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ON A MEASURABLE SET

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ABSTRACT. In this paper, a result in connection with a set of positive Lebesgue measure in N-dimensional Euclidean space is obtained.

In [2], S. Kurepa proved two theorems, the first of which runs as follows.

THEOREM. *Let $A \subseteq \mathbb{R}_N$ (N-dimensional Euclidean space) be a set of strictly positive measure. For any system of p real numbers $\alpha_1, \alpha_2, \dots, \alpha_p$ ($\alpha_k \neq 0$) there exists a ball K_r of radius r with centre at the origin, such that for any $x \in K_r$ there are vectors $a_0(x), a_1(x), \dots, a_p(x)$ in A such that*

$$x = \frac{a_1(x) - a_0(x)}{\alpha_1} = \frac{a_2(x) - a_0(x)}{\alpha_2} = \dots = \frac{a_p(x) - a_0(x)}{\alpha_p} .$$

The proof of the theorem as adopted by S. Kurepa in [2] is lengthy and involves characteristic functions. In [4], K. C. Ray has shown that both theorems of [2] admit of shorter proofs.

In [3], M. Pal proved a theorem using a bounded sequence of non-zero real numbers instead of a system of a finite number of non-zero real numbers as used by S. Kurepa in [2].

In this paper, we prove a theorem which sharpens the result as proved by M. Pal in [3].

We prove the theorem using the technique as adopted by K. C. Ray in [4] with necessary modifications.

Before going into details we state some of the properties of a convex set in $\mathbb{R}_N$.

I) For a convex set A containing the origin and for $0 < \alpha \leqq 1$, one has $\alpha A \subset A$, where

$$\alpha A = \{\alpha x : \ x \in A\} .$$

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II) For a convex set A containing the origin and for positive numbers α, β, with $\alpha < \beta$, one has $\alpha A \subset \beta A$.

III) For convex sets A and B with $B \subset A$ and for any positive number α, one has $\alpha B \subset \alpha A$.

In this context, we note a well-known result [5], viz., if T is a linear transformation in $\mathbb{R}_N$ given by

$$x'_i = \sum_{j=1}^{N} a_{ij} x_j \,, \qquad i = 1, 2, \ldots, N \,,$$

a_{ij} being real numbers, and if E is a measurable set in $\mathbb{R}_N$, then $|T(E)| = \delta |E|$, where δ is the absolute value of the determinant of T, and $|E|$ denotes the Lebesgue measure of the set E.

As a corollary of this result, it can be easily deduced that, if α is a non-zero real number and E is a measure set in $\mathbb{R}_N$, then

$$|\alpha E| = |\alpha|^N |E| \,,$$

where $|\alpha|$ denotes the absolute value of the real number α.

NOTATION.

(1) $B[c, \varrho]$ stands for the closed ball with the centre at c and the radius ϱ.

(2) If A and B are two sets, then $A \setminus B$ is the set of points of A which are not in B.

(3) For a set A and a vector a in $\mathbb{R}_N$, $A - a$ denotes the set of vectors $x - a$, where x runs over the set A.

THEOREM. *Let A be a closed convex set of positive Lebesgue measure in $\mathbb{R}_N$. Also let $\{a_n\}$ be a bounded sequence of non-zero real numbers, and let $\{\beta_n\}$ be an increasing sequence of real numbers with $\beta_n \leqq 1$ such that $\lim_{n\to\infty} \beta_n = 1$. Then there exist a sequence $\{K_n\}$ of balls with centres at the origin, a ball $K_A = B[a, r]$ with $a \in A$, and a sub-sequence $\{\beta_{N_i}\}$ of the sequence $\{\beta_n\}$ so that for every sequence $\{x_n\}$ of vectors with $x_n \in K_n$ there exist vectors*

$$a_0(x_1, x_2, \ldots; \beta_{N_1}, \beta_{N_2}, \ldots) \in A \,,$$
$$a_k(x_1, x_2, \ldots; \beta_{N_1}, \beta_{N_2}, \ldots) \in A \,,$$
$$a_k^k(x_1, x_2, \ldots; \beta_{N_1}, \beta_{N_2}, \ldots) \in K_A$$

such that

$$x_k = \frac{\frac{1}{\beta_{N_k}} a_k(x_1, x_2, \ldots; \beta_{N_1}, \beta_{N_2}, \ldots) - a_0(x_1, x_2, \ldots; \beta_{N_1}, \beta_{N_2}, \ldots)}{\alpha_k}$$

and

$$a_0(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots) - a = \frac{1}{\beta_{N_k}} a_k^k(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots) - \frac{1}{\beta_{N_k}} a\,,$$
$$k = 1, 2, \dots .$$

P r o o f. Since A is a set of positive measure, there exists a ball $B[a,r] = K_A$, $a \in A$, $a \neq 0$, such that

$$|A \cap K_A| > \frac{3}{4}|K_A|\,.$$

Let $C = B \cap K'_A$, where $B = A - a$ and $K'_A = K_A - a$. Then C is a bounded closed convex set of positive measure containing the origin. Following K e s t e l m a n , we similarly define a sequence of bounded open convex sets such that

$$U_1 \supset U_2 \supset U_3 \supset \dots \qquad \text{with} \quad \bigcap_{n=1}^{\infty} U_n = C\,,$$

and $\sum_{n=1}^{\infty} \{|U_n| - |C|\} < \frac{|C|\nu^N}{4}$, where ν is a positive number less than 1 such that $\nu^N < \frac{4}{5}$.

Since $\{\alpha_n\}$ is a bounded sequence of real numbers, there exists a positive number α such that $|\alpha_n| < \alpha$, $n = 1, 2, 3, \dots$. Let δ_r be the distance between the sets U'_r and C, U'_r being the complement of U_r $(r = 1, 2, 3, \dots)$. Then $\{\delta_r\}$ is a null sequence of positive numbers.

Since $\lim_{n\to\infty} \beta_n = 1$, there exists a sequence $\{N_i\}$ of positive integers with $N_1 < N_2 < \dots < N_n < \dots$ such that corresponding to N_r we have

$$|\beta_{N_r} - 1| < \min\left(1 - \nu,\ \frac{\delta_r}{2|a|}\right).$$

We shall show that

$$C - \alpha_i \beta_{N_i} x_i + a - \beta_{N_i} a \subset U_i\,, \qquad i = 1, 2, \dots .$$

Here

$$x_i \in K_i = B\left[0,\ \frac{\delta_i}{2\alpha}\right].$$

Now

$$|\alpha_i \beta_{N_i} x_i - a + \beta_{N_i} a| \leqq |\alpha_i||\beta_{N_i}||x_i| + |a|(1 - \beta_{N_i})$$
$$< \alpha \frac{\delta_i}{2\alpha} + |a| \frac{\delta_i}{2|a|} = \frac{\delta_i}{2} + \frac{\delta_i}{2} = \delta_i\,.$$

Since δ_i is the distance between C and U_i',

$$\delta_i \leqq |x - y| \qquad \text{for every } x \in C \text{ and every } y \in U_i' .$$

Let $x \in C$ and $\overline{y}_i = \alpha_i \beta_{N_i} x_i - a + \beta_{N_i} a$. Then $x - \overline{y}_i \in U_i$ or $x - \overline{y}_i \in U_i'$. Let $z_i = x - \overline{y}_i \in U_i'$. Then $|x - z_i| = |\overline{y}_i| < \delta_i$. This leads to a contradiction. So $x - \overline{y}_i \notin U_i'$, and hence $x - \overline{y}_i \in U_i$. Therefore

$$C - \alpha_i \beta_{N_i} x_i + a - \beta_{N_i} a \subset U_i , \qquad i = 1, 2, \dots .$$

Next we show that $\{K_n\}$ is a sequence of balls such that for every sequence $\{x_n\}$ of vectors with $x_n \in K_n$,

$$\begin{aligned} &X(x_1, x_2, \dots ; \beta_{N_1}, \beta_{N_2}, \dots) \\ &= (B \cap K_A') \cap \Big[\Big(\frac{1}{\beta_{N_1}} C - \alpha_1 x_1 + \frac{1}{\beta_{N_1}} a - a\Big) \cap \frac{1}{\beta_{N_1}} K_A'\Big] \\ &\qquad \cap \Big[\Big(\frac{1}{\beta_{N_2}} C - \alpha_2 x_2 + \frac{1}{\beta_{N_2}} a - a\Big) \cap \frac{1}{\beta_{N_2}} K_A'\Big] \cap \dots \end{aligned}$$

is a set of positive measure.

Now

$$\begin{aligned} &X(x_1, x_2, \dots ; \beta_{N_1}, \beta_{N_2}, \dots) \cap U_1 \\ &= (B \cap K_A' \cap U_1) \cap \Big[\Big(\frac{1}{\beta_{N_1}} C - \alpha_1 x_1 + \frac{1}{\beta_{N_1}} a - a\Big) \cap \frac{1}{\beta_{N_1}} K_A'\Big] \\ &\qquad \cap \Big[\Big(\frac{1}{\beta_{N_2}} C - \alpha_2 x_2 + \frac{1}{\beta_{N_2}} a - a\Big) \cap \frac{1}{\beta_{N_2}} K_A'\Big] \cap \dots \\ &= (B \cap K_A' \cap U_1) \cap \Big[\frac{1}{\beta_{N_1}} \{(C - \alpha_1 \beta_{N_1} x_1 + a - \beta_{N_1} a) \cap K_A'\}\Big] \\ &\qquad \cap \Big[\frac{1}{\beta_{N_2}} \{(C - \alpha_2 \beta_{N_2} x_2 + a - \beta_{N_2} a) \cap K_A'\}\Big] \cap \dots . \end{aligned}$$

It is evident that $\dfrac{1}{\beta_{N_i}} < \dfrac{1}{\nu}$ and $\dfrac{1}{\nu} > 1$, and, on account of (II) and (III),

$$(B \cap K_A' \cap U_1) \subset (K_A' \cap U_1) \subset \frac{1}{\nu} (K_A' \cap U_1)$$

and

$$\frac{1}{\beta_{N_i}} [(C - \alpha_i \beta_{N_i} x_i + a - \beta_{N_i} a) \cap K_A'] \subset \frac{1}{\beta_{N_i}} (U_1 \cap K_A') \subset \frac{1}{\nu} (U_1 \cap K_A') .$$

So,

$$
\begin{aligned}
&X(x_1, x_2, \ldots; \beta_{N_1}, \beta_{N_2}, \ldots) \cap U_1 \\
&= \frac{1}{\nu}(K'_A \cap U_1) \setminus \left[\bigcup_{i=1}^{\infty} \left\{ \frac{1}{\nu}(K'_A \cap U_1) \setminus \frac{1}{\beta_{N_i}} \big((C - \alpha_i \beta_{N_i} x_i + a - \beta_{N_i} a) \cap K'_A \big) \right\} \right. \\
&\qquad \left. \cup \left\{ \frac{1}{\nu}(K'_A \cap U_1) \setminus (B \cap K'_A \cap U_1) \right\} \right] \\
&= \frac{1}{\nu}(K'_A \cap U_1) \setminus \left[\bigcup_{i=1}^{\infty} \left[\left\{ \frac{1}{\nu}(K'_A \cap U_1) \setminus \frac{1}{\beta_{N_i}}(K'_A \cap U_i) \right\} \right. \right. \\
&\qquad \left. \cup \left\{ \frac{1}{\beta_{N_i}}(K'_A \cap U_1) \setminus \frac{1}{\beta_{N_i}} \big(K'_A \cap (C - \alpha_i \beta_{N_i} x_i + a - \beta_{N_i} a) \big) \right\} \right] \\
&\qquad \left. \cup \left\{ \frac{1}{\nu}(K'_A \cap U_1) \setminus (B \cap K'_A) \right\} \right] \\
&= \left[\frac{1}{\nu}(K'_A \cap U_1) \setminus \bigcup_{i=1}^{\infty} \left\{ \frac{1}{\nu}(K'_A \cap U_1) \setminus \frac{1}{\beta_{N_i}}(K'_A \cap U_i) \right\} \right] \\
&\qquad \setminus \left[\bigcup_{i=1}^{\infty} \left\{ \frac{1}{\beta_{N_i}}(K'_A \cap U_i) \setminus \frac{1}{\beta_{N_i}} \big(K'_A \cap (C - \alpha_i \beta_{N_i} x_i + a - \beta_{N_i} a) \big) \right\} \right. \\
&\qquad \left. \cup \left\{ \frac{1}{\nu}(K'_A \cap U_1) \setminus (B \cap K'_A) \right\} \right] \\
&= \bigcap_{i=1}^{\infty} \frac{1}{\beta_{N_i}}(K'_A \cap U_i) \setminus \left[\bigcup_{i=1}^{\infty} \left\{ \frac{1}{\beta_{N_i}}(K'_A \cap U_i) \right. \right. \\
&\qquad \left. \setminus \frac{1}{\beta_{N_i}} \big(K'_A \cap (C - \alpha_i \beta_{N_i} x_i + a - \beta_{N_i} a) \big) \right\} \\
&\qquad \left. \cup \left\{ \frac{1}{\nu}(K'_A \cap U_1) \setminus (B \cap K'_A) \right\} \right] \\
&\supset \bigcap_{i=1}^{\infty} (K'_A \cap U_i) \setminus \left[\left\{ \bigcup_{i=1}^{\infty} \frac{1}{\beta_{N_i}} \Big[K'_A \cap \big(U_i \setminus (C - \alpha_i \beta_{N_i} x_i + a - \beta_{N_i} a) \big) \Big] \right\} \right. \\
&\qquad \left. \cup \left\{ \frac{1}{\nu}(K'_A \cap U_1) \setminus (B \cap K'_A) \right\} \right]
\end{aligned}
$$

$$\supset C \setminus \left[\left\{ \bigcup_{i=1}^{\infty} \frac{1}{\beta_{N_i}} [U_i \setminus (C - \alpha_i \beta_{N_i} x_i + a - \beta_{N_i} a)] \right\} \right.$$

$$\left. \cup \left\{ \frac{1}{\nu} (K'_A \cap U_1) \setminus (B \cap K'_A) \right\} \right] .$$

So,

$$\begin{aligned}
&|X(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots)| \\
&\geqq |C| - \sum_{i=1}^{\infty} \frac{1}{\beta_{N_i}^N} \left[|U_i| - |C - \beta_{N_i} \alpha_i x_i + a - \beta_{N_i} a| \right] \\
&\qquad - \left[\frac{1}{\nu^N} |K'_A \cap U_1| - |C| \right] \\
&= |C| - \sum_{i=1}^{\infty} \frac{1}{\beta_{N_i}^N} \left[|U_i| - |C| \right] - \left[\frac{1}{\nu^N} |K'_A \cap U_1| - |C| \right] \\
&\geqq |C| - \frac{1}{\nu^N} \sum_{i=1}^{\infty} \left[|U_i| - |C| \right] - \frac{1}{\nu^N} |U_1| + |C| \\
&= |C| - \frac{1}{\nu^N} \sum_{i=1}^{\infty} \left[|U_i| - |C| \right] - \frac{1}{\nu^N} \left[|U_1| - |C| \right] + \left(1 - \frac{1}{\nu^N} \right) |C| \\
&= |C| - \frac{1}{\nu^N} \sum_{i=1}^{\infty} \left[|U_i| - |C| \right] - \frac{1}{\nu^N} \left[|U_1| - |C| \right] - \left(\frac{1}{\nu^N} - 1 \right) |C| \\
&> |C| - \frac{|C|}{4} - \frac{|C|}{4} - \frac{|C|}{4} = \frac{|C|}{4} > 0 .
\end{aligned}$$

Consequently, the set $X(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots)$ is a set of positive measure for every sequence $\{x_n\}$ of vectors with $x_n \in K_n$.

Hence there are vectors

$$\begin{aligned}
&a_0(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots) \in A , \\
&a_k(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots) \in A , \\
&a_k^k(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots) \in K_A , \qquad k = 1, 2, \dots
\end{aligned}$$

such that

$$\begin{aligned}
&a_0(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots) - a \\
&= \frac{1}{\beta_{N_1}} \left[a_1(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots) - a\right] - \alpha_1 x_1 + \frac{1}{\beta_{N_1}} a - a \\
&= \frac{1}{\beta_{N_2}} \left[a_2(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots) - a\right] - \alpha_2 x_2 + \frac{1}{\beta_{N_2}} a - a \\
&= \dots \\
&= \frac{1}{\beta_{N_1}} a_1^1(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots) - \frac{1}{\beta_{N_1}} a \\
&= \frac{1}{\beta_{N_2}} a_2^2(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots) - \frac{1}{\beta_{N_2}} a \\
&= \dots,
\end{aligned}$$

i.e.

$$x_k = \frac{\frac{1}{\beta_{N_k}} a_k(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots) - a_0(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots)}{\alpha_k}$$

and

$$a_0(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots) - a = \frac{1}{\beta_{N_k}} a_k^k(x_1, x_2, \dots; \beta_{N_1}, \beta_{N_2}, \dots) - \frac{1}{\beta_{N_k}} a, \quad k = 1, 2, \dots .$$

This completes the proof. □

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