Jozef Džurina

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OSCILLATION OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENT

JOZEF DŽURINA

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ABSTRACT. Our aim in this paper is to present a sufficient condition for the oscillation of the second order differential equation with advanced argument

$$\left(\frac{1}{r(t)}u'(t)\right)' + p(t)u(t) + q(t)u(\tau(t)) = 0 \tag{$*$}$$

by comparing $(*)$ with the first order advanced equation of the form

$$z'(t) - q_2(t)z(\tau(t)) = 0 .$$

We consider the second order functional differential equation with advanced argument

$$\left(\frac{1}{r(t)}u'(t)\right)' + p(t)u(t) + q(t)u(\tau(t)) = 0 , \tag{1}$$

where $r, p, q, \tau \in C([t_0, \infty))$, $r(t)$ and $q(t)$ are positive, $p(t)$ is nonnegative and $\tau(t) \geqslant t$.

Let us denote

$$L_0u(t) = u(t) , \qquad L_1u(t) = \frac{1}{r(t)}u'(t) , \qquad L_2u(t) = (L_1u(t))' .$$

Then equation (1) can be rewritten as

$$L_2u(t) + p(t)u(t) + q(t)u(\tau(t)) = 0 .$$

For convenience and further references, we introduce the notation

$$R(t) = \int_{t_0}^{t} r(s) \, ds , \qquad t \geqslant t_0 .$$

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We always assume that $R(t) \to \infty$ as $t \to \infty$.

In the sequel, we shall restrict our attention to nontrivial solutions of the equations considered. Such a solution is called oscillatory if the set of its zeros is unbounded. Otherwise, it is said to be nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory.

In this paper, we have been motivated by the observation that there are very few effective criteria for equation (1) involving advanced argument ($\tau(t) \geqslant t$) to be oscillatory. For some typical results on the subject we refer to the papers [4], and [5]. On the other hand, differential equations with advanced argument can be used to discuss properties of ordinary equations (without deviating argument) as we can see in [2].

THEOREM 1. *Let us define for all $t \geqslant t_1$ ($\geqslant t_0$)*

$$p_1(t) = p(t)\big(R(t) - R(t_1)\big)\,,$$
$$q_1(t) = q(t)\big(R(t) - R(t_1)\big)\,.$$

If the differential inequality

$$y'(t)\operatorname{sgn} y(t) - p_1(t)|y(t)| - q_1(t)\big|y\big(\tau(t)\big)\big| \geqslant 0 \tag{2}$$

is oscillatory, then so is equation (1).

P r o o f . For the sake of contradiction, let us suppose that $u(t)$ is a positive solution of (1) on $[t_0, \infty)$. Since $L_2u(t) < 0$, then according to a lemma of Kiguradze ([3]), there exists some $t_1 \geqslant t_0$, such that $L_1u(t) > 0$ for all $t \geqslant t_1$. An integration of (1) from t to ∞ leads to

$$L_1u(t) \geqslant \int_t^\infty \big[p(s)u(s) + q(s)u\big(\tau(s)\big)\big]\,\mathrm{d}s\,, \qquad t \geqslant t_1\,.$$

Now, integrating the last inequality from t to t_1 one obtains

$$\begin{aligned} u(t) &\geqslant \int_{t_1}^t r(x)\int_x^\infty \big[p(s)u(s) + q(s)u\big(\tau(s)\big)\big]\,\mathrm{d}s\,\mathrm{d}x\,,\\ &\geqslant \int_{t_1}^t \big[p(s)u(s) + q(s)u\big(\tau(s)\big)\big]\int_{t_1}^s r(x)\,\mathrm{d}x\,\mathrm{d}s\,, \qquad t \geqslant t_1\,. \end{aligned}$$

Hence

$$u(t) \geqslant \int_{t_1}^t \big[p_1(s)u(s) + q_1(s)u\big(\tau(s)\big)\big]\,\mathrm{d}s\,, \qquad t \geqslant t_1\,. \tag{3}$$

Let us denote the right hand side of (3) by $y(t)$. Then $y(t) > 0$, $t \geqslant t_1$ and

$$y'(t) - p_1(t)u(t) - q_1(t)u\big(\tau(t)\big) = 0\,, \qquad t \geqslant t_1\,.$$

Since $u(t) \geqslant y(t)$ and $u\big(\tau(t)\big) \geqslant y\big(\tau(t)\big)$, then $y(t)$ is a positive solution of the differential inequality

$$y'(t) - p_1(t)y(t) - q_1(t)y\big(\tau(t)\big) \geqslant 0\,, \qquad t \geqslant t_1\,,$$

which contradicts the hypotheses. □

Now, we transform differential inequality (2) into a simple differential inequality. Let

$$y(t) = \exp\left(\int_{t_1}^{t} p_1(x)\,\mathrm{d}x\right) z(t)\,, \qquad t \geqslant t_1\,. \tag{4}$$

Then (2) becomes

$$z'(t)\operatorname{sgn} z(t) - q_1(t)\exp\left(\int_{t}^{\tau(t)} p_1(x)\,\mathrm{d}x\right)\big|z\big(\tau(t)\big)\big| \geqslant 0 \qquad \text{for} \quad t \geqslant t_1\,. \tag{5}$$

Setting

$$q_2(t) = q_1(t)\exp\left(\int_{t}^{\tau(t)} p_1(x)\,\mathrm{d}x\right),$$

we find that (5) becomes

$$z'(t)\operatorname{sgn} z(t) - q_2(t)\big|z\big(\tau(t)\big)\big| \geqslant 0\,. \tag{6}$$

We see that transformation (4) preserves oscillation. Therefore, we can apply results holding for (6) also to (2). For example, we have the following theorem.

THEOREM 2. *Let $\tau(t) > t$. Define a function $f = f(\lambda)$ for $0 \leqslant \lambda \leqslant 1/\mathrm{e}$ by*

$$f\,\mathrm{e}^{-\lambda f} = 1\,, \qquad 1 \leqslant f \leqslant \mathrm{e}\,.$$

Assume that either

$$d = \liminf_{t\to\infty} \int_{t}^{\tau(t)} q_2(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}\,,$$

or

$$c = \limsup_{t\to\infty} \int_{t}^{\tau(t)} q_2(s)\,\mathrm{d}s > 1\,,$$

or, when $0 < d \leqslant 1/\mathrm{e}$ *and* $c \leqslant 1$, *the following condition hold:*

$$f(d)\big(1-\sqrt{1-c}\,\big)^2 > 1 .$$

Then equation (1) *is oscillatory.*

P r o o f. By [1; Corollary 4'] or by [6; Theorems 2.4.1 and 2.4.3] differential inequality (6) is oscillatory (and as well (2)). Our assertion follows from Theorem 1. □

The following illustrative example serves to compare our results with those in [4].

E x a m p l e 1. Consider the differential equation

$$y''(t) + \frac{1}{4t^2} y(\lambda t) = 0 , \qquad t > 1 . \tag{7}$$

It is easy to verify that (7) is not oscillatory for $\lambda = 1$. Note that for (7), $p_1(t) = 0$ and $q_2(t) = q_1(t) = \frac{1}{4t^2}(t - t_1)$. By Theorem 2, equation (7) is oscillatory if

$$\lambda > \mathrm{e}^{\frac{4}{\mathrm{e}}} \approx 4.356 .$$

On the other hand, by [4; Corollary 1], equation (5) is oscillatory if $\lambda\,\mathrm{e}^{\frac{1}{\lambda}} > \mathrm{e}^5 \approx 148.4$, and with respect to [4; Corollary 2], equation (5) is oscillatory if $\lambda > \mathrm{e}^4 \approx 54.6$. Note that [5; Theorem 3] cannot be applied to (7).

LEMMA 1. *Differential inequality* (6) *is oscillatory if and only if the differential equation*

$$z'(t) - q_2(t) z\big(\tau(t)\big) = 0 \tag{8}$$

is oscillatory.

P r o o f. It is sufficient to prove that, if (6) has a positive solution, so does (8). To prove this, we can use the same arguments as in [7; Lemma 2]. □

COROLLARY 1. *If equation* (8) *is oscillatory, then equation* (1) *is oscillatory.*

We can easily extend our previous results to more general nonlinear differential equation of the form

$$\left(\frac{1}{r(t)} u'(t)\right)' + f\big(t, u(t), u(\tau(t))\big) = 0 , \tag{9}$$

where the functions r and τ are the same as in (1), $f\colon [t_0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuos, and $yf(t,x,y) \geqslant 0$ for $xy > 0$.

THEOREM 3. *Let* p_1 *and* q_1 *are defined as in Theorem* 1. *Let*

$$|f(t,x,y)| \geqslant p(t)|x| + q(t)|y|\,, \qquad xy > 0\,. \tag{10}$$

If differential inequality (2) *is oscillatory, then so is* (9).

Proof. For the sake of contradiction, we assume that $u(t)$ is a positive solution of (9). Since $L_2u(t) < 0$, then, by a generalization of a lemma of Kiguradze ([8]), $L_1u(t) > 0$ for all large t, say $t \geqslant t_1$. An integration of (9) yields with help of (10)

$$\begin{aligned} L_1u(t) &\geqslant \int_t^\infty f\bigl(s, u(s), u(\tau(s))\bigr)\,\mathrm{d}s \\ &\geqslant \int_t^\infty \bigl[p(s)u(s) + q(s)u(\tau(s))\bigr]\,\mathrm{d}s\,, \qquad t \geqslant t_1\,. \end{aligned}$$

Then arguing exactly as in the proof of Theorem 1 we get that differential inequality (2) has a positive solution, which contradicts the hypotheses.

By Theorem 3 and transformation (6), we have the following corollary:

COROLLARY 2. *Let* (10) *hold. If equation* (8) *is oscillatory, then equation* (9) *is oscillatory.*

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Department of Mathematical Analysis
Faculty of Sciences
Šafárik University
Jesenná 5
SK-041 54 Košice
SLOVAKIA