 0$. Therefore $P_1(\eta y_k^{2l}) = \eta\, y_k^{2l}$. Using this fact and (4.2) we get

$$K_P P_1(\eta\, y_k^{2l}) = K_P(\eta\, y_k^{2l})$$
$$= \eta K_P\left(\left[\sum_{j=m,n}(A_j^k \cos jt + B_j^k \sin jt)\right]^{2l}\right) + O(\beta_k^{4l-1}), \qquad k \to \infty .$$

With this estimate, (4.2) can be written as

$$A^{\lambda_k}(y_k)(t)$$
$$= P_0(y_k)(t) - P_0(\eta\, y_k^{2l})(t) - (1-\lambda_k)\eta K_{P_0}\left(\left[\sum_{j=m,n}(A_j^k \cos jt + B_j^k \sin jt)\right]^{2l}\right)$$
$$- \lambda_k \eta P_w \circ K_{P_0}\left(\left[\sum_{j=m,n}(A_j^k \cos jt + B_j^k \sin jt)\right]^{2l}\right) + O(\beta_k^{4l-1}) . \quad (4.6)$$

Now apply $I - P_0$ to the equation $y_k = A^{\lambda_k} y_k$, where $A^{\lambda_k} y_k$ is given by (4.6) to obtain

$$z_k(t) = -(1-\lambda_k)\eta K_{P_0}\left(\left[\sum_{j=m,n}(A_j^k \cos jt + B_j^k \sin jt)\right]^{2l}\right)$$
$$- \lambda_k \eta P_w \circ K_{P_0}\left(\left[\sum_{j=m,n}(A_j^k \cos jt + B_j^k \sin jt)\right]^{2l}\right) + O(\beta_k^{4l-1}) .$$
$$(4.7)$$

Using (4.7) in (4.5) we get

$$y_k^{2l}(t) = \left[\sum_{j=m,n}(A_j^k \cos jt + B_j^k \sin jt)\right]^{2l} - 2l\left[\sum_{j=m,n}(A_j^k \cos jt + B_j^k \sin jt)\right]^{2l-1} \times$$
$$\times \left[(1-\lambda_k)\eta K_{P_0}\left(\left[\sum_{j=m,n}(A_j^k \cos jt + B_j^k \sin jt)\right]^{2l}\right)\right.$$
$$\left. + \lambda_k \eta P_w \circ K_{P_0}\left(\left[\sum_{j=m,n}(A_j^k \cos jt + B_j^k \sin jt)\right]^{2l}\right) + O(\beta_k^{4l-1})\right]$$
$$+ O(\beta_k^{6l-2}), \qquad k \to \infty .$$
$$(4.8)$$

Apply P_0 to (4.8) and by the Lemma 3.1 and $\beta_k \to 0$ as $k \to \infty$ we see that

$$P_0\Bigg(\eta\, 2l\Big[\sum_{j=m,n}(A_j^k \cos jt + B_j^k \sin jt)\Big]^{2l-1} \times$$
$$\times \Bigg[(1-\lambda_k)K_{P_0}\bigg(\Big[\sum_{j=m,n}(A_j^k \cos jt + B_j^k \sin jt)\Big]^{2l}\bigg)$$
$$+\lambda_k P_w \circ K_{P_0}\bigg(\Big[\sum_{j=m,n}(A_j^k \cos jt + B_j^k \sin jt)\Big]^{2l}\bigg)\Bigg]\Bigg)$$
$$= o(\beta_k^{4l-1}), \qquad k \to \infty. \tag{4.9}$$

Divide (4.9) by β_k^{4l-1}. From (3.4) and $\beta_k = c_k\alpha_k$, $1 \le c_k \le c_0$, $c_0 > 1$ we have

$$P_0\Bigg(\eta\,\frac{2l}{c_k^{4l-1}}\left[\frac{\sum_{j=m,n}(A_j^k \cos jt + B_j^k \sin jt)}{\alpha_k}\right]^{2l-1} \times$$
$$\times \Bigg[(1-\lambda_k)K_{P_0}\left(\left[\frac{\sum_{j=m,n}(A_j^k \cos jt + B_j^k \sin jt)}{\alpha_k}\right]^{2l}\right)$$
$$+\lambda_k P_w \circ K_{P_0}\left(\left[\frac{\sum_{j=m,n}(A_j^k \cos jt + B_j^k \sin jt)}{\alpha_k}\right]^{2l}\right)\Bigg]\Bigg)$$
$$= o(1), \qquad k \to \infty. \tag{4.10}$$

The sequence $\{c_k\}_{k=1}^{\infty}$ is bounded. So there exists such a subsequence $\{c_{k_p}\}_{p=1}^{\infty}$ that $\lim_{p\to\infty} c_{k_p} = c$, $1 \le c \le c_0$. In the sequence of the indices $\{k_p\}_{p=1}^{\infty}$ there exists such a subsequence $\{k_{p_r}\}_{r=1}^{\infty}$ that Proposition 3.1 is valid. The sequence $\{\lambda_{k_{p_r}}\}_{r=1}^{\infty}$ is bounded, therefore there exists a subsequence which approaches to λ_0, where $0 \le \lambda_0 \le 1$. The operators P_0, K_P are continuous, therefore we can do the passage to the limit in (4.10) and we obtain

$$P_0\Bigg(\eta\,\frac{2l}{c}\Big[\sum_{j=m,n}(a_j \cos jt + b_j \sin jt)\Big]^{2l-1}\Bigg[(1-\lambda_0)K_{P_0}\bigg(\Big[\sum_{j=m,n}(a_j \cos jt + b_j \sin jt)\Big]^{2l}\bigg)$$
$$+\lambda_0 P_w \circ K_{P_0}\bigg(\Big[\sum_{j=m,n}(a_j \cos jt + b_j \sin jt)\Big]^{2l}\bigg)\Bigg]\Bigg) = 0.$$

It is true that $L(w) = (a_m \cos mt + b_m \sin mt + a_n \cos nt + b_n \sin nt)^{2l}$. Therefore

$$w = K_{P_0}\left(\left[\sum_{j=m,n}(a_j \cos jt + b_j \sin jt)\right]^{2l}\right)$$
$$= P_w \circ K_{P_0}\left(\left[\sum_{j=m,n}(a_j \cos jt + b_j k \sin jt)\right]^{2l}\right).$$

We get

$$P_0\left(\eta\,\frac{2l}{c}\left[\sum_{j=m,n}(a_j \cos jt + b_j \sin jt)\right]^{2l-1}\cdot w(t)\right) = 0\,.$$

This contradicts the hypothesis that

$$\left[\sum_{j=m,n}(a_j^k \cos jt + a_j^k \sin jt)\right]^{2l-1}\cdot w(t) \notin R(L)\,.$$

Denote the linear span of the functions $x_1,\dots,x_p$ as $\langle x_1,\dots,x_p\rangle$. Now we show that $d(I-A^1,\,\Delta,\,0) \neq 0$ for small Δ. Note that A^1 is already an operator of finite rank; in fact $R(A^1) \subset \langle\varphi_1,\,\varphi_2,\,\varphi_3,\,\varphi_4,w\rangle$. Hence we must compute $d(S,\,\overline{\Delta},\,0)$, where $\overline{\Delta} = \Delta \cap \langle\varphi_1,\,\varphi_2,\,\varphi_3,\,\varphi_4,w\rangle$ and

$$S = (I - A^1)|_{\langle\varphi_1,\varphi_2,\varphi_3,\varphi_4,w\rangle} = (I - P_0)y + P_0(\eta\, y^{2l}) + P_w K_P P_1(\eta\, y^{2l})\,.$$

$y \in \langle\varphi_1,\,\varphi_2,\,\varphi_3,\,\varphi_4,w\rangle$. We can write the function y in the form

$$y = \sum_{i=1}^{4} t_i\varphi_i(t) + t_5\cdot w(t)\,.$$

Hence

$$S\colon \begin{bmatrix} t_1\\ t_2\\ t_3\\ t_4\\ t_5\end{bmatrix} \to \begin{bmatrix} s_1(t_1,t_2,t_3,t_4,t_5)\\ s_2(t_1,t_2,t_3,t_4,t_5)\\ s_3(t_1,t_2,t_3,t_4,t_5)\\ s_4(t_1,t_2,t_3,t_4,t_5)\\ s_5(t_1,t_2,t_3,t_4,t_5)\end{bmatrix} = \begin{bmatrix} \eta\frac{1}{\pi}\int_0^{2\pi}\left(\sum_{i=1}^{4} t_i\varphi_i(t) + t_5 w(t)\right)^{2l}\varphi_1 t\,\mathrm{d}t\\ \eta\frac{1}{\pi}\int_0^{2\pi}\left(\sum_{i=1}^{4} t_i\varphi_i(t) + t_5 w(t)\right)^{2l}\varphi_2 t\,\mathrm{d}t\\ \eta\frac{1}{\pi}\int_0^{2\pi}\left(\sum_{i=1}^{4} t_i\varphi_i(t) + t_5 w(t)\right)^{2l}\varphi_3 t\,\mathrm{d}t\\ \eta\frac{1}{\pi}\int_0^{2\pi}\left(\sum_{i=1}^{4} t_i\varphi_i(t) + t_5 w(t)\right)^{2l}\varphi_4 t\,\mathrm{d}t\\ t_5 + \eta\, P_w K_p P_1\left(\left[\sum_{i=1}^{4} t_i\varphi_i(t) + t_5\, w(t)\right]^{2l}\right)\end{bmatrix}$$

Note, $s_5(t_1,t_2,t_3,t_4,t_5)$ is a scalar multiple of the function w. We replace the assumption that there is no solution $v \in BC$ of the equation $Lv(t) = (a_m + \cos mt + b_m \sin mt + a_n \cos nt + b_n \sin nt)^{2l-1} w(t)$ by the assumption that all integrals

$$\int_0^{2\pi} [c_1 \cos mt + c_2 \sin mt + c_3 \cos nt + c_4 \sin nt]^{2l-1} w(t)\varphi_j(t)\ dt \neq 0\,,$$

$j = 1,2,3,4$, if at least one of $c_i \in \mathbb{R}$, $c_i \neq 0$. From that assumption it follows that there is no solution $v \in BC$ of the equation $Lv(t) = (c_1 \cos mt + c_2 \sin mt + c_3 \cos nt + c_4 \sin nt)^{2l-1} w(t)$. So Lemma 4.3 holds with new assumption. Make the change of variables $T(x_1,\dots,x_5) = (t_1,\dots,t_5)$, where $t_i = x_i$, $i = 1,2,3,4$ and $t_5 = \left(\sum_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} x_5 + \left(\sum_{i=1}^{4} x_i^2\right)^2$. Thus if $(x_1,\dots,x_5)$ tend to zero and at least one of $x_j \neq 0$, $j = 1,2,3,4$ and $x_5 \neq 0$, then

$$\begin{aligned}
&s_j(T(x_1,\dots,x_5)) \\
&= 0 + \left[\left(\sum_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} x_5 + \left(\sum_{i=1}^{4} x_i^2\right)^2\right] 2l\eta \frac{1}{\pi} \int_0^{2\pi} \left[\sum_{i=1}^{4} x_i\varphi_i(t)\right]^{2l-1} w(t)\varphi_j(t)\ dt \\
&\quad + o\left(\left[\left(\sum_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} |x_5| + \left(\sum_{i=1}^{4} x_i^2\right)^2\right]^2 \left(\sum_{i=1}^{5} x_i^2\right)^{l-2}\right),
\end{aligned}$$

and

$$\begin{aligned}
s_5(T(x_1,\dots,x_5)) &= \left(\sum_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} x_5 + \left(\sum_{i=1}^{4} x_i^2\right)^2 \\
&\quad + \eta P_w K_P P_1 \left(\left[\sum_{i=1}^{4} x_i\varphi_i(t) + \left(\left(\sum_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} x_5 + \left(\sum_{i=1}^{4} x_i^2\right)^2\right) w(t)\right]^{2l}\right) \\
&= \left(\sum_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} x_5 + \left(\sum_{i=1}^{4} x_i^2\right)^2 + \eta P_w K_P P_1 \left(\left[\sum_{i=1}^{4} x_i\varphi_i(t)\right]^{2l}\right) \\
&\qquad + o\left(\left[\left(\sum_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} |x_5| + \left(\sum_{i=1}^{4} x_i^2\right)^2\right] \left(\sum_{i=1}^{5} x_i^2\right)^{l-1}\right).
\end{aligned}$$

Again, using homotopy invariance, we simplify the operator $S \circ T$ before calculating its degree. Define

$$\Omega_t : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \to \begin{bmatrix} \left[\left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} x_5 + \left(\sum\limits_{i=1}^{4} x_i^2\right)^2\right] \cdot 2l\eta\frac{1}{\pi}\int\limits_0^{2\pi}\left[\sum\limits_{i=1}^{4} x_i\varphi_i(t)\right]^{2l-1} w(t)\varphi_1(t)\,\mathrm{d}t \\ \left[\left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} x_5 + \left(\sum\limits_{i=1}^{4} x_i^2\right)^2\right] \cdot 2l\eta\frac{1}{\pi}\int\limits_0^{2\pi}\left[\sum\limits_{i=1}^{4} x_i\varphi_i(t)\right]^{2l-1} w(t)\varphi_2(t)\,\mathrm{d}t \\ \left[\left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} x_5 + \left(\sum\limits_{i=1}^{4} x_i^2\right)^2\right] \cdot 2l\eta\frac{1}{\pi}\int\limits_0^{2\pi}\left[\sum\limits_{i=1}^{4} x_i\varphi_i(t)\right]^{2l-1} w(t)\varphi_3(t)\,\mathrm{d}t \\ \left[\left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} x_5 + \left(\sum\limits_{i=1}^{4} x_i^2\right)^2\right] \cdot 2l\eta\frac{1}{\pi}\int\limits_0^{2\pi}\left[\sum\limits_{i=1}^{4} x_i\varphi_i(t)\right]^{2l-1} w(t)\varphi_4(t)\,\mathrm{d}t \\ \left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} x_5 + (1-t)\left[\left(\sum\limits_{i=1}^{4} x_i^2\right)^2 + \eta P_w K_P P_1\left(\left[\sum\limits_{i=1}^{4} x_i\varphi_i(t)\right]^{2l}\right)\right] \end{bmatrix}$$

$$+ \begin{bmatrix} (1-t)\,o\left(\left[\left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} |x_5| + \left(\sum\limits_{i=1}^{4} x_i^2\right)^2\right]^2 \left(\sum\limits_{i=1}^{5} x_i^2\right)^{l-2}\right) \\ (1-t)\,o\left(\left[\left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} |x_5| + \left(\sum\limits_{i=1}^{4} x_i^2\right)^2\right]^2 \left(\sum\limits_{i=1}^{5} x_i^2\right)^{l-2}\right) \\ (1-t)\,o\left(\left[\left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} |x_5| + \left(\sum\limits_{i=1}^{4} x_i^2\right)^2\right]^2 \left(\sum\limits_{i=1}^{5} x_i^2\right)^{l-2}\right) \\ (1-t)\,o\left(\left[\left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} |x_5| + \left(\sum\limits_{i=1}^{4} x_i^2\right)^2\right]^2 \left(\sum\limits_{i=1}^{5} x_i^2\right)^{l-2}\right) \\ (1-t)\,o\left(\left[\left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} |x_5| + \left(\sum\limits_{i=1}^{4} x_i^2\right)^2\right] \left(\sum\limits_{i=1}^{5} x_i^2\right)^{l-1}\right) \end{bmatrix}$$

In the next lemma we show that the equation $\Omega_t x = 0$ has the trivial solution for $t \in [0,1]$ which is separated, i.e., in its sufficient small neighbourhood there is no other solution of the equation $\Omega_t x = 0$.

LEMMA 4.4. *The equation $\Omega_t x = 0$ has the trivial solution in $\overline{\Delta}$ for $t \in [0,1]$ which is separated.*

Proof. If this were not the case, then for all $\delta = \dfrac{1}{n}$, $n \in \mathbb{N}$ there would exist such a vector $(x_{1_n}, \dots, x_{5_n})$ that $0 < |x_{i_n}| < \delta = \dfrac{1}{n}$, $i = 1, 2, 3, 4, 5$

and such a $t_n \in [0,1]$ that $\Omega_{t_n} x_n = 0$, i.e.,

$$\left[\left(\sum_{i=1}^{5} x_{i_n}^2\right)^{\frac{5}{2}} x_{5_n} + \left(\sum_{i=1}^{4} x_{i_n}^2\right)^2\right] \cdot 2l\eta\frac{1}{\pi}\int_0^{2\pi}\left[\sum_{i=1}^{4} x_{i_n}\varphi_i(t)\right]^{2l-1} w(t)\varphi_j(t)\, dt$$
$$= (1-t_n)\, o\left(\left[\left(\sum_{i=1}^{5} x_{i_n}^2\right)^{\frac{5}{2}} |x_{5_n}| + \left(\sum_{i=1}^{4} x_{i_n}^2\right)^2\right]^2 \cdot \left(\sum_{i=1}^{5} x_{i_n}^2\right)^{l-2}\right),$$
$$j = 1,2,3,4, \tag{4.11}$$

and

$$\left(\sum_{i=1}^{5} x_{i_n}^2\right)^{\frac{5}{2}} x_{5_n} + (1-t_n)\left[\left(\sum_{i=1}^{4} x_{i_n}^2\right)^2 + \eta P_w K_p P_1\left(\left[\sum_{i=1}^{4} x_{i_n}\varphi_i(t)\right]^{2l}\right)\right]$$
$$= (1-t_n)\, o\left(\left[\left(\sum_{i=1}^{5} x_{i_n}^2\right)^{\frac{5}{2}} |x_{5_n}| + \left(\sum_{i=1}^{4} x_{i_n}^2\right)^2\right]\left(\sum_{i=1}^{5} x_{i_n}^2\right)^{l-1}\right). \tag{4.12}$$

Consider the following cases.

1. Let there exist such a subsequence $\{x_{i_{n_k}}\} \subset \{x_{i_n}\}$ that

$$|x_{j_{0_{n_k}}}| = \max\{|x_{i_{n_k}}|,\ i = 1,2,3,4,5\} \qquad \text{for all } n_k \text{ and } j_0 \in \{1,2,3,4\}.$$

Rewrite $x_{i_{n_k}} = x_{i_n}$. Then there exists such $z_{i_n} \in \mathbb{R}$, $|z_{i_n}| \le 1$ that $x_{i_n} = z_{i_n} \cdot x_{j_{0_n}}$, $i = 1,2,3,4,5$. Consider the expression $\left(\sum\limits_{i=1}^{5} x_{i_n}^2\right)^{\frac{5}{2}} x_{5_n} + \left(\sum\limits_{i=1}^{4} x_{i_n}^2\right)^2$. It is true that $x_{5_n}^2 \le x_{j_{0_n}}^2 \le \sum\limits_{i=1}^{4} x_{i_n}^2$. If $\left(\sum\limits_{i=1}^{5} x_{i_n}^2\right)^{\frac{5}{2}} x_{5_n} + \left(\sum\limits_{i=1}^{4} x_{i_n}^2\right)^2 = 0$, then

$$-x_{5_n} = \frac{\left(\sum\limits_{i=1}^{4} x_{i_n}^2\right)^2}{\left(\sum\limits_{i=1}^{5} x_{i_n}^2\right)^{\frac{5}{2}}} > \frac{\left(\sum\limits_{i=1}^{4} x_{i_n}^2\right)^2}{\left(\sum\limits_{i=1}^{5} x_{i_n}^2\right)^2} = \frac{\left(\sum\limits_{i=1}^{4} x_{i_n}^2\right)^2}{\left(\sum\limits_{i=1}^{4} x_{i_n}^2\right)^2 + 2\sum\limits_{i=1}^{4} x_{i_n}^2 x_{5_n}^2 + x_{5_n}^4} \ge \frac{1}{1+2+1} = \frac{1}{4}$$

This case cannot hold for x_{5_n} sufficiently small. Dividing (4.11) by

$\left(\sum_{i=1}^{5} x_{i_n}^2\right)^{\frac{5}{2}} x_{5_n} + \left(\sum_{i=1}^{4} x_{i_n}^2\right)^2$ and letting $j = j_0$ we get

$$2l\eta\frac{1}{\pi}\int_0^{2\pi}\left[\sum_{i=1}^{4} x_{i_n}\varphi_i(t)\right]^{2l-1} w(t)\varphi_j(t)\,\mathrm{d}t$$
$$= o\left(\left[\left(\sum_{i=1}^{5} x_{i_n}^2\right)^{\frac{5}{2}}|x_{5_n}| + \left(\sum_{i=1}^{4} x_{i_n}^2\right)^2\right]\cdot\left(\sum_{i=1}^{5} x_{i_n}^2\right)^{l-2}\right), \qquad j = 1,2,3,4\,. \tag{4.13}$$

Writing $x_{i_n} = z_{i_n} x_{j_{0_n}}$ in (4.13) we have

$$x_{j_{0_n}}^{2l-1}\cdot A_{j_{0_n}} = o\left(\left[\left(\sum_{i=1}^{5} z_{i_n}^2\right)^{\frac{5}{2}}|z_{5_n}|\cdot x_{j_{0_n}}^6 + \left(\sum_{i=1}^{4} z_{i_n}^2\right)^2 x_{j_{0_n}}^4\right]\left(\sum_{i=1}^{5} z_{i_n}\right)^{l-2} x_{j_{0_n}}^{2l-4}\right)$$
$$= o\left(\left[\left(\sum_{i=1}^{5} z_{i_n}^2\right)^{\frac{5}{2}}|z_{5_n}|\cdot x_{j_{0_n}}^2 + \left(\sum_{i=1}^{4} z_{i_n}^2\right)^2\right]\left(\sum_{i=1}^{5} z_{i_n}\right)^{l-2} x_{j_{0_n}}^{2l}\right). \tag{4.14}$$

Dividing (4.14) by $x_{j_{0_n}}^{2l-1}$ we get

$$A_{j_{0_n}} = o\left(\left[\left(\sum_{i=1}^{5} z_{i_n}^2\right)^{\frac{5}{2}}|z_{5_n}|\cdot x_{j_{0_n}}^2 + \left(\sum_{i=1}^{4} z_{i_n}^2\right)^2\right]\left(\sum_{i=1}^{5} z_{i_n}\varphi_i(t)\right)^{l-2}|x_{j_{0_n}}|\right), \tag{4.15}$$

where

$$A_{j_{0_n}} = 2l\eta\frac{1}{\pi}\int_0^{2\pi}\left[\varphi_{j_0}(t) + \sum_{\substack{i=1\\ i\neq j_0}}^{4} z_{i_n}\varphi_i(t)\right]^{2l-1} w(t)\varphi_{j_0}(t)\,\mathrm{d}t\,.$$

Consider such a subsequence $\{x_{i_{n_k}}\}$ of the sequence $\{x_{i_n}\}$ that $z_{i_{n_k}} \to z_{i_0}$ as $n_k \to \infty$. Then from the Lebesgue dominant convergence theorem it follows that

$$A_{j_{0_n}} \to 2l\eta\frac{1}{\pi}\int_0^{2\pi}\left[\varphi_{j_0}(t) + \sum_{\substack{i=1\\ i\neq j_0}}^{4} z_{i_0}\varphi_i(t)\right]^{2l-1} w(t)\varphi_{j_0}(t)\,\mathrm{d}t \neq 0\,.$$

By the equation (4.15) we get the contradiction, therefore $\sum_{i=1}^{5} z_{i_{n_k}}^2$ is bounded.

2. Let there exist such a subsequence $\{x_{i_{n_k}}\} \subset \{x_{i_n}\}$ that

$$|x_{5_{n_k}}| = \max\{|x_{i_{n_k}}|,\ i = 1,2,3,4,5 \text{ for all } n_k \in \mathbb{N}\}\,.$$

Rewrite $x_{i_{n_k}} = x_{i_n}$. Then there exists such $z_{i_n} \in \mathbb{R}$, $|z_{i_n}| \le 1$ that $x_{i_n} = z_{i_n} \cdot x_{5_n}$. Using this fact in (4.12) we write

$$\left(\sum_{i=1}^{4} z_{i_n}^2 + 1\right)^{\frac{5}{2}} x_{5_n}^6 + (1-t_n)\left[\left(\sum_{i=1}^{4} z_{i_n}^2\right)^2 x_{5_n}^4 + x_{5_n}^{2l} \eta P_w K_P P_1 \left(\left[\sum_{i=1}^{4} z_{i_n}\varphi_i(t)\right]^{2l}\right)\right]$$
$$= o\left(\left[\left(\sum_{i=1}^{4} z_{i_n}^2 + 1\right)^{\frac{5}{2}} x_{5_n}^6 + \left(\sum_{i=1}^{4} z_{i_n}^2\right)^2 x_{5_n}^4\right]\left(\sum_{i=1}^{5} z_{i_n}^2\right)^{l-1} x_{5_n}^{2l-2}\right). \tag{4.16}$$

The sequence $\{t_n\}$ is bounded and hence we can use the subsequence $t_{n_k} \to t_0$ for $n_k \to \infty$ and $t_0 \in [0,1]$. Dividing (4.16) by $x_{5_n}^6$ we get

$$\left(\sum_{i=1}^{4} z_{i_n}^2 + 1\right) + (1-t_n)\left[\left(\sum_{i=1}^{4} z_{i_n}^2\right)^2 \frac{1}{x_{5_n}^2} + x_{5_n}^{2l-6} \eta P_w K_P P_1 \left(\left[\sum_{i=1}^{4} z_{i_n}\varphi_i(t)\right]^{2l}\right)\right]$$
$$= o\left(\left[\left(\sum_{i=1}^{4} z_{i_n}^2 + 1\right)^{\frac{5}{2}} x_{5_n}^2 + \left(\sum_{i=1}^{4} z_{i_n}^2\right)^2\right] \cdot \left(\sum_{i=1}^{5} z_{i_n}^2\right)^{l-1} x_{5_n}^{2l-4}\right).$$

It is true that $1 < \sum\limits_{i=1}^{4} z_i^2 + 1 \le 5$, $0 \le (1-t_n)\left(\sum\limits_{i=1}^{4} z_{i_n}^2\right)\frac{1}{x_{5_n}^2}$. The other expressions are sufficiently small and this gives the contradiction.

Using the homotopy invariance and Lemma 4.4 we obtain that $d(\Omega_0, \overline{\Delta}, 0) = d(\Omega_1, \overline{\Delta}, 0)$. Now we compute the degree of the mapping

$$\Omega_1 : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \to \begin{bmatrix} A_1 \left[\left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} x_5 + \left(\sum\limits_{i=1}^{4} x_i^2\right)^2\right] \\ A_2 \left[\left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} x_5 + \left(\sum\limits_{i=1}^{4} x_i^2\right)^2\right] \\ A_3 \left[\left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} x_5 + \left(\sum\limits_{i=1}^{4} x_i^2\right)^2\right] \\ A_4 \left[\left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} x_5 + \left(\sum\limits_{i=1}^{4} x_i^2\right)^2\right] \\ \left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{5}{2}} x_5 \end{bmatrix},$$

where

$$A_j = 2l\eta\frac{1}{\pi}\int_0^{2\pi}\left[\sum_{j=1}^{4} x_i\varphi_i(t)\right]^{2l-1} w(t)\varphi_j(t)\,\mathrm{d}t\,, \qquad j = 1,2,3,4\,.$$

Using the Borsuk theorem in generalized form ([1, p. 46–47]), we compute the degree of the mapping Ω_1. It is necessary to show that $\dfrac{\Omega_1(x)}{|\Omega_1(x)|} \neq \dfrac{\Omega_1(-x)}{|\Omega_1(-x)|}$ and it is equivalent with $\Omega_1(x) \neq k \cdot \Omega_1(-x)$, where $k > 0$ and $k = \dfrac{|\Omega_1(x)|}{|\Omega_1(-x)|}$. By the fifth component of the mapping Ω_1 we see that this condition is fulfilled. Therefore $d(\Omega_1, \overline{\Delta}, 0) \neq 0$. We have shown that

$$d(S \circ T, \overline{\Delta}, 0) \neq 0 \qquad \text{for sufficiently small } \overline{\Delta}. \tag{4.17}$$

To return to the original operator S we can use the multiplication theorem for the degree. Note the Jacobian

$$J_T = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \rho x_1 & \rho x_2 & \rho x_3 & \rho x_4 & \rho^{\frac{3}{2}}\left[5x_5^2 + \sigma\right] \end{vmatrix} = \left(\sum_{i=1}^{5} x_i^2\right)^{\frac{3}{2}} \left[5x_5^2 + \sum_{i=1}^{5} x_i^2\right] \geq 0,$$

where ρ denotes the sum $5\left(\sum\limits_{i=1}^{5} x_i^2\right)^{\frac{3}{2}} x_5 + 4\sum\limits_{i=1}^{4} x_i^2$ and σ the sum $\sum\limits_{i=1}^{5} x_i^2$, respectively. It is easily verified that T is a homeomorphism from $\mathbb{R}^5$ to $\mathbb{R}^5$ which takes $(0,0,0,0,0)$ to $(0,0,0,0,0)$. By the definition of degree in finite-dimensional normed space it follows that for any open subset $D \subset \langle \varphi_1, \varphi_2, \varphi_3, \varphi_4, w \rangle$ and any $p \in T(D)$ is $d(T, D, p) = 1$. In particular

$$d(T, D, p) = 1 \qquad \text{for all } p \in T(\overline{\Delta}). \tag{4.18}$$

Let M be an open ball containing $T(\partial\overline{\Delta})$ and so that the equation $Sy = 0$ has no solutions in M except $y = 0$. By Lemma 4.4 and from the fact that T is a homeomorphism with $T(0,0,0,0,0) = (0,0,0,0,0)$ it follows that such an M exists. The multiplication theorem for the degree of mapping tells us that

$$d(S \circ T, \overline{\Delta}, 0) = \sum_{\Delta_j} d(S, \Delta_j, 0) \cdot d(T, \overline{\Delta}, \Delta_j), \tag{4.19}$$

where Δ_j are the components of $M \setminus T(\partial\overline{\Delta})$. Since T is a homeomorphism, $M \setminus T(\partial\overline{\Delta})$ has only two components. Let Δ_1 be the component which does not contain the origin and Δ_2 the complementary component. Observe that $Sy = 0$ has no solutions in Δ_1; therefore $d(S, \Delta_1, 0) = 0$. By the definition of S and (4.17), (4.18), (4.19) it follows that

$$0 \neq d(S \circ T, \overline{\Delta}, 0) = d(S, \Delta_2, 0) \cdot 1.$$

Therefore $d(I - A_1, \Delta, 0) \neq 0$. We summarize our result in the next theorem.

THEOREM 4.1. *Suppose that the integrals*

$$\int_0^{2\pi} [c_1 \cos mt + c_2 \sin mt + c_3 \cos nt + c_4 \sin nt]^{2l-1} \cdot w(t)\, \varphi_j(t)\ \mathrm{d}t \neq 0\,,$$

$$\text{for} \quad j = 1, 2, 3, 4\,,$$

where $c_1, \dots, c_4$ *are arbitrary constants such that at least one of* $c_i \neq 0$, $\varphi_1(t) = \cos mt$, $\varphi_2(t) = \sin mt$, $\varphi_3(t) = \cos nt$, $\varphi_4(t) = \sin nt$, $w \in NS(L)^\perp$ *is the solution of the equation*

$$Lw(t) = (a_m \cos mt + b_m \sin mt + a_n \cos nt + b_n \sin nt)^{2l}\,,$$

for arbitrary constants a_i, b_i, $i = m, n$, *satisfying* (3.5), $\eta = \pm 1$,

$$\Delta = \left\{ y \in BC:\ \|y\|_\infty < \left(\frac{1}{c - \varepsilon} \|f\|_1 \right)^{\frac{1}{4l-1}} < \delta \right\},$$

and f *is sufficiently small. Then the equation* $\mathcal{L}y(t) = Ly(t) + \eta y^{2l}(t) = f(t)$, $l \geq 4$ *has at least one solution in* $\Delta \subset BC$.

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