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Mathematica Slovaca, Vol. 44 (1994), No. 5, 569--574

Persistent URL: http://dml.cz/dmlcz/136629

Mathematica Slovaca
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Slovak Academy of Sciences

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APPROXIMATE COUNTING VIA EULER TRANSFORM

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(Communicated by Stanislav Jakubec)

ABSTRACT. In this short note we would like to emphasize how some elements of "q-analysis" (basic hypergeometric functions) allow some shortcuts in the enumerative part.

Approximate Counting might be described as follows. There is a counter C which is initially set to 1 and incremented randomly depending on the counter value. If this value is k, the probability that the counter will be increased by 1, is 2^{-k}. Otherwise, the counter value will stay the same. The idea is that after n random increments the counter should have a value close to $\log_2 n$. It is convenient to replace $\frac{1}{2}$ by q.

There is another useful way to imagine this procedure: There are the states $1, 2, \dots$, and in one step one may either advance from state i to state $i+1$ with probability q^i, or stay in the state i with probability $1 - q^i$. The interesting parameter is the state that one reaches after n random steps, starting in state 1. This is clearly the value of the counter C.

The original analysis was performed by Flajolet [2] and consists of an enumerative (or algebraic) part and an asymptotic (or analytic) part. The latter was done by the Mellin transform. In [5], some additional manipulations allowed to rewrite the sought quantities in such a way that an alternative asymptotic technique (Rice's method) could be used. See also the related papers [6], [7].

In this short note we would like to emphasize how some elements of "q-analysis" (basic hypergeometric functions) allow some shortcuts in the enumerative part. Let $H_l(x)$ be the generating function, where the coefficient of x^n is the probability that n random steps have led to state l. We find a rather explicit form for the bivariate generating function $\sum_{l\geq 1} H_l(x)y^l$, which is interesting in

AMS Subject Classification (1991): Primary 11B68.
Key words: Generating function, Euler transform.

itself and might lead to additional insight. We only use it here to find representations for the expectation and the second factorial moment as alternating sums, with binomial coefficients and some simple quantities, which is essential for the use of Rice's method. One could avoid to use such explicit formulae and, using the (new) generating functions, derive the asymptotics directly by the ingenious method in [4]. However, here, we only deal with the enumerative part.

We need a few concepts from q–analysis which are taken from [1].

q-Pochhammer symbol:

$$(a)_n := (1-a)(1-aq)\dots(1-aq^{n-1})\,, \qquad (a)_0 = 1\,, \qquad (a)_\infty = \lim_{n\to\infty}(a)_n\,.$$

Cauchy's formula:

$$\sum_{n\ge0}\frac{(a)_n t^n}{(q)_n} = \frac{(at)_\infty}{(t)_\infty}\,.$$

Heine's transformation:

$$\sum_{n\ge0}\frac{(a)_n(b)_n t^n}{(q)_n(c)_n} = \frac{(b)_\infty(at)_\infty}{(c)_\infty(t)_\infty}\sum_{n\ge0}\frac{(c/b)_n(t)_n b^n}{(q)_n(at)_n}\,.$$

Euler's transformation: If

$$f(x) = \sum_{n\ge0} a_n x^n\,,$$

then

$$\frac{1}{1-x}f\Big(\frac{x}{x-1}\Big) = \sum_{n\ge0}\Big(\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k\Big)x^n\,.$$

This is very easily computed directly and was used with great success in [3] and [4].

The generating function $H_l(x)$ was computed in [2]. Using a decomposition of a path from 1 to l into stages, it is not hard to see that

$$H_l(x) = \frac{x^{l-1}q^{\binom{l}{2}}}{\prod_{i=1}^{l}\big(1-x(1-q^i)\big)} = \frac{\frac{1}{x}\Big(\frac{x}{1-x}\Big)^l q^{\binom{l}{2}}}{\Big(\frac{xq}{x-1}\Big)_l}\,.$$

The coefficient of x^n in $\sum_{l\geq 1} H_l(x)$ must be 1 for all n, since each path of length n must simply lead somewhere. Let us see how we find the formula

$$\sum_{l\geq 1} H_l(x) = \frac{1}{1-x}$$

by some properties of "q-analysis". It is equivalent to showing

$$\sum_{l\geq 1} \frac{(-z)^l q^{\binom{l}{2}}}{(qz)_l} = -z\,,$$

with $z = \dfrac{x}{x-1}$. Let us show that

$$\sum_{l\geq 0} \frac{(-z)^l q^{\binom{l}{2}}}{(qz)_l} = 1 - z\,.$$

First, write

$$(-1)^l q^{\binom{l}{2}} = (0-1)(0-q)\dots(0-q^{l-1}) = \lim_{\varepsilon\to 0} \varepsilon^l (1/\varepsilon)_l\,.$$

Then we can use the transformation of Heine and compute

$$\sum_{l\geq 0} \frac{(1/\varepsilon)_l (\varepsilon z)^l}{(qz)_l} \cdot \frac{(q)_l}{(q)_l} = \frac{(q)_\infty (z)_\infty}{(qz)_\infty (\varepsilon z)_\infty} \sum_{n\geq 0} \frac{(z)_n (\varepsilon z)_n q^n}{(q)_n (z)_n}\,.$$

Now we can perform the limit $\varepsilon \to \infty$ on the right hand side without problems and obtain

$$\frac{(q)_\infty (z)_\infty}{(qz)_\infty} \sum_{n\geq 0} \frac{q^n}{(q)_n}\,.$$

We use the trivial fact $(z)_\infty = (1-z)(qz)_\infty$ for the first factor and Cauchy's identity (the special case which is attributed to Euler), which evaluates the sum to $1/(q)_\infty$, which gives us the desired $1-z$.

Now let us attack the generating function of the expectations,

$$\sum_{l\geq 1} l H_l(x)\,.$$

For that, we consider the bivariate generating function

$$H(x,y)=\sum_{l\geq 0} H_l(x)y^l\,,$$

differentiate it w.r.t. y and evaluate at $y=1$. We obtain

$$H(x,y)=\frac{1}{x}\sum_{l\geq 0}\frac{(-yz)^l q^{\binom{l}{2}}}{(qz)_l}\,,\qquad z=\frac{x}{x-1}\,.$$

Using Heine's transformation as before, we get

$$\sum_{l\geq 0}\frac{(-yz)^l q^{\binom{l}{2}}}{(qz)_l}=\frac{(q)_\infty (yz)_\infty}{(qz)_\infty}\sum_{n\geq 0}\frac{(z)_n q^n}{(q)_n (yz)_n}\,.$$

For $y=1$ we could evaluate the sum immediately. Now we transform the sum again by "Heine", with $a=0$, $b=z$, $c=yz$ and $t=q$ and find (for the sum only)

$$\frac{(z)_\infty}{(yz)_\infty (q)_\infty}\sum_{n\geq 0}\frac{(y)_n (q)_n z^n}{(q)_n}\,,$$

which gives

$$H(x,y)=\frac{1}{x}(1-z)\sum_{n\geq 0}(y)_n z^n\,,\qquad z=\frac{x}{x-1}\,.$$

Now observe that

$$\frac{d}{dy}(y)_n\bigg|_{y=1}=-(q)_{n-1}\,,\qquad n\geq 1\,,$$

so that the generating function of the expectations $E(x)$ is

$$E(x)=-\frac{1}{x}(1-z)\sum_{n\geq 0}(q)_n z^{n+1}=(1-z)^2\sum_{n\geq 0}(q)_n z^n\,.$$

According to Euler's transformation we have to extract the coefficient of z^n in

$$\frac{1}{1-z}\cdot(1-z)^2\sum_{n\geq 0}(q)_n z^n=(1-z)\sum_{n\geq 0}(q)_n z^n\,,$$

which is

$$(q)_n - (q)_{n-1} = -q^n (q)_{n-1}$$

for $n \geq 1$ and 1 for $n = 0$. Hence the expected value E_n is

$$E_n = 1 - \sum_{k=1}^{n} \binom{n}{k} (-1)^k q^k (q)_{k-1} \,,$$

a formula already reported in [5].

To obtain the generating function $E_2(x)$ for the second factorial moments we have to differentiate twice w.r.t. y and evaluate at $y = 1$. Observe that for $n \geq 2$

$$\frac{d^2}{dy^2} (y)_n^- \Big|_{y=1} = 2(q)_{n-1} \sum_{k=1}^{n-1} \frac{q^k}{1 - q^k} \,.$$

Let us abbreviate

$$T_n = \sum_{k=1}^{n} \frac{q^k}{1 - q^k} \,.$$

Then

$$E_2(x) = \frac{1}{x}(1 - z) \cdot 2 \sum_{n \geq 1} (q)_n T_n z^{n+1} \,.$$

As before, we have to extract the coefficient of z^n in

$$\frac{1}{1 - z} E_2(x) = -2(1 - z) \sum_{n \geq 1} (q)_n T_n z^n \,,$$

which is

$$-2\big((q)_n T_n - (q)_{n-1} T_{n-1}\big) = 2q^n (q)_{n-1} (T_{n-1} - 1) \,.$$

Hence the second factorial moment $E_n^{(2)}$ is given by

$$E_n^{(2)} = \sum_{k=1}^{n} \binom{n}{k} (-1)^k \cdot 2q^k (q)_{k-1} (T_{k-1} - 1) \,.$$

This is equivalent to the formula given in [5].

As stated before, asymptotics follow, as demonstrated in [5], by Rice's method. For the sake of completeness we cite the expectation E_n ('the average value of the counter C after n random increments') and refer for the variance to the literature.

$$E_n = \log_2 n + \frac{\gamma}{\log 2} + \frac{1}{2} - \alpha + \delta(\log_2 n) + \mathcal{O}\Big(\frac{1}{n}\Big) \,,$$

where $\gamma = 0.577\ldots$ is Euler's constant,

$$\alpha = \sum_{k\geq 1} \frac{1}{2^k - 1} = 1.606695\ldots$$

and $\delta(x)$ is a periodic function of period 1, mean 0, small amplitude and known Fourier expansion.

We would also like to mention that, since we have the explicit forms of the generating functions of the moments, an approach like the one in [4] would give the asymptotics even a little bit more directly. We do not work it out, because the computations are somehow similar to those which occur with Rice's method.

Acknowledgement

I thank Christian Krattenthaler for his help with q-hypergeometric functions.

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Received November 25, 1994

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