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Dedicated to Academician Štefan Schwarz
on the occasion of his 80th birthday

SUBSEMIGROUPS OF COMPLETELY SIMPLE SEMIGROUPS III

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ABSTRACT. We shall give a set of implications which determines the class of all semigroups which are embeddable in completely simple semigroups. No finite set of implications is sufficient to ensure that a semigroup is embeddable in a completely simple semigroup. We derive a set of implications which determines the class of group embeddable semigroups.

1. Introduction

A quasivariety is a class consisting of algebras of the same type and definable by implications. Alternatively, a class of algebras of the same type is a quasivariety if and only if it is closed under the formation of isomorphic images, products, subalgebras and direct limits (see e.g. [13], [19]). Therefore the class of all semigroups embeddable in completely simple semigroups constitutes a quasivariety $\mathcal{C}S_s$ (see e.g. [13; p. 216, Corollary 5]). For the necessary background on semigroups and completely simple semigroups in particular, we refer to [4], and for information concerning the variety $\mathcal{C}S$ of completely simple semigroups, when considered as unary semigroups, the reader may consult [3], [16], [17]. For some particular embeddings of semigroups into completely simple semigroups and related results, we refer to [1], [5], [6], [7], [8].

Let X be a fixed countably infinite set of variables. All the implications that will be considered will be implications with variables in X. If $\Lambda(\mathcal{C}S_s)$ (resp. $\Lambda(\mathcal{C}S)$) denotes the set of all semigroup implications on X satisfied by all the members of $\mathcal{C}S_s$ (resp. $\mathcal{C}S$), then of course $\Lambda(\mathcal{C}S_s) = \Lambda(\mathcal{C}S)$. For $\Lambda \subseteq \Lambda(\mathcal{C}S_s)$

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we say that Λ is a basis of $\Lambda(\mathcal{CS}_s)$ if the class of all semigroups satisfying all the implications of Λ coincides with $\mathcal{CS}_s$ or equivalently, if the implications of $\Lambda(\mathcal{CS}_s)$ are all derivable (in the sense of [18]) from the implications of Λ. In this paper, we shall indicate how to recursively list the implications of such a basis of $\Lambda(\mathcal{CS}_s)$. The main tool to be used is a model of the free completely simple semigroup on a given semigroup. The method is comparable to the building of a model for the free group on a given semigroup, as outlined in [4; Section 12.4]. We then show that $\Lambda(\mathcal{CS}_s)$ does not have a finite basis.

We refer to [4; Chapter 12] for a discussion on the embeddability of semigroups into groups, and for the relevant references to the work of Lambek, Malcev and Pták. Implications for semigroups embeddable into a member of a given variety $\mathcal{V}$ of orthocryptogroups which contains the variety of all groups were given in [2]. Using a result of Malcev [11], [12], it was shown in [2] that for such a variety $\mathcal{V}$ the set of implications $\Lambda(\mathcal{V})$ cannot have a finite basis.

2. The free completely simple semigroup on a given semigroup

If $\mathcal{V}$ is a variety of unary semigroups and S a given semigroup, then the pair (C,γ) is called *free in $\mathcal{V}$ on the semigroup S* if

(i) $C \in \mathcal{V}$ and $\gamma \colon S \to C$ is a homomorphism,

(ii) C is generated as a unary semigroup by $S\gamma$,

(iii) if (H,δ) is such that $H \in \mathcal{V}$ and $\delta\colon S \to H$ is a homomorphism, then there exists a (necessarily unique) homomorphism $\psi\colon C \to H$ such that

$$\begin{tikzcd} S \arrow[r, "\gamma"] \arrow[dr, "\delta"'] & C \arrow[d, "\psi"] \\ & H \end{tikzcd} \qquad (1)$$

is a commutative diagram.

Such objects (C,γ) exist of course: indeed, if (F,ι) is free in $\mathcal{V}$ on the set S, and σ is the congruence relation on F generated by the set

$$\{(ab,c) \mid a,b,c \in S,\ ab = c \text{ in } S\},$$

then $(F/\sigma,\ \iota\sigma^\natural)$ is free in $\mathcal{V}$ on the semigroup S. In particular, a free completely simple semigroup on a semigroup S exists; moreover, such free objects are isomorphic (in the obvious sense). Though we have a good understanding of free completely simple semigroups (see e.g. [3], [17]), we shall find it useful to give a model of a free completely simple semigroup on a given semigroup which already satisfies certain implications.

For a semigroup S and $a, b \in S$ we put $a\,\rho\,b$ if there exists $n \geq 1$ and elements $d_1, \ldots, d_n$, $e_1, \ldots, e_n$, $c_1, \ldots, c_{n-1}$ of S such that

$$ad_1 = c_1e_1\,, \quad c_1d_2 = c_2e_2\,, \quad \ldots\,, \quad c_{n-2}d_{n-1} = c_{n-1}e_{n-1}\,, \quad c_{n-1}d_n = be_n\,.$$

If $n = 1$, we assume that $ad_1 = be_1$ for some d_1 and e_1. It is easy to see that ρ is the least left zero semigroup congruence on S ([9]). The relation λ on S is defined in a dual way: λ is the least right zero semigroup congruence on S. We use the notation $\eta = \lambda \cap \rho$. Thus η is the least rectangular band congruence on S ([9]). For any $i, n \geq 1$, and $x, y \in X$, $x\,\rho_{i,n}\,y$ will stand for the following conjunction of (formal) equalities on the free semigroup X^+ on the set of variables X:

$$\begin{aligned} xv_{i,1} = p_{i,1}w_{i,1}\,, \quad p_{i,1}v_{i,2} = p_{i,2}w_{i,2}\,, \quad \ldots \\ \ldots\,, \quad p_{i,n-2}v_{i,n-1} = p_{i,n-1}w_{i,n-1}\,, \quad p_{i,n-1}v_{i,n} = yw_{i,n}\,. \end{aligned} \tag{2}$$

If $n = 1$, we assume that (2) reduces to $xv_{i,1} = yw_{i,1}$. Dually, $x\,\lambda_{i,n}\,y$ stands for the conjunction of

$$\begin{aligned} s_{i,n}x = t_{i,1}q_{i,1}\,, \quad s_{i,2}q_{i,1} = t_{i,2}q_{i,2}\,, \quad \ldots \\ \ldots\,, \quad s_{i,n-1}q_{i,n-2} = t_{i,n-1}q_{i,n-1}\,, \quad s_{i,n}q_{i,n-1} = t_{i,n}y\,, \end{aligned} \tag{3}$$

where again for $n = 1$ we assume that (3) reduces to $s_{i,1}x = t_{i,1}y$, and $x\,\eta_{i,n}\,y$ denotes the two sequences combined, with the understanding that the $v_{i,\ell}$, $w_{i,\ell}$, $s_{i,\ell}$, $t_{i,\ell}$, $1 \leq \ell \leq n$ and $p_{i,j}$, $q_{i,j}$, $i \leq j \leq n-1$, are distinct variables of the set X.

With the notation introduced above, we thus have that $a\,\rho\,b$ in the semigroup S if and only if there exists $n \geq 1$ such that a substitution of the variables involved in (2), with x substituted by a and y substituted by b, yields from (2) a sequence of true equalities in S. In this case we write $a\,\rho_n\,b$. The definitions for the relations λ_n and η_n on S are analogous. It should be noted that if $a\,\rho_n\,b$, then $a\,\rho_m\,b$ for every $m \geq n$, and the same remark applies for λ_n and η_n. From this it also follows that $a\,\eta\,b$ if and only if, for some $n \geq 1$, $a\,\eta_n\,b$.

Lemma 1. *Let $\mathcal{A}$ be the quasivariety defined by the implications $A_j \longrightarrow B_j$, $j \in J$, where A_j is a conjunction of equations, and B_j, an equation. Let $\{x_1, \ldots, x_{n_j}\}$ be the set of variables involved in $A_j \longrightarrow B_j$ for $j \in J$. Then a semigroup is a rectangular band of members of $\mathcal{A}$ if and only if it satisfies the implications*

$$\left.\begin{array}{l} A_j \\ x_1\,\eta_{i,n}\,x_i\,, \quad 1 < i \le n_j \end{array}\right\} \longrightarrow B_j \tag{4}$$

for every $j \in J$ and every $n \geq 1$.

P r o o f . Let S be a rectangular band of semigroups which belong to $\mathcal{A}$. Thus, there exists a rectangular band congruence δ on S such that each δ-class belongs to $\mathcal{A}$. The least rectangular band congruence η is contained in δ, whence each η-class is contained in some δ-class. Let $j \in J$, $n \geq 1$, and let a substitution in A_j and $x_1\,\eta_{i,n}\,x_i$ of the variables $x_1, \ldots, x_{n_j}, \ldots$ by the elements $a_1, \ldots, a_{n_j}, \ldots$ of S yield a collection of true equalities in S. Then the elements $a_1, \ldots, a_{n_j}$ of S are η-related and belong to some δ-class T of S. Since T satisfies $A_j \longrightarrow B_j$, the substitution in B_j of $x_1, \ldots, x_{n_j}$ by $a_1, \ldots, a_{n_j}$ yields a true equality in S. Therefore S satisfies the implications (4).

Conversely, assume that S satisfies the implications (4). Let T be an η-class, $j \in J$, and let a substitution in A_j of $x_1, \ldots, x_{n_j}$ by elements $a_1, \ldots, a_{n_j}$ of T yield a collection of true equalities. In particular, $a_1, \ldots, a_{n_j}$ are η_n-related for some $n \geq 1$. Since S satisfies (4), the substitution in B_j by $a_1, \ldots, a_{n_j}$ yields a true equality. Hence T belongs to $\mathcal{A}$ and S is a rectangular band of members of $\mathcal{A}$.

Let us denote the variety of all rectangular bands by $\mathcal{RB}$. For any quasivariety $\mathcal{A}$ of semigroups, the quasivariety consisting of the semigroups which are rectangular bands of members of $\mathcal{A}$ is precisely $\mathcal{A} \circ \mathcal{RB}$, the Malcev product of $\mathcal{A}$ and $\mathcal{RB}$. Let $\mathcal{G}$ be the variety of all groups, and $\mathcal{G}_s$ the quasivariety of all semigroups which are group embeddable. Thus $\mathcal{G}_s \circ \mathcal{RB}$ is the quasivariety consisting of the semigroups which are rectangular bands of group embeddable semigroups. Every semigroup which is embeddable into a completely simple semigroup is a rectangular band of group embeddable semigroups: $\mathcal{CS}_s \subseteq \mathcal{G}_s \circ \mathcal{RB}$. Lemma 1 and, for instance, Malcev's basis of implications [11] for the set $\Lambda(\mathcal{G}_s)$ of all semigroup implications on X satisfied by all the members of $\mathcal{G}_s$, give us a means to find a basis of the set $\Lambda(\mathcal{G}_s \circ \mathcal{RB})$ of the semigroup implications satisfied by the members of $\mathcal{G}_s \circ \mathcal{RB}$. Since we ultimately want to find a basis for $\Lambda(\mathcal{CS}_s)$, an affirmative answer to the following would end our quest.

PROBLEM 2. Does $\mathcal{CS}_s$ coincide with $\mathcal{G}_s \circ \mathcal{RB}$?

Another consequence of Lemma 1 is given by

LEMMA 3. *A semigroup is a rectangular band of cancellative semigroups if and only for every $n \geq 1$ it satisfies the implication*

$$(x_1x_3 = x_2x_3,\ x_1\,\eta_{1,n}\,x_2,\ x_2\,\eta_{2,n}\,x_3) \longrightarrow x_1 = x_2 \qquad (C_n)$$

and its dual (C_n^*).

COROLLARY 4. $\Lambda(\mathcal{C}S_s)$ *contains the implications (C_n) and (C_n^*) for every n.*

P r o o f. A semigroup which is embeddable into a completely simple semigroup is a rectangular band of cancellative semigroups.

Recall that groups satisfy the following so called quotient condition [10] (see [4; Section 12.4]):

$$\left.\begin{aligned} xw &= yz \\ xp &= yq \\ uw &= vz \end{aligned}\right\} \longrightarrow up = vq\,. \qquad (Q)$$

LEMMA 5. $\Lambda(\mathcal{C}S_s)$ *contains the implications*

$$\left.\begin{aligned} xw &= yz\,, & w\,\eta_{1,n}\,y \\ xp &= yq\,, & p\,\eta_{2,n}\,y \\ uw &= vz\,, & w\,\eta_{3,n}\,v \end{aligned}\right\} \longrightarrow up = vq \qquad (Q_n)$$

for every $n \geq 1$.

P r o o f. Since $\Lambda(\mathcal{C}S) = \Lambda(\mathcal{C}S_s)$, it suffices to show that every Rees matrix semigroup $\mathcal{M} = \mathcal{M}(G; I, \Lambda; P)$ over a group G satisfies (Q_n) for every n.

For $t \in X$, let $\bar{t} = (i_t, g_t, \lambda_t) \in \mathcal{M}$, such that $\bar{x}\bar{w} = \bar{y}\bar{z}$, $\bar{x}\bar{p} = \bar{y}\bar{q}$, $\bar{u}\bar{w} = \bar{v}\bar{z}$, $\bar{w}\,\eta_n\,\bar{y}$, $\bar{p}\,\eta_n\,\bar{y}$ and $\bar{w}\,\eta_n\,\bar{v}$. From the last three conditions we infer that $\bar{w}$, $\bar{y}$, $\bar{p}$ and $\bar{v}$ belong to the same maximal subgroup of $\mathcal{M}$. Hence, there exist $i \in I$ and $\lambda \in \Lambda$ such that $i = i_w = i_y = i_p = i_v = i_x = i_u$ and $\lambda = \lambda_w = \lambda_y = \lambda_p = \lambda_v = \lambda_z = \lambda_q$. From $\bar{x}\bar{w} = \bar{y}\bar{z}$, $\bar{x}\bar{p} = \bar{y}\bar{q}$, $\bar{u}\bar{w} = \bar{v}\bar{z}$, we thus obtain, respectively, $g_x\,p_{\lambda_x i}\,g_w = g_y\,p_{\lambda i_z}\,g_z$, $g_x\,p_{\lambda_x i}\,g_p = g_y\,p_{\lambda i_q}\,g_q$, $g_u\,p_{\lambda_u i}\,g_w = g_v\,p_{\lambda i_z}\,g_z$. In other words, if in the formal equalities $xw = yz$, $xp = yq$ and $uw = vz$ the variables x, y, w, z, p, q, u, v are substituted respectively by the elements $g_x\,p_{\lambda_x i}$, g_y, g_w, $p_{\lambda i_z}\,g_z$, g_p, $p_{\lambda i_q}\,g_q$, $g_u\,p_{\lambda_u i}$, g_v of the group G, then we obtain true equalities in G. Hence, since G satisfies (Q), it follows that $g_u\,p_{\lambda_u i}\,g_p = g_v\,p_{\lambda i_q}\,g_q$ is true in G. Consequently

$$\bar{u}\bar{p} = (i,\ g_u\,p_{\lambda_u i}\,g_p,\ \lambda) = \big(i,\ g_v\,p_{\lambda i_q}\,g_q,\ \lambda\big) = \bar{v}\bar{q}\,.$$

Therefore $\mathcal{M}$ satisfies (Q_n), as required.

We set out to construct a model for the free completely simple semigroup on a semigroup S which satisfies the implications (Q_n) for every $n \geq 1$.

CONSTRUCTION. Let S be any semigroup. For $a, b \in S$ we define

$$a/b = \{(g,h) \mid \; ga = hb\,, \;\; a\,\eta\,h\}$$

if this set is nonempty, and otherwise we say that a/b does not exist. Whenever a/b exists, we call a/b a right quotient of S. Let Q be the set of right quotients of S, and let T be the free semigroup with Q as its set of free generators. Let θ be the congruence relation on T generated by pairs of the form $\big((a/b)(bc/d),\, (ac/d)\big)$, where $a, b, d \in S$, $c \in S^1$ such that a/b, bc/d and ac/d exist. Let $\theta^\natural$ be the canonical homomorphism of T onto $C = T/\theta$. For any $a \in S$, a^2/a exists since $(a, a^2) \in a^2/a$, and so we may define a mapping $\kappa\colon S \to T$, $a \mapsto a^2/a$. Then $\gamma = \kappa\theta^\natural$ maps S into C.

LEMMA 6. *If the semigroup S satisfies the implications (Q_n) for every $n \geq 1$, then*

(i) *for every $a, b, c, d \in S$, $(a/b) \cap (c/d) \neq \emptyset$ if and only if $a/b = c/d$;*

(ii) *if for $a, b \in S$, a/b exists, then $ab/b = a^2/a$.*

If S satisfies in addition the implications (C_n), (C_n^*), $n \geq 1$, then κ is one-to-one.

P r o o f. The proof of (i) follows immediately from the fact that S satisfies (Q_n), $n \geq 1$.

To prove (ii), let $a, b \in S$ such that $(g,h) \in a/b$. Therefore $ga = hb$ and $a\,\eta\,h$. Thus $ab\,\eta\,hb$ and $g(ab) = (hb)b$, whence $(g, hb) \in ab/b$. From the foregoing also follows that $b\,\lambda\,h$, whence $hb\,\eta\,h\,\eta\,a\,\eta\,a^2$ and $ga^2 = (hb)a$, thus $(g, hb) \in a^2/a$. By (i), we may thus conclude that $ab/b = a^2/a$.

As for the last statement, assume that $a^2/a = a\kappa = b\kappa = b^2/b$. Since then $(a, a^2) \in a^2/a = b^2/b$, we find $ab^2 = a^2b$ and $b^2\,\eta\,a^2$, whence $a\,\eta\,b$. Since the η-classes are cancellative, we thus have $a = b$.

THEOREM 7. *Let S be a semigroup which satisfies the implication (Q_n) for every $n \geq 1$. Then (C, γ) is a free completely simple semigroup on the semigroup S.*

P r o o f. For any $a, b \in S$ we have

$$a(aba) = a^2(ba)\,, \qquad aba\,\eta\,a\,\eta\,a^2\,,$$

hence $(a, a^2) \in (aba/ba) \cap (a^2/a)$, and so by Lemma 6 (i), $aba/ba = a^2/a = a\kappa$. Similarly, $bab/ab = b^2/b = b\kappa$. Therefore

$$\begin{aligned}(a\gamma)(b\gamma) &= (aba/ba)\theta\,(bab/ab)\theta\\ &= (abab/ab)\theta\\ &= (ab)\gamma\end{aligned}$$

because $\big((aba/ba)(bab/ab),\,(abab/ab)\big)$ is one of the pairs that generate θ. We proved that γ is a homomorphism of S into C.

From the definition of θ it follows that for every $a \in S$,

$$(a/a)\theta\,(a^2/a)\theta = (a^2/a)\theta = (a^2/a)\theta\,(a/a)\theta\,, \tag{5}$$

$$\big((a/a)\theta\big)^2 = (a/a)\theta\,, \tag{6}$$

$$(a^2/a)\theta\,(a/a^2)\theta = (a^2/a^2)\theta = (a/a)\theta = (a/a^2)\theta\,(a^2/a)\theta\,, \tag{7}$$

where, by Lemma 6 (i), $a^2/a^2 = a/a$ because $(a,a) \in (a/a) \cap (a^2/a^2)$. Hence every element $(a^2/a)\theta = a\gamma$ belongs to a maximal subgroup of C with identity element $(a/a)\theta$, and the inverse of $(a^2/a)\theta$ within this subgroup of C is $(a/a^2)\theta$. Using this and the fact that γ is a homomorphism, we find that for $a, c \in S$,

$$\begin{aligned}(ca/ca)\theta\,(a/a)\theta &= (ca/caca)\theta\,(caca/ca)\theta\,(a/a)\theta\\ &= (ca/caca)\theta\,(ca)\gamma\,(a/a)\theta\\ &= (ca/caca)\theta\,c\gamma\,a\gamma\,(a/a)\theta\\ &= (ca/caca)\theta\,c\gamma\,a\gamma\\ &\ \ \vdots\\ &= (ca/ca)\theta\,.\end{aligned} \tag{8}$$

For all $a, c \in S$, a/ca exists because $(ac, a) \in a/ca$, and furthermore by Lemma 6 (ii),

$$(a/ca)\theta\,(caca/ca)\theta = (aca/ca)\theta = (a^2/a)\theta\,,$$

and consequently by (5),

$$\begin{aligned}(a/a)\theta\,(ca/ca)\theta &= (a/a^2)\theta\,(a^2/a)\theta\,(ca/ca)\theta\\ &= (a/a^2)\theta\,(a/ca)\theta\,(caca/ca)\theta\,(ca/ca)\theta\\ &= (a/a^2)\theta\,(a/ca)\theta\,(caca/ca)\theta\\ &= (a/a^2)\theta\,(a^2/a)\theta\\ &= (a/a)\theta\,.\end{aligned} \tag{9}$$

From (8) and (9), we thus have that the idempotents $(a/a)\theta$ and $(ca/ca)\theta$ of C are $\mathcal{L}$-related. From this a more general statement follows:

$$a\,\lambda\,b \text{ in } S \implies (a/a)\theta\;\mathcal{L}\;(b/b)\theta \text{ in } C\,. \tag{10}$$

Clearly, for all $a, c \in S$

$$(a/a)\theta\,(ac/ac)\theta = (ac/ac)\theta\,. \tag{11}$$

Also, $a/a = aca/aca$ by Lemma 6 (i) since $(a, a) \in (a/a) \cap (aca/aca)$, and therefore

$$\begin{aligned}(ac/ac)\theta\,(a/a)\theta &= (ac/ac)\theta\,(aca/aca)\theta \\ &= (aca/aca)\theta = (a/a)\theta\,.\end{aligned} \tag{12}$$

From (11) and (12), we thus have that the idempotents $(a/a)\theta$ and $(ac/ac)\theta$ of C are $\mathcal{R}$-related. From this a more general statement follows:

$$a\,\rho\,b \text{ in } S \implies (a/a)\theta\,\mathcal{R}\,(b/b)\theta \text{ in } C\,. \tag{13}$$

Let E be the set of idempotents of C of the form $(a/a)\theta$, $a \in S$. For any $(a/a)\theta$, $(b/b)\theta \in E$ we have

$$(a/a)\theta\,\mathcal{R}\,(ab/ab)\theta\;\;\mathcal{L}\,(b/b)\theta$$

and

$$(a/a)\theta\,\mathcal{L}\,(ba/ba)\theta\,\mathcal{R}\,(b/b)\theta$$

in C. It follows that E is biorder isomorphic (in the sense of [14] or [15]) to the biordered set of a rectangular band. Therefore the subsemigroup of C generated by E is an idempotent generated completely simple semigroup ([14], [15]). Let D be the union of the maximal subgroups of C whose identity elements belong to E. Obviously, D is a completely simple semigroup.

Let $a/b \in Q$. Then by Lemma 6 (ii), $ab/b = a^2/a$, whence

$$\begin{aligned}(a/b)\theta &= (a/b)\theta\,(b/b)\theta \\ &= (a/b)\theta\,(b^2/b)\theta\,(b/b^2)\theta \\ &= (ab/b)\theta\,(b/b^2)\theta \\ &= (a^2/a)\theta\,(b/b^2)\theta\,,\end{aligned} \tag{14}$$

and since by (5), (6) and (7), $(a^2/a)\theta, (b/b^2)\theta \in D$, we have from (14) that $(a/b)\theta \in D$. Since C is the semigroup generated by $Q\theta^{\natural}$, it follows that $C = D$. Therefore C is a completely simple semigroup. From (7), we know that for all $b \in S$, $(b/b^2)\theta = (b\gamma)^{-1}$, and so by (14), we have that for all $a/b \in Q$,

$$(a/b)\theta = (a\gamma)\,(b\gamma)^{-1}\,. \tag{15}$$

Therefore C is generated, as a unary semigroup, by $S\gamma$.

Assume that H is a completely simple semigroup and $\delta\colon S \to H$ is a homomorphism. If $a/b = c/d$ exist, then there exists $(g,h) \in a/b = c/d$, whence

$$ga = hb\,, \qquad gc = hd\,, \qquad a\,\eta\,h\,\eta\,c\,,$$

and so

$$(g\delta)(a\delta) = (h\delta)(b\delta)\,, \qquad (g\delta)(c\delta) = (h\delta)(d\delta)\,,$$
$$g\delta\ \mathcal{R}\ a\delta\ \mathcal{H}\ h\delta\ \mathcal{H}\ c\delta\ \mathcal{L}\ b\delta\ \mathcal{L}\ d\delta\,.$$

From this it follows that

$$(a\delta)(b\delta)^{-1} = (g\delta)^{-1}(h\delta) = (c\delta)(d\delta)^{-1}\,.$$

Consequently, the mapping

$$\chi\colon Q \to H\,, \quad a/b \mapsto (a\delta)(b\delta)^{-1}$$

is well-defined. Since T is free on Q, there exists a homomorphism $\varphi\colon T \to H$ extending χ. If a/b, bc/d and ac/d exist, then, since $a\lambda b$ and thus $(a\delta)\ \mathcal{L}\ (b\delta)$,

$$\begin{aligned}
\big((a/b)(bc/d)\big)\varphi &= (a/b)\varphi\,(bc/d)\varphi \\
&= (a/b)\chi\,(bc/d)\chi \\
&= (a\delta)(b\delta)^{-1}(bc)\delta\,(d\delta)^{-1} \\
&= (a\delta)(b\delta)^{-1}(b\delta)(c\delta)(d\delta)^{-1} \\
&= (a\delta)(c\delta)(d\delta)^{-1} \\
&= (ac)\delta\,(d\delta)^{-1} \\
&= (ac/d)\varphi\,.
\end{aligned}$$

It follows that $\theta \subseteq \varphi\varphi^{-1}$, and consequently there exists $\psi\colon C \to H$ such that $\theta^\natural\psi = \varphi$. Since for any $a \in S$

$$a\gamma\psi = (a^2/a)\theta\psi = (a^2/a)\varphi = (a^2/a)\chi = (a\delta)^2(a\delta)^{-1} = a\delta\,,$$

the diagram (1) commutes. Hence (C,γ) is a free completely simple semigroup on S.

For the record we note

THEOREM 8. *A semigroup S can be embedded into a completely simple semigroup if and only if S satisfies (Q_n) for all $n \geq 1$, and $\gamma\colon S \to C$ is one-to-one.*

For γ to be one-to-one, we need κ and $\theta^\natural$ to be one-to-one. By Lemma 6, κ is guaranteed to be one-to-one if the semigroup S satisfies the implications (C_n) and (C_n^*) for all $n \geq 1$. As we shall see in the next section, these implications (C_n), (C_n^*) need not be present in a basis for $\Lambda(\mathcal{CS}_s)$.

3. Implications for semigroups embeddable in completely simple semigroups

In this section we indicate how to list the implications of a basis of $\Lambda(\mathcal{C}S_s)$ and we show that $\Lambda(\mathcal{C}S_s)$ cannot have a finite basis.

For $m \geq 1$ let $\mathbb{N}$ be the set of positive integers, $A = \{-1, 1\}$, $B = \{0, 1\}$, and $A_m = \{1, \ldots, m\}$. Let $g = (g_1, g_2, g_3)$ be a mapping of A_m into $A \times \mathbb{N} \times B$ such that, with $m_k = 1 + \sum_{i=1}^{k-1} g_1(i)$ for all $1 < k \leq m$ and $m_1 = 1$.

(i) $1 \leq g_2(k) \leq m_k$ for all $1 \leq k \leq m$;
(ii) if $g_2(k) = 1$, then $g_1(k) = -1$;
(iii) $\sum_{i=1}^{m} g_1(i) = 0$, $g_2(m) = 1$.

With such a mapping g we associate an implication $(I_{g,n})$ for all n such that $m \leq n$ in the following way. If for $1 \leq k \leq m$, $g_1(k) = 1$, $g_3(k) = 1$, then I_k stands for the formal equalities over X:

$$
\begin{aligned}
&p_k x_{g_2(k),k} = q_k\, x'_{g_2(k),k}\,, \quad p_k u_k = q_k v_k\,,\\
&q_k\ \eta_{5k+1,n}\ x_{g_2(k),k}\,, \quad q_k\ \eta_{5k+2,n}\ u_k\,,\\
&r_k x_{g_2(k)+1,k} = s_k x'_{g_2(k)+1,k}\,, \quad r_k v_k w_k = s_k z_k\,,\\
&s_k\ \eta_{5k+3,n}\ x_{g_2(k)+1,k}\,, \quad s_k\ \rho_{5k+4,n}\ v_k\,, \quad s_k\ \lambda_{5k+4,n}\ w_k\,,\\
&t_k u_k w_k = y_k z_k\,, \quad y_k\ \rho_{5k+5,n}\ u_k\,, \quad y_k\ \lambda_{5k+5,n}\ w_k\,,\\
&x_{i,k+1} = x_{i,k} \quad x'_{i,k+1} = x'_{i,k} \qquad \text{for all} \quad 1 \leq i < g_2(k)\,,\\
&x_{g_2(k),k+1} = u_k w_k\,, \quad x'_{g_2(k),k+1} = z_k\,,\\
&x_{i,k+1} = x_{i+1,k}\,, \quad x'_{i,k+1} = x'_{i+1,k} \qquad \text{for all} \quad g_2(k) < i < m_k\,.
\end{aligned}
$$

If $g_1(k) = -1$ and $g_3(k) = 0$, then I_k stands for the variant of I_k just described, where w_k is the empty symbol, and where $s_k\ \eta_{5k+4}\ w_k$ and $y_k\ \eta_{5k+5,n}\ u_k$, whereas the unmeaningful $s_k\ \lambda_{5k+4,n}\ w_k$ and $y_k\ \lambda_{5k+5,n}\ w_k$ are to be ignored.

If $g_1(k) = 1$, $g_3(k) = 1$, then I_k stands for

$$
\begin{aligned}
& t_k x_{g_2(k),k} = y_k\, x'_{g_2(k),k}\,, \quad t_k u_k w_k = y_k z_k\,,\\
& y_k\, \eta_{5k+1,n}\, x_{g_2(k),k}\,, \quad y_k\, \rho_{5k+2,n}\, u_k\,, \quad y_k\, \lambda_{5k+2,n}\, w_k\\
& p_k u_k = q_k v_k\,, \quad a_k\, \eta_{5k+3,n}\, u_k\,,\\
& r_k v_k w_k = s_k z_k\,, \quad s_k\, \rho_{5k+4,n}\, v_k,\ s_k\, \lambda_{5k+4,n}\, w_k\,,\\
& x_{i,k+1} = x_{i,k}\,, \quad x'_{i,k+1} = x'_{i,k} \qquad \text{for all} \quad 1 \le i < g_2(k)\,,\\
& x_{g_2(k),k+1} = u_k\,, \quad x'_{g_2(k),k+1} = v_k\,,\\
& x_{g_2(k)+1,k+1} = v_k w_k\,, \quad x'_{g_2(k)+1,k+1} = z_k\,,\\
& x_{i,k+1} = x_{i-1,k}\,, \quad x'_{i,k+1} = x'_{i-1,k} \qquad \text{for all} \quad g_2(k)+1 < i \le m_k + 1\,.
\end{aligned}
$$

If $g_1(k) = 1$ and $g_3(k) = 0$, then I_k stands for the variant of I_k just described, with w_k the empty symbol. Here $y_k\, \eta_{5k+2,n}\, u_k$ and $s_k\, \eta_{5k+4,n}\, v_k$, whereas the unmeaningful $y_k\, \lambda_{5k+2,n}\, w_k$ and $s_k\, \eta_{5k+4,n}\, w_k$ are to be ignored.

Then $(I_{g,n})$ is the implication

$$
\left.\begin{array}{cc}
x x_{1,1} = x^2 x'_{1,1}\,, & x\, \eta_{0,n}\, x_{1,1},\\
I_1, \ldots, I_m\,, & \\
y\, x_{1,m+1} = y^2 x'_{1,m+1}\,, & y\, \eta_{5m+6,n}\, x_{1,m+1}
\end{array}\right\} \longrightarrow x = y\,. \tag{$I_{g,n}$}
$$

THEOREM 9. *A semigroup can be embedded in a completely simple semigroup if and only if it satisfies the implications (Q_n) and $(I_{g,n})$.*

Proof. We already proved that $\Lambda(\mathcal{CS}_s)$ contains the implications (Q_n). In order to show that $\Lambda(\mathcal{CS}) = \Lambda(\mathcal{CS}_s)$ contains $(I_{g,n})$, let S be a completely simple semigroup, and for any $t \in X$, we take $\bar{t} \in S$ such that the formal equalities in the left-hand side of $(I_{g,n})$ yield, after substitution, true equalities in S. We then must prove that $\bar{x} = \bar{y}$.

If $g_1(k) = -1$, then with the notation introduced above,

$$
\bar{p}_k\, \bar{x}_{g_2(k),k} = \bar{q}_k\, \bar{x}'_{g_2(k),k}\,, \qquad \bar{p}_k\, \bar{u}_k = \bar{q}_k\, \bar{v}_k\,,
$$
$$
\bar{q}_k\; \mathcal{H}\; \bar{x}_{g_2(k),k}\; \mathcal{H}\; \bar{u}_k\,,
$$

thus

$$
\bar{x}_{g_2(k),k}\, \bar{x}'^{\,-1}_{g_2(k),k} = \bar{p}_k^{-1}\, \bar{q}_k = \bar{u}_k\, \bar{v}_k^{-1}\,.
$$

Similarly,

$$
\bar{x}_{g_2(k)+1,k}\, \bar{x}'^{\,-1}_{g_2(k)+1,k} = \bar{r}_k^{-1}\, \bar{s}_k = \bar{v}_k \bar{w}_k \bar{z}_k^{-1}\,,
$$

where $\bar{w}_k = 1 \in S^1$ if w_k is the empty symbol, that is, if $g_3(k) = 0$. Therefore

$$\begin{aligned}\bar{x}_{g_2(k),k}\,\bar{x}'^{\,-1}_{g_2(k),k}\,\bar{x}_{g_2(k)+1,k}\,\bar{x}'^{\,-1}_{g_2(k)+1,k} &= \bar{u}_k\,\bar{v}_k^{-1}\,\bar{v}_k\bar{w}_k\,\bar{z}_k^{-1}\\ &= \bar{u}_k\bar{w}_k\bar{z}_k^{-1} \qquad (\text{since}\quad \bar{u}_k\ \mathcal{L}\ \bar{v}_k\ \text{ in } S)\\ &= \bar{x}_{g_2(k),k+1}\,\bar{x}'^{\,-1}_{g_2(k),k+1}\,,\end{aligned}$$

so that

$$\bar{x}_{1,k}\,\bar{x}'^{\,-1}_{1,k}\ldots\bar{x}_{m_k,k}\,\bar{x}'^{\,-1}_{m_k,k} = \bar{x}_{1,k+1}\,\bar{x}'^{\,-1}_{1,k+1}\ldots\bar{x}_{m_{k+1},k+1}\,\bar{x}'^{\,-1}_{m_{k+1},k+1}\,. \tag{16}$$

Similarly, if $g_1(k) = 1$, then (16) holds. Therefore (16) holds for all $1 \le k \le m$, and we find

$$\bar{x}_{1,1}\,\bar{x}'^{\,-1}_{1,1} = \bar{x}_{1,m+1}\,\bar{x}'^{\,-1}_{1,m+1}\,. \tag{17}$$

Since also

$$\bar{x}\bar{x}_{1,1} = \bar{x}^2\bar{x}'_{1,1} \qquad \text{and} \qquad \bar{x}\ \mathcal{H}\ \bar{x}_{1,1}\,,$$

we have that

$$\bar{x} = \bar{x}_{1,1}\,\bar{x}'^{\,-1}_{1,1}\,, \tag{18}$$

and similarly we find that

$$\bar{y} = \bar{x}_{1,m+1}\,\bar{x}'^{\,-1}_{1,m+1}\,. \tag{19}$$

Thus from (17), (18) and (19), we have that $\bar{x} = \bar{y}$, as required.

Conversely, assume that a semigroup S satisfies the implications (Q_n) and $(I_{g,n})$. By Theorem 7, with the notation adopted in the previous section, (C,γ) is a free completely simple semigroup on S. We must prove that γ is one-to-one. Assume that $\bar{x}, \bar{y} \in S$ such that $\bar{x}\gamma = \bar{y}\gamma$. That is, $(\bar{x}^2/\bar{x})\ \theta\ (\bar{y}^2/\bar{y})$. Since θ is generated by pairs of the form $\big((a/b)(bc/d),\,(ac/d)\big)$, where $a, b, d \in S$, $c \in S^1$ such that a/b, bc/d and ac/d exist, there exists a proof in T of the form

$$\begin{aligned}(\bar{x}^2/\bar{x}) = (\bar{x}_{1,1}/\bar{x}'_{1,1}) \longrightarrow \cdots \longrightarrow (\bar{x}_{1,k}/\bar{x}'_{1,k})\ldots(\bar{x}_{m_k,k}/\bar{x}'_{m_k,k})\\ \longrightarrow \cdots \longrightarrow (\bar{x}_{1,m+1}/\bar{x}'_{1,m+1}) = (\bar{y}^2/\bar{y})\ .\end{aligned}$$

where for each $1 \le k \le m$, there exists $1 \le g_2(k) \le m_k$ such that

$$(\bar{x}_{g_2(k),k}/\bar{x}'_{g_2(k),k}) = (\bar{u}_k/\bar{v}_k)\ , \qquad (\bar{x}_{g_2(k)+1,k}/\bar{x}'_{g_2(k)+1,k}) = (\bar{v}_k\bar{w}_k/\bar{z}_k)\ .$$

and

$$\bar{u}_k\bar{w}_k = \bar{x}_{g_2(k),k+1}\,, \qquad \bar{z}_k = \bar{x}_{g_2(k),k+1}\,,$$

while

$$\bar{x}_{i,k+1} = \bar{x}_{i,k}\,, \quad \bar{x}'_{i,k+1} = \bar{x}'_{i,k} \qquad \text{for all} \quad 1 \le i < g_2(k)\,,$$

and

$$\bar{x}_{i,k+1} = \bar{x}_{i+1,k}\,, \quad \bar{x}'_{i,k+1} = \bar{x}'_{i+1,k} \qquad \text{for all} \quad g_2(k) < i < m_k\,.$$

Or otherwise,

$$\begin{gathered}
\left(\bar{x}_{g_2(k),k}/\bar{x}'_{g_2(k),k}\right) = (\bar{u}_k\bar{w}_k/\bar{z}_k)\,,\\
\bar{u}_k = \bar{x}_{g_2(k),k+1}\,, \qquad \bar{v}_k = \bar{x}'_{g_2(k),k+1}\,,\\
\bar{v}_k\bar{w}_k = \bar{x}_{g_2(k)+1,k+1}\,, \qquad \bar{z}_k = \bar{x}'_{g_2(k)+1,k+1}\,,
\end{gathered}$$

while

$$\bar{x}_{i,k+1} = \bar{x}_{i,k}\,, \quad \bar{x}'_{i,k+1} = \bar{x}'_{i,k} \qquad \text{for all} \quad 1 \le i < g_2(k)\,,$$

and

$$\bar{x}_{i,k+1} = \bar{x}_{i-1,k}\,, \quad \bar{x}'_{i,k+1} = \bar{x}'_{i-1,k} \qquad \text{for all} \quad g_2(k)+1 < i \le m_k + 1\,.$$

In the former case, we define $g_1(k) = -1$ and $g_3(k) = 0$ if $\bar{w}_k = 1 \in S^1$, and $g_3(k) = 1$ if $\bar{w}_k \in S$. In the latter case we put $g_1(k) = 1$ and $g_3(k) = 0$ if $\bar{w}_k = 1 \in S^1$, and $g_3(k) = 1$ if $\bar{w}_k \in S$. It is now easy to see that $g = (g_1, g_2, g_3)$ satisfies the required conditions. From the above and the definition of the right quotients it follows that for a sufficiently large n there exists a substitution of the letters t of X by corresponding elements $\bar{t}$ of S such that after substitution the formal equalities in the left-hand side of $(I_{g,n})$ yield true equalities in S. Since S satisfies $(I_{g,n})$, we have $\bar{x} = \bar{y}$. We proved that $\bar{x}\gamma = \bar{y}\gamma$ implies $\bar{x} = \bar{y}$, and so γ embeds S into the completely simple semigroup C.

We next show that the result obtained in Theorem 9 yields a basis of implications for the quasivariety $\mathcal{G}_s$ consisting of all semigroups which are group embeddable.

We shall denote the identity element of the free monoid X^* by 1 and the equality relation in X^* by $\equiv$. For any finite subset A of X and $w \in X^*$, let w_A be the word which results from w by deleting all the occurrences of elements of A. In particular, $w_A \equiv 1$ if all the elements of X which occur in w belong to A. Given any set $E = \{p_1 = q_1, \ldots, p_k = q_k\}$ of formal equalities on the free semigroup X^+, let E_A be the set of formal equalities on X^+, where $v = w$ belongs to E_A if and only if one of the following cases occur:

(i) $v = w$ is of the form $(p_i)_A = (q_i)_A$,
where $1 \le i \le k$, $(p_i)_A \not\equiv 1 \not\equiv (q_i)_A$,

(ii) $v \equiv y \equiv w$ if for some $1 \le i \le k$ we have $(p_i)_A \equiv 1 \equiv (q_i)_A$,

(iii) $v = w$ is of the form $(p_i)_A\, y = y$, where $1 \le i \le k$, $(p_i)_A \not\equiv 1 \equiv (q_i)_A$,

(iv) $v = w$ is of the form $(q_i)_A\, y = y$, where $1 \le i \le k$, $(p_i)_A \equiv 1 \not\equiv (q_i)_A$, where y is some fixed variable of X. Given any set Λ of semigroup implications, we denote by Λ^* the set of semigroup implications defined by the following:

$$v_1 = w_1, \dots, v_\ell = w_\ell \longrightarrow v = w \tag{20}$$

belongs to Λ^* if and only if there exists an implication

$$p_1 = q_1, \dots, p_k = q_k \longrightarrow p = q \tag{21}$$

in Λ and a finite subset A of X such that

$$\{v_1 = w_1, \dots, v_\ell = w_\ell\} = \{p_1 = q_1, \dots , p_k = q_k\}_A \tag{22}$$

and

$$(v = w) \in \{p = q\}_A \,. \tag{23}$$

We remark that Λ is finite if and only if Λ^* is finite, and that $\Lambda \subseteq \Lambda^*$.

For any semigroup S let S^1 be the semigroup S with an extra identity element adjoined if S does not have an identity element, and $S = S^1$ otherwise.

THEOREM 10. *Let Λ be a set of semigroup implications, and S be a cancellative semigroup. Then S^1 satisfies the implications of Λ if and only if S satisfies the implications of Λ^*.*

P r o o f. Assume that S satisfies the implications of Λ^*. If $S = S^1$, then S^1 satisfies the implications of Λ, since $\Lambda \subseteq \Lambda^*$. We now assume that $S \neq S^1$. Let (21) be any implication of Λ and for any $t \in X$, let $\bar{t} \in S^1$ such that after substitution the formal equalities in the left-hand side of (21) yield true equalities in S. Let $A = \{t \in X \mid \bar{t} = 1\}$. Since S satisfies the implication (20), where (22) and (23) are satisfied, it follows that $v = w$ yields after substitution a true equality $\bar{v} = \bar{w}$ in S. Since S does not have an identity element, either $v = w$ is of the form $y = y$, in which case $p_A \equiv q_A$, and then $\bar{p} = \bar{q}$, or $v \equiv p_A$ and $w \equiv q_A$, and then again $\bar{p} = \bar{q}$ because $\bar{v} = \bar{w}$. Thus S^1 satisfies the implications of Λ.

Assume that S^1 satisfies the implications of Λ. Let (20) be any implication of Λ^*, derived from the implication (21) of Λ by (22) and (23) for some finite subset A of X, and for any variable $t \in X$ which occurs in (20), let $\bar{t} \in S$, so that after substitution the formal equalities in the left-hand side of (20) yield true equalities in S. For any $t \in A$, let $\bar{t} = 1 \in S^1$. Extending the previous assignment in this way, after substituting the formal equalities in the left-hand side of (21), then yields true equalities in S^1: here we use the fact that cancellative semigroups satisfy $xy = y \longrightarrow xz = z$ and $xy = y \longrightarrow zx = z$. Since S^1 satisfies the implication (21), the considered assignment yields a true equality $\bar{p} = \bar{q}$ in S^1, and thus also a true equality $\bar{v} = \bar{w}$ in S. Hence S satisfies the implications of Λ^*.

COROLLARY 11. *A semigroup is embeddable in a group if and only if it is cancellative and satisfies the implications of* Λ^* *, where* Λ *is the set of implications of Theorem 9.*

P r o o f . This is an immediate consequence of Theorems 9 and 10 since a semigroup S is embeddable in a group if and only if S^1 is cancellative and embeddable in a completely simple semigroup.

COROLLARY 12. *The set of implications* $\Lambda(\mathcal{C}S_s)$ *does not have a finite basis.*

P r o o f . If $\Lambda(\mathcal{C}S_s)$ has a finite basis Λ , then the implications of Λ^* together with the cancellative laws constitute a finite basis for $\Lambda(\mathcal{G}_s)$, where $\mathcal{G}_s$ is the class of all group embeddable semigroups. This, however, is impossible by the results of [12].

REFERENCES

[1] ANTONIPPILLAI, A.—PASTIJN, F.: *Subsemigroups of completely simple semigroups II*, Semigroup Forum **47** (1993), 126–129.

[2] BYLEEN, K.—PASTIJN, F.: *Implications for semigroups embeddable in orthocryptogroups*, Rocky Mountain J. Math. **17** (1987), 463–478.

[3] CLIFFORD, A. H.: *The free completely regular semigroup on a set*, J. Algebra **59** (1979), 434–451.

[4] CLIFFORD, A. H.—PRESTON, G. B.: *The Algebraic Theory of Semigroups. Vol. I, II. Math. Surveys Monographs 7*, Amer. Math. Soc., Providence, 1961, 1967.

[5] FOUNTAIN, J. B.—PETRICH, M.: *Completely* 0*-simple semigroups of quotients*, J. Algebra **101** (1986), 365–402.

[6] FOUNTAIN, J.—PETRICH, M.: *Completely* 0*-simple semigroups of quotients III*, Math. Proc. Cambridge Philos. Soc. **105** (1989), 263–275.

[7] GOULD, V.: *Left orders in regular* $\mathcal{H}$*-semigroups I*, J. Algebra **141** (1991), 11–35.

[8] GOULD, V.: *Left orders in regular* $\mathcal{H}$*-semigroups II*, Glasgow Math. J. **32** (1990), 95–108.

[9] LALLEMENT, G.—PETRICH, M.: *Décompositions I-matricielles d'un demi-groupe*, J. Math. Pures Appl. **45** (1966), 67–117.

[10] MALCEV, A. I.: *On the immersion of an algebraic ring into a field*, Math. Ann. **113** (1937), 686–691.

[11] MALCEV, A. I.: *On embedding of associative systems into groups* (Russian), Mat. Sb. (N.S.) **6** (1939), 331–336.

[12] MALCEV, A. I.: *On embedding of associative systems into groups* (Russian), Mat. Sb. (N.S.) **8** (1940), 251–264.

[13] MALCEV, A. I.: *Algebraic Systems*, Springer-Verlag, Berlin, 1973.

[14] PASTIJN, F.: *Idempotent-generated completely* 0*-simple semigroups*, Semigroup Forum **15** (1977), 41–50.

[15] PASTIJN, F.: *The biorder on the partial groupoid of idempotents of a semigroup*, J. Algebra **65** (1980), 147–187.

[16] PETRICH, M.—REILLY, N. R.: *Varieties of groups and completely simple semigroups*, Bull. Austral. Math. Soc. **23** (1981), 339–359.

[17] RASIN, V. V.: *Svobodnye vpolne prostye polugruppy.* Issledovanniya po Sovremennoĭ Algebre. Mat. Zapiski (Sverdlovsk) (1979), 140–151.

[18] SELMAN, A.: *Completeness of calculi for axiomatically defined classes of algebras.* Algebra Universalis **2** (1972), 20–32.

[19] TAYLOR, W.: *Equational logic.* In: Universal Algebra (G. Grätzer, ed.), Springer-Verlag, New York, 1979, pp. 378–400.

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