Lubomír Kubáček
Linear statistical models with constraints revisited

Mathematica Slovaca, Vol. 45 (1995), No. 3, 287--307

Persistent URL: http://dml.cz/dmlcz/136650

Math. Slovaca, **45** (1995), No. 3, 287–307

LINEAR STATISTICAL MODELS WITH CONSTRAINTS REVISITED

LUBOMÍR KUBÁČEK[1]

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. In the practice, two classes of basic linear models with constraints occur, i.e., models with observations of a complete first order vector parameter and models with observations of an incomplete first order vector parameter. In the former class, the locally best linear unbiased estimators are known explicitly. In the latter class, they are given by algorithms.

In the paper, explicit formulae for these estimators are given, the effects of constraints on the estimators in the form of suitable projection operators correcting estimators non respecting these constraints are investigated in both of the classes.

Introduction

Let $\boldsymbol{Y}$ be an n-dimensional random vector characterized by the class of distribution functions $\{F(\cdot\,,\boldsymbol{\beta}) : \boldsymbol{\beta} \in \mathcal{V}\}$ possessing the properties

$$E(\boldsymbol{Y} \mid \boldsymbol{\beta}) = \int_{\mathbb{R}^n} \boldsymbol{u}\, \mathrm{d}F(\boldsymbol{u},\boldsymbol{\beta}) = \mathbf{X}\boldsymbol{\beta}\,, \qquad \boldsymbol{\beta} \in \mathcal{V}\,,$$

(the vector parameter $\boldsymbol{\beta}$ is called the first order parameter since it occurs in the first statistical moments) and

$$\mathrm{Var}(\boldsymbol{Y} \mid \boldsymbol{\beta}) = \int_{\mathbb{R}^n} (\boldsymbol{u} - \mathbf{X}\boldsymbol{\beta})(\boldsymbol{u} - \mathbf{X}\boldsymbol{\beta})'\, \mathrm{d}F(\boldsymbol{u},\boldsymbol{\beta}) = \boldsymbol{\Sigma}$$

(independence of the covariance matrix of the first order vector parameter $\boldsymbol{\beta}$); in the following, the matrix $\boldsymbol{\Sigma}$ is assumed to be known.

Here $\mathbb{R}^n$ is an n-dimensional Euclidean space and $\mathbf{X}$ is a known $n \times k$ matrix and $\mathcal{V}$ is a linear manifold in $\mathbb{R}^k$.

AMS Subject Classification (1991): Primary 62J05. Secondary 62F10.

Key words: linear model with constraints, singular linear model.

[1] Supported by the grant of the Slovak Academy of Sciences No. 366.

The described model is recorded as $(\boldsymbol{Y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$, $\boldsymbol{\beta} \in \mathcal{V}$. If $\mathcal{V} \neq \mathbb{R}^k$, then it is a model with constraints. The locally best linear unbiased estimator (LBLUE) $\hat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$ in the case $\mathcal{V} = \mathbb{R}^k$ (without constraints) differs from the LBLUE $\hat{\hat{\boldsymbol{\beta}}}$ of $\boldsymbol{\beta}$ in the case $\mathcal{V} \subset \mathbb{R}^k$, $\mathcal{V} \neq \mathbb{R}^k$.

The aim of the paper is to investigate the effect of the constraints on $\hat{\boldsymbol{\beta}}$ and to characterize it geometrically, i.e., by projection matrices.

1. Notation and auxiliary statements

Let $\mathbf{A}$ be a $k \times n$ matrix; $\mathcal{M}(\mathbf{A}) = \{\mathbf{A}\boldsymbol{u} : \boldsymbol{u} \in \mathbb{R}^n\}$ is the column space of the matrix $\mathbf{A}$ and $\mathcal{K}er(\mathbf{A}) = \{\boldsymbol{u} : \mathbf{A}\boldsymbol{u} = \boldsymbol{O}\}$ its null space. The norm of a vector $\boldsymbol{y} \in \mathbb{R}^n$ in an n-dimensional Hilbert space $(\mathbb{R}^n, \|\cdot\|_{\mathbf{W}})$, where $\mathbf{W}$ is an $n \times n$ positive definite (p.d.) matrix, is defined by the relation $\|\boldsymbol{y}\|_{\mathbf{W}} = \sqrt{\boldsymbol{y}'\mathbf{W}\boldsymbol{y}}$ ($'$ means the transposition). If the norm in $\mathbb{R}^n$ is not specified, then $\mathbf{W} = \mathbf{I}$ (identity matrix). $\mathbf{P}_{\mathbf{B}}^{\mathbf{W}}$ denotes the projection matrix of the space $(\mathbb{R}^n, \|\cdot\|_{\mathbf{W}})$ on the subspace $\mathcal{M}(\mathbf{B}_{n,s})$; further $\mathbf{M}_{\mathbf{B}}^{\mathbf{W}} = \mathbf{I} - \mathbf{P}_{\mathbf{B}}^{\mathbf{W}}$.

The symbol $\mathbf{P}_{\mathbf{B}}^{\mathbf{W}}$ is used also in the case when $\mathbf{W}$ is only positive semidefinite (p.s.d.), when simultaneously $\mathcal{M}(\mathbf{B}) \subset \mathcal{M}(\mathbf{W})$.

The notation $\tilde{\boldsymbol{\beta}} \sim (\boldsymbol{\beta}, \mathbf{W})$ means that the mean value of the random vector $\tilde{\boldsymbol{\beta}}$ is $\boldsymbol{\beta}$ and its covariance matrix is $\mathbf{W}$.

If $\boldsymbol{u} \in \mathbb{R}^k$, $\mathbf{A}$ is a $k \times l$ matrix and $\mathbf{W}$ is a $k \times k$ p.s.d. matrix, then the symbol $\boldsymbol{u} \perp_{\mathbf{W}} \mathcal{M}(\mathbf{A})$ means $\boldsymbol{u}'\mathbf{W}\mathbf{A} = \boldsymbol{O}$.

A generalized inverse (g-inverse) of a matrix $\mathbf{A}$ is denoted as $\mathbf{A}^-$ (any matrix satisfying the relation $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$). A special kind of this inverse is the Moore-Penrose g-inverse denoted as $\mathbf{A}^+$ (the matrix satisfying the relations $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$, $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$, $\mathbf{A}\mathbf{A}^+ = (\mathbf{A}\mathbf{A}^+)'$ and $\mathbf{A}^+\mathbf{A} = (\mathbf{A}\mathbf{A}^+)'$); the theory of g-inverses is given in [5].

If $\mathbf{W}$ is p.s.d., then the symbol $\mathbf{P}_{\mathcal{K}er(\mathbf{A})}^{\mathbf{W}}$ is used in the case $\mathcal{M}(\mathbf{A}') \subset \mathcal{M}(\mathbf{W})$ (cf. the following lemma).

Lemma 1.1. *Let $\mathbf{W}$ be an $n \times n$ p.s.d. matrix and $\mathbf{A}$ any $k \times n$ matrix satisfying the inclusion $\mathcal{M}(\mathbf{A}') \subset \mathcal{M}(\mathbf{W})$. Then*

$$\mathbf{P}_{\mathcal{K}er(\mathbf{A})}^{\mathbf{W}} = \mathbf{I} - \mathbf{W}^+\mathbf{A}'(\mathbf{A}\mathbf{W}^-\mathbf{A}')^-\mathbf{A} = \left(\mathbf{M}_{\mathbf{A}'}^{\mathbf{W}^+}\right)'$$

is a projection matrix on $\mathcal{K}er(\mathbf{A})$ in $(\mathbb{R}^n, \|\cdot\|_{\mathbf{W}})$ and $\mathbf{M}_{\mathbf{A}'}^{\mathbf{W}^+}$ a projection matrix in $(\mathbb{R}^n, \|\cdot\|_{\mathbf{W}^+})$.

Proof. The matrix $\mathbf{P} = \mathbf{P}_{\mathcal{K}er(\mathbf{A})}^{\mathbf{W}}$ is obviously idempotent and satisfies the relationships $\mathbf{O} = \mathbf{A}\mathbf{P} \implies \mathcal{M}(\mathbf{P}) \subset \mathcal{K}er(\mathbf{A})$ and $r(\mathbf{P}) = \mathrm{Tr}(\mathbf{P}) = n - \mathrm{Tr}\left[(\mathbf{A}\mathbf{W}^-\mathbf{A}')\mathbf{A}\mathbf{W}^-\mathbf{A}'\right] = n - r(\mathbf{A}) = \dim\left[\mathcal{K}er(\mathbf{A})\right]$, thus $\mathcal{M}(\mathbf{P}) = \mathcal{K}er(\mathbf{A})$. Further $\mathbf{W}\mathbf{P} = \mathbf{P}'\mathbf{W}$.

Analogously, $\mathbf{M}_{\mathbf{A}'}^{\mathbf{W}^+} = \mathbf{I} - \mathbf{P}_{\mathbf{A}'}^{\mathbf{W}^+}$, where $\mathbf{P}_{\mathbf{A}'}^{\mathbf{W}^+} = \mathbf{A}'(\mathbf{A}\mathbf{W}^-\mathbf{A}')^-\mathbf{A}\mathbf{W}^+$ is idempotent and satisfies the relationships $\mathcal{M}(\mathbf{P}_{\mathbf{A}'}^{\mathbf{W}^+}) = \mathcal{M}(\mathbf{A}')$ and $\mathbf{W}^+\mathbf{P}_{\mathbf{A}'}^{\mathbf{W}^+} = (\mathbf{P}_{\mathbf{A}'}^{\mathbf{W}^+})'\mathbf{W}^+$. □

Here $r(\cdot)$ and $\mathrm{Tr}(\cdot)$ denote the rank and the trace of a matrix, respectively.

Lemma 1.2. *Let* $\mathbf{B}$ *be any* $q\times k$ *matrix and* $\mathbf{W}$ *be any* $k\times k$ *p.s.d. matrix. Then*

$$(\mathbf{M}_{\mathbf{B}'}\mathbf{W}\mathbf{M}_{\mathbf{B}'})^+ = \begin{cases} \mathbf{W}^{-1} - \mathbf{W}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{W}^{-1}\mathbf{B}')^-\mathbf{B}\mathbf{W}^{-1} & \textit{if } \mathbf{W} \textit{ is regular,} \\ \mathbf{W}^+ - \mathbf{W}^+\mathbf{B}'(\mathbf{B}\mathbf{W}^-\mathbf{B}')^-\mathbf{B}\mathbf{W}^+ & \textit{if } \mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{W}), \\ (\mathbf{W}+\mathbf{B}'\mathbf{B})^+ - (\mathbf{W}+\mathbf{B}'\mathbf{B})^+\mathbf{B}'\cdot & \\ \cdot\big[\mathbf{B}(\mathbf{W}+\mathbf{B}'\mathbf{B})^-\mathbf{B}'\big]^-\mathbf{B}(\mathbf{W}+\mathbf{B}'\mathbf{B})^+ & \textit{in general.} \end{cases}$$

Proof. It is sufficient to verify the properties of the Moore-Penrose g-inverse. □

Corollary 1.3. *If* $\mathbf{B}$ *and* $\mathbf{W}$ *are matrices from Lemma* 1.2, *then*

$$\begin{aligned}(\mathbf{M}_{\mathbf{B}'}\mathbf{W}\mathbf{M}_{\mathbf{B}'})^+ &= \mathbf{M}_{\mathbf{B}'}(\mathbf{M}_{\mathbf{B}'}\mathbf{W}\mathbf{M}_{\mathbf{B}'})^+ = (\mathbf{M}_{\mathbf{B}'}\mathbf{W}\mathbf{M}_{\mathbf{B}'})^+\mathbf{M}_{\mathbf{B}'} \\ &= \mathbf{M}_{\mathbf{B}'}(\mathbf{M}_{\mathbf{B}'}\mathbf{W}\mathbf{M}_{\mathbf{B}'})^+\mathbf{M}_{\mathbf{B}'}\,.\end{aligned}$$

Lemma 1.4. *Let* $\mathbf{W}$ *be an* $n\times n$ *p.s.d. matrix and* $\mathbf{A}$ *any* $k\times n$ *matrix. Then any matrix from the class*

$$\mathcal{A}^-_{m(\mathbf{W})} = \left\{\mathbf{A}^-_{m(\mathbf{W})} :\ \mathbf{A}\mathbf{A}^-_{m(\mathbf{W})}\mathbf{A} = \mathbf{A},\ \ \mathbf{W}\mathbf{A}^-_{m(\mathbf{W})}\mathbf{A} = \mathbf{A}'\big(\mathbf{A}^-_{m(\mathbf{W})}\big)'\mathbf{W}\right\}$$

has the property

$$\forall\{y\in\mathcal{M}(\mathbf{A})\}\,\forall\{x_y :\ \mathbf{A}x_y = y\}\,\left\{\|\mathbf{A}^-_{m(\mathbf{W})}y\|_{\mathbf{W}} \le \|x_y\|_{\mathbf{W}}\,,\ \ \mathbf{A}\mathbf{A}^-_{m(\mathbf{W})}y = y\right\}.$$

$\mathcal{A}^-_{m(\mathbf{W})}$ *is the class of all such matrices.*

Proof. Cf. [5]. □

Lemma 1.5. *Let* $\mathbf{W}$ *and* $\mathbf{A}$ *be matrices from Lemma* 1.4. *Then*

$$\mathbf{W}^-\mathbf{A}'(\mathbf{A}\mathbf{W}^-\mathbf{A}')^- \in \mathcal{A}^-_{m(\mathbf{W})} \qquad \textit{if}\ \ \mathcal{M}(\mathbf{A}') \subset \mathcal{M}(\mathbf{W})$$

and

$$(\mathbf{W}+\mathbf{A}'\mathbf{A})^-\mathbf{A}'\big[\mathbf{A}(\mathbf{W}+\mathbf{A}'\mathbf{A})^-\mathbf{A}'\big]^- \in \mathcal{A}^-_{m(\mathbf{W})} \qquad \textit{in general.}$$

Proof. Cf. [5]. □

Remark 1.6. The notations

$$\begin{aligned} &\mathbf{C} = \mathbf{X}'\Sigma^{-}\mathbf{X} && \text{if } \mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\Sigma), \\ &\mathbf{D} = \mathbf{B}\mathbf{C}^{-}\mathbf{B}' && \text{if } \mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{X}') \text{ and } \mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\Sigma), \\ &\mathbf{G} = \mathbf{C} + \mathbf{B}'\mathbf{B}, \\ &\mathbf{H} = \mathbf{B}\mathbf{G}^{-}\mathbf{B}', \\ &\mathbf{L} = \mathbf{X}'\big(\mathbf{X}\mathbf{C}_4\mathbf{X}' + \mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\big)^{-}\mathbf{X} + \mathbf{B}'\mathbf{B} \end{aligned}$$

will be used in Section 2.

The notations

$$\begin{aligned} &\mathbf{E} = \mathbf{B}_1\mathbf{C}^{-}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2' && \text{if } \mathcal{M}(\mathbf{B}_1') \subset \mathcal{M}(\mathbf{X}') = \mathcal{M}(\mathbf{X}'\Sigma^{-}\mathbf{X}), \\ &\mathbf{F} = \mathbf{B}_2'\mathbf{E}^{-}\mathbf{B}_2, \\ &\mathbf{D}_1 = \mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' && \text{if } \Sigma \text{ and } \mathbf{X}'\Sigma^{-1}\mathbf{X} \text{ are regular}, \\ &\mathbf{K} = \mathbf{C} + \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1, \\ &\mathbf{Q} = \mathbf{X}'\big(\mathbf{X}\mathbf{C}_4\mathbf{X}' + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'\big)^{-}\mathbf{X} + \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1 \end{aligned}$$

will be used in Section 3.

2. Indirect observation of the complete first order vector parameter with constraints

DEFINITION 2.1. Let $\boldsymbol{Y} \sim (\mathbf{X}\beta, \Sigma)$, $\beta \in \mathcal{V} = \{\boldsymbol{u}: \ \boldsymbol{u} \in \mathbb{R}^k, \ \boldsymbol{b} + \mathbf{B}\boldsymbol{u} = \boldsymbol{O}\}$ (a model of an indirect observation of the complete vector parameter $\boldsymbol{\beta}$ with constraints). This model is *regular* if $r(\mathbf{X}_{n,k}) = k < n$, $r(\mathbf{B}_{q,k}) = q < k$ and $r(\Sigma) = n$.

LEMMA 2.2. *In the model from Definition 2.1 a linear function $p(\beta) = p_0 + \boldsymbol{p}'\beta$, $\beta \in \mathcal{V}$, is unbiasedly estimable if and only if $\boldsymbol{p} \in \mathcal{M}(\mathbf{X}', \mathbf{B}')$.*

Proof. Cf. [2; Theorem 2.2]. □

LEMMA 2.3. *Let $\mathbf{W}$ be a $k \times k$ p.s.d. matrix and $\mathbf{X}$ be any $n \times k$ matrix. If $\mathcal{M}(\mathbf{W}) = \mathcal{M}(\mathbf{X}')$, then $\mathbf{X}'(\mathbf{X}\mathbf{W}^{-}\mathbf{X}')^{-}\mathbf{X} = \mathbf{W}$.*

Proof. Let $\mathbf{F}_1\boldsymbol{\delta}\mathbf{F}_1'$ be a spectral decomposition of the matrix

$$\mathbf{X}'(\mathbf{X}\mathbf{W}^{-}\mathbf{X}')^{-}\mathbf{X},$$

i.e., $\mathbf{F}_1'\mathbf{F}_1 = \mathbf{I}$ and δ is p.d. Let $\mathbf{J}\boldsymbol{\Delta}\mathbf{J}'$ be spectral decomposition of the matrix $\mathbf{W}^{+}$, i.e., $\mathbf{J}'\mathbf{J} = \mathbf{I} = \mathbf{F}_1'\mathbf{F}_1$ and $\boldsymbol{\Delta}$ is p.d. The fact that $\mathcal{M}(\mathbf{X}') = \mathcal{M}(\mathbf{W})$ implies that there exists a matrix $\mathbf{U}$ such that $\mathbf{X}' = \mathbf{J}\mathbf{U}$ and an orthogonal matrix $\mathbf{Q}$ such that $\mathbf{F}_1 = \mathbf{J}\mathbf{Q}$. The expression $\mathbf{X}'(\mathbf{X}\mathbf{W}^{-}\mathbf{X}')^{-}\mathbf{X}$ is invariant with respect to the choice of the g-inverse of the matrix $\mathbf{W}$ and also the matrix $\mathbf{X}\mathbf{W}^-\mathbf{X}'$ ($\Longleftarrow \mathcal{M}(\mathbf{X}') = \mathcal{M}(\mathbf{W})$), and therefore, in the following, the matrix $\mathbf{J}\Delta\mathbf{J}' = \mathbf{W}^+$ can be used instead of $\mathbf{W}^-$. Thus

$$\mathbf{X}'(\mathbf{X}\mathbf{W}^-\mathbf{X}')^-\mathbf{X} = \mathbf{J}\mathbf{U}(\mathbf{U}'\mathbf{J}'\mathbf{J}\Delta\mathbf{J}'\mathbf{J}\mathbf{U})^-\mathbf{U}'\mathbf{J} = \mathbf{J}\mathbf{U}(\mathbf{U}'\Delta\mathbf{U})^-\mathbf{U}'\mathbf{J}' = \mathbf{F}_1\delta\mathbf{F}_1'$$
$$= \mathbf{J}\mathbf{Q}\delta\mathbf{Q}'\mathbf{J}' \implies \mathbf{Q}\delta\mathbf{Q}' = \mathbf{U}(\mathbf{U}'\Delta\mathbf{U})^-\mathbf{U}' \text{ is regular}$$

and

$$\Delta^{\frac{1}{2}}\mathbf{Q}\delta\mathbf{Q}'\Delta^{\frac{1}{2}} = \Delta^{\frac{1}{2}}\mathbf{U}(\mathbf{U}'\Delta\mathbf{U})^-\mathbf{U}'\Delta^{\frac{1}{2}}$$

is regular and idempotent, i.e.,

$$\Delta^{\frac{1}{2}}\mathbf{Q}\delta\mathbf{Q}'\Delta\mathbf{Q}\delta\mathbf{Q}'\Delta^{\frac{1}{2}} = \Delta^{\frac{1}{2}}\mathbf{Q}\delta\mathbf{Q}'\Delta^{\frac{1}{2}}$$
$$\implies \delta\mathbf{Q}'\Delta\mathbf{Q}\delta = \delta \implies \mathbf{Q}'\Delta\mathbf{Q} = \delta^{-1} \implies \delta = \mathbf{Q}'\Delta^{-1}\mathbf{Q}.$$

Thus

$$\mathbf{X}'(\mathbf{X}\mathbf{W}^-\mathbf{X}')^-\mathbf{X} = \mathbf{F}_1\delta\mathbf{F}_1' = \mathbf{F}_1\mathbf{Q}'\Delta^{-1}\mathbf{Q}\mathbf{F}_1' = \mathbf{J}\Delta^{-1}\mathbf{J}' = \mathbf{W}.$$

□

LEMMA 2.4. *Let Σ be an $n \times n$ p.s.d. matrix, $\mathbf{X}$ an $n \times k$ matrix, and $\mathbf{B}$ a $q \times k$ matrix. The matrices $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ and $\mathbf{C}_4$ are matrices given by the relationship*

$$\begin{pmatrix} \Sigma, & \mathbf{X} \\ \mathbf{X}', & \mathbf{O} \end{pmatrix}^- = \begin{pmatrix} \mathbf{C}_1, & \mathbf{C}_2 \\ \mathbf{C}_3, & -\mathbf{C}_4 \end{pmatrix}.$$

Then a version of the matrix $\left[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\Sigma)}\right]'$ is

$$\mathbf{A} = \left[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\right]'\mathbf{X}\left[(\mathbf{X}')^-_{m(\Sigma)}\right]'.$$

P r o o f . With respect to Lemma 1.4, it is necessary and sufficient to prove the validity of the following two relationships

$$\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\mathbf{A}'\mathbf{M}_{\mathbf{B}'}\mathbf{X}' = \mathbf{M}_{\mathbf{B}'}\mathbf{X}'$$

and

$$\Sigma\mathbf{A}'\mathbf{M}_{\mathbf{B}'}\mathbf{X}' = \mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{A}\Sigma.$$

The validity of the first is obvious.

With respect to the properties of the Pandora-Box matrix (cf. [4] and [1; p. 176])

$$\mathbf{X}\mathbf{C}_3\Sigma = \Sigma\mathbf{C}_3'\mathbf{X}' = \mathbf{X}\mathbf{C}_4\mathbf{X}',$$

and $\mathbf{C}_3'$ can be chosen as $(\mathbf{X}')^-_{m(\Sigma)}$. Now

$$\Sigma\mathbf{A}'\mathbf{M}_{\mathbf{B}'}\mathbf{X}' = \Sigma(\mathbf{X}')^-_{m(\Sigma)}\mathbf{X}'(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\mathbf{M}_{\mathbf{B}'}\mathbf{X}'$$
$$= \mathbf{X}\mathbf{C}_4\mathbf{X}'(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\mathbf{M}_{\mathbf{B}'}\mathbf{X}',$$

what is, with respect to Lemma 1.4, a symmetric matrix. □

This lemma characterizes geometrically the estimation in the sample space $\mathbb{R}^n$. Let in the model from Definition 2.1, $\boldsymbol{b} = \boldsymbol{O}$ and $\boldsymbol{\Sigma}$ be regular. Then the $\boldsymbol{\Sigma}$-LBLUE of $\mathbf{X}\boldsymbol{\beta}$ is

$$\mathbf{X}\mathbf{M}_{\mathbf{B}'}\big[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\big]'\boldsymbol{Y},$$

where $\mathbf{X}\mathbf{M}_{\mathbf{B}'}\big[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\big]'$ is the projection matrix in $\big(\mathbb{R}^n, \|\cdot\|_{\boldsymbol{\Sigma}^{-1}}\big)$ on the subspace $\mathcal{M}(\mathbf{X}\mathbf{M}_{\mathbf{B}'})$. The resulting estimator can be obtained as a projection, by the projection matrix $\mathbf{X}\mathbf{M}_{\mathbf{B}'}\big[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\big]'$, of the estimator $\mathbf{X}\big[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\big]'\boldsymbol{Y}$, which does not respect the constraints $\mathcal{V}$. (The matrix $\mathbf{X}\big[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\big]'$ is the projection matrix on $\mathcal{M}(\mathbf{X})$ in $\big(\mathbb{R}^n, \|\cdot\|_{\boldsymbol{\Sigma}^{-1}}\big)$.)

Corollary 2.7 characterizes geometrically the estimation in the case when $\boldsymbol{\beta}$ is unbiasedly estimable.

LEMMA 2.5. *Let $\boldsymbol{\beta}_0^*$ be within the model from Definition* 2.1 *any fixed solution of the equation $\boldsymbol{b} + \mathbf{B}\boldsymbol{\beta} = \boldsymbol{O}$. Then for each $\boldsymbol{\beta} \in \{\boldsymbol{u}:\ \boldsymbol{b} + \mathbf{B}\boldsymbol{u} = \boldsymbol{O}\}$ one has*

$$\mathbf{X}\boldsymbol{\beta} - \mathbf{X}\mathbf{M}_{\mathbf{B}'}\big[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\big]'\mathbf{X}\boldsymbol{\beta} = \mathbf{X}\boldsymbol{\beta}_0^* - \mathbf{X}\mathbf{M}_{\mathbf{B}'}\big[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\big]'\mathbf{X}\boldsymbol{\beta}_0^*.$$

P r o o f. It is sufficient to realize that $\boldsymbol{\beta} = \boldsymbol{\beta}_0^* + \mathbf{M}_{\mathbf{B}'}\boldsymbol{\kappa}$. □

The notation $\mathbf{X}\boldsymbol{u}^*$ is used for $\mathbf{X}\boldsymbol{\beta}_0^* - \mathbf{X}\mathbf{M}_{\mathbf{B}'}\big[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\big]'\mathbf{X}\boldsymbol{\beta}_0^*$.

THEOREM 2.6. *In the model from Definition* 2.1

(i) *the $\boldsymbol{\Sigma}$-LBLUE of $\mathbf{X}\boldsymbol{\beta}$ is*

$$\widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}} = \mathbf{X}\mathbf{P}^{\mathbf{L}}_{\mathcal{K}er(\mathbf{B})}\hat{\boldsymbol{\beta}} + \mathbf{X}\boldsymbol{u}^*,$$

where $\widehat{\mathbf{X}\boldsymbol{\beta}} = \mathbf{X}\big[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\big]'\boldsymbol{Y}$ (the $\boldsymbol{\Sigma}$-LBLUE of $\mathbf{X}\boldsymbol{\beta}$ in the model $\boldsymbol{Y} \sim (\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$, $\boldsymbol{\beta} \in \mathbb{R}^k$), $\hat{\boldsymbol{\beta}} = \big[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\big]'\boldsymbol{Y}$ (one version of the $\boldsymbol{\Sigma}$-LBLUE of $\boldsymbol{\beta}$ in the same model; if the model is regular, then there exists only one such estimator) and $\boldsymbol{u}^* = -\mathbf{B}^-_{m(\mathbf{L})}\boldsymbol{b}$,

(ii)

$$\boldsymbol{u}^* \in \mathcal{V} = \{\boldsymbol{u}:\ \boldsymbol{b} + \mathbf{B}\boldsymbol{u} = \boldsymbol{O}\} \quad \textit{and} \quad \boldsymbol{u}^* \perp_{\mathbf{L}} \mathcal{K}er(\mathbf{B}),$$

(iii)

$$\operatorname{Var}\Big(\widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}} \mid \boldsymbol{\Sigma}\Big) = \mathbf{X}\big[(\mathbf{M}_{\mathbf{B}'}\mathbf{L}\mathbf{M}_{\mathbf{B}'})^+ - \mathbf{M}_{\mathbf{B}'}\big]\mathbf{X}',$$

$$\operatorname{Var}\Big(\widehat{\mathbf{X}\boldsymbol{\beta}} \mid \boldsymbol{\Sigma}\Big) = \mathbf{X}\big\{\big[\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{X}')^-\mathbf{X}\big]^+ - \mathbf{I}\big\}\mathbf{X}',$$

$$\operatorname{Var}\Big(\widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}} \mid \boldsymbol{\Sigma}\Big) \leq_{\mathbf{L}} \operatorname{Var}\big(\widehat{\mathbf{X}\boldsymbol{\beta}} \mid \boldsymbol{\Sigma}\big)$$

($\leq_{\mathbf{L}}$ *denotes the Loevner ordering p.s.d. matrices*).

Proof.

(i) Let β_0 be any fixed solution of the equation $\boldsymbol{b} + \mathbf{B}\boldsymbol{\beta} = \boldsymbol{O}$, and $\mathbf{K_B}$ be a $k \times [k - r(\mathbf{B})]$ matrix satisfying the equality $\mathcal{M}(\mathbf{K_B}) = \mathcal{K}er(\mathbf{B})$ $(= \mathcal{M}(\mathbf{M_{B'}}))$. Then the model

$$\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0 \sim (\mathbf{X}\mathbf{K_B}\boldsymbol{\gamma}, \boldsymbol{\Sigma}), \qquad \boldsymbol{\gamma} \in \mathbb{R}^{k-r(\mathbf{B})},$$

is equivalent to the model from Definition 2.1, and $\mathbf{X}\boldsymbol{\beta} = \mathbf{X}\boldsymbol{\beta}_0 + \mathbf{X}\mathbf{K_B}\boldsymbol{\gamma}$. Thus

$$\widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}} = \mathbf{X}\boldsymbol{\beta}_0 + \mathbf{X}\mathbf{K_B}\Big\{\big[(\mathbf{X}\mathbf{K_B})'\big]^-_{m(\boldsymbol{\Sigma})}\Big\}'(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_0).$$

As $\mathcal{M}(\mathbf{X}\mathbf{K_B}) = \mathcal{M}(\mathbf{X}\mathbf{M_{B'}})$, then, with respect to Lemma 1.4,

$$\mathbf{X}\mathbf{K_B}\Big\{\big[(\mathbf{X}\mathbf{K_B})'\big]^-_{m(\boldsymbol{\Sigma})}\Big\}' = \mathbf{X}\mathbf{M_{B'}}\Big\{\big[(\mathbf{X}\mathbf{M_{B'}})'\big]^-_{m(\boldsymbol{\Sigma})}\Big\}'.$$

In accordance with Lemma 2.4,

$$\begin{aligned}\widehat{\widehat{\mathbf{X}\boldsymbol{\beta}}} = {}& \mathbf{X}\mathbf{M_{B'}}\big[(\mathbf{M_{B'}}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\big]'\mathbf{X}\big[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\big]'\boldsymbol{Y}\\ &+ \mathbf{X}\Big\{\mathbf{I} - \mathbf{M_{B'}}\big[(\mathbf{M_{B'}}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\big]'\mathbf{X}\Big\}\boldsymbol{\beta}_0.\end{aligned}$$

The expression $\mathbf{M_{B'}}\big[(\mathbf{M_{B'}}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\big]'\mathbf{X}$ gives the whole class of matrices, however $\mathbf{X}\mathbf{M_{B'}}\big[(\mathbf{M_{B'}}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\big]'\mathbf{X}$ is only one matrix; we show that it can be expressed in the form $\mathbf{X}\mathbf{P}^{\mathbf{L}}_{\mathcal{K}er(\mathbf{B})}$.

Let $\mathbf{X}\mathbf{P} = \mathbf{X}\mathbf{M_{B'}}\big[(\mathbf{M_{B'}}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\big]'\mathbf{X}$. According to Lemma 1.5,

$$\mathbf{X}\mathbf{P} = \mathbf{X}\mathbf{M_{B'}}\big[\mathbf{M_{B'}}\mathbf{X}'(\mathbf{X}\mathbf{C}_4\mathbf{X}' + \mathbf{X}\mathbf{M_{B'}}\mathbf{X}')^+\mathbf{X}\mathbf{M_{B'}}\big]^+\mathbf{M_{B'}}\mathbf{X}'(\mathbf{X}\mathbf{C}_4\mathbf{X}' + \mathbf{X}\mathbf{M_{B'}}\mathbf{X}')^+\mathbf{X}.$$

Lemma 1.2 implies

$$\begin{aligned}\mathbf{X}\mathbf{P} &= \mathbf{X}\mathbf{M_{B'}}(\mathbf{M_{B'}}\mathbf{L}\mathbf{M_{B'}})^+(\mathbf{L} - \mathbf{B}'\mathbf{B})\\ &= \mathbf{X}\mathbf{M_{B'}}\big[\mathbf{L}^+ - \mathbf{L}^+\mathbf{B}'(\mathbf{B}\mathbf{L}^+\mathbf{B}')^+\mathbf{B}\mathbf{L}^+\big](\mathbf{L} - \mathbf{B}'\mathbf{B})\\ &= \mathbf{X}\mathbf{M_{B'}}\mathbf{P}^{\mathbf{L}}_{\mathcal{K}er(\mathbf{B})} = \mathbf{X}\mathbf{P}^{\mathbf{L}}_{\mathcal{K}er(\mathbf{B})}.\end{aligned}$$

Thus

$$\mathbf{X}\mathbf{K_B}\Big\{\big[(\mathbf{X}\mathbf{K_B})'\big]^-_{m(\boldsymbol{\Sigma})}\Big\}'\boldsymbol{Y} = \mathbf{X}\mathbf{P}^{\mathbf{L}}_{\mathcal{K}er(\mathbf{B})}\big[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\big]'\boldsymbol{Y}.$$

Finally,

$$\mathbf{X}\Big\{\mathbf{I}-\mathbf{M}_{\mathbf{B}'}\big[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\big]'\mathbf{X}\Big\}\beta_0$$

$$=\mathbf{X}\Big\{\mathbf{I}-\mathbf{M}_{\mathbf{B}'}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{X}'(\mathbf{X}\mathbf{C}_4\mathbf{X}'+\mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^+\mathbf{X}\mathbf{M}_{\mathbf{B}'}\big]^+\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\cdot$$

$$\cdot(\mathbf{X}\mathbf{C}_4\mathbf{X}'+\mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^+\mathbf{X}\Big\}\beta_0$$

$$=\mathbf{X}\{\mathbf{I}-[\mathbf{L}^+-\mathbf{L}^+\mathbf{B}'(\mathbf{B}\mathbf{L}^+\mathbf{B}')^+\mathbf{B}\mathbf{L}^+](\mathbf{L}-\mathbf{B}'\mathbf{B})\}\beta_0$$

$$=\{\mathbf{X}-\mathbf{X}[\mathbf{L}^+\mathbf{L}-\mathbf{L}^+\mathbf{B}'\mathbf{B}-\mathbf{L}^+\mathbf{B}'(\mathbf{B}\mathbf{L}^+\mathbf{B}')^+\mathbf{B}\mathbf{L}^+\mathbf{L}+\mathbf{L}^+\mathbf{B}'(\mathbf{B}\mathbf{L}^+\mathbf{B}')^+\mathbf{B}\mathbf{L}^+\mathbf{B}'\mathbf{B}]\}\beta_0$$

$$=[\mathbf{X}-\mathbf{X}+\mathbf{X}\mathbf{L}^+\mathbf{B}'\mathbf{B}+\mathbf{X}\mathbf{L}^+\mathbf{B}'(\mathbf{B}\mathbf{L}^+\mathbf{B}')^+\mathbf{B}-\mathbf{X}\mathbf{L}^+\mathbf{B}'(\mathbf{B}\mathbf{L}^+\mathbf{B}')^+\mathbf{B}\mathbf{L}^+\mathbf{B}'\mathbf{B}]\beta_0$$

$$=\mathbf{X}\mathbf{L}^+\mathbf{B}'(\mathbf{B}\mathbf{L}^+\mathbf{B}')^+\mathbf{B}\beta_0=-\mathbf{X}\mathbf{B}^-_{m(\mathbf{L})}\boldsymbol{b}$$

since $\mathbf{X}\mathbf{L}^+\mathbf{L}=\mathbf{X}$, $\mathbf{B}\mathbf{L}^+\mathbf{L}=\mathbf{B}$ and $\mathbf{X}\mathbf{L}^+\mathbf{B}'(\mathbf{B}\mathbf{L}^+\mathbf{B}')^+\mathbf{B}\mathbf{L}^+\mathbf{B}'=\mathbf{X}\mathbf{L}^+\mathbf{B}'$.

(ii) As $\boldsymbol{b}\in\mathcal{M}(\mathbf{B})$, obviously $\boldsymbol{b}+\mathbf{B}(-\mathbf{B}^-\boldsymbol{b})=\boldsymbol{O}$. Further, $\mathcal{K}er(\mathbf{B})=\mathcal{M}(\mathbf{I}-\mathbf{B}^-\mathbf{B})$ and $\boldsymbol{b}=\mathbf{B}\boldsymbol{s}$ for some vector $\boldsymbol{s}\in\mathbb{R}^q$; thus

$$\boldsymbol{u}^*\mathbf{L}(\mathbf{I}-\mathbf{B}^-\mathbf{B})=-\boldsymbol{b}'\big(\mathbf{B}^-_{\mathbf{M}(\mathbf{L})}\big)'\mathbf{L}+\boldsymbol{b}'\big(\mathbf{B}^-_{m(\mathbf{L})}\big)'\mathbf{L}\mathbf{B}^-\mathbf{B}$$

$$=-\boldsymbol{s}'\mathbf{B}'\big(\mathbf{B}^-_{m(\mathbf{L})}\big)'\mathbf{L}(\mathbf{I}-\mathbf{B}\mathbf{B}^-)=-\boldsymbol{s}'\mathbf{L}\mathbf{B}^-_{m(\mathbf{L})}\mathbf{B}(\mathbf{I}-\mathbf{B}^-\mathbf{B})=\boldsymbol{O}$$

(cf. Lemma 1.4).

(iii) Thanks to Lemma 1.4, for any $n\times k$ matrix $\mathbf{A}$ we have

$$\operatorname{Var}\Big\{\mathbf{A}\big[(\mathbf{A}')^-_{m(\Sigma)}\big]'\boldsymbol{Y}\mid\Sigma\Big\}=\mathbf{A}\big[(\mathbf{A}')^-_{m(\Sigma)}\big]'\Sigma(\mathbf{A}')^-_{m(\Sigma)}\mathbf{A}'$$

$$=\mathbf{A}\big[(\mathbf{A}')^-_{m(\Sigma)}\big]'\Big\{\mathbf{A}\big[(\mathbf{A}')^-_{m(\Sigma)}\big]'\Big\}\Sigma=\mathbf{A}\big[(\mathbf{A}')^-_{m(\Sigma)}\big]'\Sigma\,.$$

Thus, Lemmas 1.4 and 2.4 together with (i) imply

$$\operatorname{Var}\Big(\widehat{\widehat{\mathbf{X}\beta}}\mid\Sigma\Big)=\mathbf{X}\mathbf{M}_{\mathbf{B}'}\Big[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\Big]'\mathbf{X}\mathbf{C}_4\mathbf{X}'$$

$$=\mathbf{X}\mathbf{M}_{\mathbf{B}'}\big[\mathbf{M}_{\mathbf{B}'}\mathbf{X}'(\mathbf{X}\mathbf{C}_4\mathbf{X}'+\mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^+\mathbf{X}\mathbf{M}_{\mathbf{B}'}\big]^-\mathbf{M}_{\mathbf{B}'}\mathbf{X}'(\mathbf{X}\mathbf{C}_4\mathbf{X}'+\mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^+\mathbf{X}\mathbf{C}_4\mathbf{X}'$$

$$=\mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{L}\mathbf{M}_{\mathbf{B}'})^+\mathbf{M}_{\mathbf{B}'}\mathbf{X}'(\mathbf{X}\mathbf{C}_4\mathbf{X}'+\mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^+(\mathbf{X}\mathbf{C}_4\mathbf{X}'+\mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}'-\mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}')$$

$$=\mathbf{X}(\mathbf{M}_{\mathbf{B}'}\mathbf{L}\mathbf{M}_{\mathbf{B}'})^+\mathbf{X}'-\mathbf{X}\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\,.$$

The expression for $\operatorname{Var}\big(\widehat{\mathbf{X}\beta}\mid\Sigma\big)$ is given in [1; Theorem 5.2.5]. Furthermore

$$\mathbf{O}\le_{\mathbf{L}}\Big\{\mathbf{I}-\mathbf{X}\mathbf{M}_{\mathbf{B}'}\big[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\big]'\Big\}\mathbf{X}\big[(\mathbf{X}')^-_{m(\Sigma)}\big]'\Sigma\,\cdot$$

$$\cdot\Big\{\mathbf{I}-(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\mathbf{M}_{\mathbf{B}'}\mathbf{X}'\Big\}$$

$$=\operatorname{Var}\Big\{\mathbf{X}\big[(\mathbf{X}')^-_{m(\Sigma)}\big]'\boldsymbol{Y}\mid\Sigma\Big\}-\operatorname{Var}\Big\{\mathbf{X}\mathbf{M}_{\mathbf{B}'}\big[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\Sigma)}\big]'\boldsymbol{Y}\mid\Sigma\Big\};$$

here the facts that $\mathbf{X}\big[(\mathbf{X}')^-_{m(\Sigma)}\big]'\Sigma = \mathbf{X}\mathbf{C}_4\mathbf{X}'$ and

$$\mathbf{X}\mathbf{M}_{\mathbf{B}'}\Big[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\Big]'\mathbf{X}\Big[(\mathbf{X}')^-_{m(\Sigma)}\Big]'\Sigma$$
$$=\mathbf{X}\mathbf{M}_{\mathbf{B}'}\Big[(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\Big]'\mathbf{X}\Big[(\mathbf{X}')^-_{m(\Sigma)}\Big]'\Sigma(\mathbf{M}_{\mathbf{B}'}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\mathbf{M}_{\mathbf{B}'}\mathbf{X}'$$

were utilized. □

Corollaries 2.7, 2.8 and 2.9 can be proved analogously.

COROLLARY 2.7. *Let the model from Definition* 2.1 *be regular. Then*

(i) *the Σ-LBLUE of $\boldsymbol\beta$ is $\hat{\hat{\boldsymbol\beta}}$, where*

$$\mathbf{X}\hat{\hat{\boldsymbol\beta}} = \mathbf{X}\boldsymbol\beta_0 + \mathbf{P}^{\Sigma^{-1}}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}(\boldsymbol Y - \mathbf{X}\boldsymbol\beta_0) = \mathbf{X}\boldsymbol\beta_0 + \mathbf{P}^{(\mathbf{X}\mathbf{C}^{-1}\mathbf{X}')^+}_{\mathbf{X}\mathbf{M}_{\mathbf{B}'}}\big(\mathbf{X}\hat{\boldsymbol\beta} - \mathbf{X}\boldsymbol\beta_0\big)$$
$$= \mathbf{X}\Big[\boldsymbol\beta_0 + \mathbf{P}^{\mathbf{X}'(\mathbf{X}\mathbf{C}^{-1}\mathbf{X}')^-\mathbf{X}}_{\mathcal{K}er(\mathbf{B})}(\hat{\boldsymbol\beta} - \boldsymbol\beta_0)\Big] = \mathbf{X}\big(\boldsymbol u^* + \mathbf{P}^{\mathbf{C}}_{\mathcal{K}er(\mathbf{B})}\hat{\boldsymbol\beta}\big),$$

$\hat{\boldsymbol\beta} = \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\boldsymbol Y$ (*the Σ-LBLUE of $\boldsymbol\beta$ in the model $Y \sim (\mathbf{X}\boldsymbol\beta, \Sigma)$, $\boldsymbol\beta \in \mathbb{R}^k$*),

(ii) $\hat{\hat{\boldsymbol\beta}} = \big[\mathbf{I} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\big]\hat{\boldsymbol\beta} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\boldsymbol b$,

(iii) $\boldsymbol u^* = -\mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\boldsymbol b \in \mathcal{V}$, $\boldsymbol u^* \perp_{\mathbf{C}} \mathcal{K}er(\mathbf{B})$,

(iv) $\operatorname{Var}\big(\hat{\hat{\boldsymbol\beta}} \mid \Sigma\big) = \mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{B}'\mathbf{D}^{-1}\mathbf{B}\mathbf{C}^{-1} = \mathbf{P}\mathbf{C}^{-1}\mathbf{P}' = \mathbf{P}\mathbf{C}^{-1} = \mathbf{C}^{-1}\mathbf{P}'$,

where $\mathbf{C}^{-1} = \operatorname{Var}\big(\hat{\boldsymbol\beta} \mid \Sigma\big)$ *and* $\mathbf{P} = \mathbf{P}^{\mathbf{C}}_{\mathcal{K}er(\mathbf{B})}$.

COROLLARY 2.8. *If $\mathbf{X} = \mathbf{I}$, then the model from Definition* 2.1 *represents a direct observation of an incomplete vector parameter with constraints* (*cf.* [3]). *If this model is regular, then* $\hat{\boldsymbol\beta} = \boldsymbol Y$, $\operatorname{Var}(\hat{\boldsymbol\beta} \mid \Sigma) = \Sigma$, $\hat{\hat{\boldsymbol\beta}} = \mathbf{P}^{\Sigma^{-1}}_{\mathcal{K}er(\mathbf{B})}\hat{\boldsymbol\beta} + \boldsymbol u^*$, $\mathbf{P}^{\Sigma^{-1}}_{\mathcal{K}er(\mathbf{B})} = \mathbf{I} - \Sigma\mathbf{B}'(\mathbf{B}\Sigma\mathbf{B}')^{-1}\mathbf{B}$, $\boldsymbol u^* = -\Sigma\mathbf{B}'(\mathbf{B}\Sigma\mathbf{B}')^{-1}\boldsymbol b = -\mathbf{B}^-_{m(\Sigma^{-1})}\boldsymbol b$, $\boldsymbol u^* \in \mathcal{V}$, $\boldsymbol u^* \perp_{\Sigma^{-1}} \mathcal{K}er(\mathbf{B})$.

COROLLARY 2.9. *The Σ-LBLUE of $\mathbf{X}\boldsymbol\beta$ in the model from Definition* 2.1 *is*

(i)

$$\widehat{\widehat{\mathbf{X}\boldsymbol\beta}} = \mathbf{X}\mathbf{P}^{\mathbf{C}}_{\mathcal{K}er(\mathbf{B})}\mathbf{C}^-\mathbf{X}'\Sigma^-\boldsymbol Y + \mathbf{X}\boldsymbol u^*,$$
$$\boldsymbol u^* = -\mathbf{B}^-_{m(\mathbf{C})}\boldsymbol b$$

if, simultaneously, $\mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{X}')$ and $\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\Sigma)$,

(ii)

$$\widehat{\widehat{\mathbf{X}\boldsymbol\beta}} = \mathbf{X}\mathbf{P}^{\mathbf{G}}_{\mathcal{K}er(\mathbf{B})}\mathbf{C}^-\mathbf{X}'\Sigma^-\boldsymbol Y + \mathbf{X}\boldsymbol u^*$$
$$= \mathbf{X}\mathbf{P}^{\mathbf{G}}_{\mathcal{K}er(\mathbf{B})}\mathbf{G}^-\mathbf{X}'\Sigma^-\boldsymbol Y + \mathbf{X}\boldsymbol u^*$$
$$\boldsymbol u^* = -\mathbf{B}^-_{m(\mathbf{G})}\boldsymbol b$$

if $\mathcal{M}(\mathbf{B}') \not\subset \mathcal{M}(\mathbf{X}')$ and $\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\Sigma)$.

THEOREM 2.10. *Consider the estimators* $\mathbf{X}\hat{\beta} = \mathbf{X}\big[(\mathbf{X}')^-_{m(\Sigma)}\big]'\boldsymbol{Y}$, $\mathbf{X}\mathbf{P}^{\mathbf{C}}_{\mathcal{K}er(\mathbf{B})}\hat{\beta} - \mathbf{X}\mathbf{B}^-_{m(\mathbf{C})}\boldsymbol{b}$, $\mathbf{X}\tilde{\tilde{\beta}} = \mathbf{X}\mathbf{G}^-\mathbf{X}'\Sigma^-\boldsymbol{Y} - \mathbf{X}\mathbf{G}^-\mathbf{B}'\boldsymbol{b}$ *and* $\mathbf{X}\mathbf{P}^{\mathbf{G}}_{\mathcal{K}er(\mathbf{B})}\tilde{\tilde{\beta}} - \mathbf{X}\mathbf{G}^-\mathbf{B}'\mathbf{H}^-\boldsymbol{b}$ *from Corollary* 2.8 *(all these estimators are unbiased). Then*

(i) $\operatorname{Var}(\mathbf{X}\hat{\beta} \mid \Sigma) = \mathbf{X}\mathbf{C}^-\mathbf{X}'$,

(ii) $\operatorname{Var}(\mathbf{X}\tilde{\tilde{\beta}} \mid \Sigma) = \mathbf{X}\mathbf{G}^-\mathbf{X}' - \mathbf{X}\mathbf{G}^-\mathbf{B}'\mathbf{B}\mathbf{G}^-\mathbf{X}'$,

(iii) $\operatorname{Var}(\mathbf{X}\mathbf{P}^{\mathbf{C}}_{\mathcal{K}er(\mathbf{B})}\hat{\beta} \mid \Sigma) = \operatorname{Var}(\mathbf{X}\hat{\beta} \mid \Sigma) - \mathbf{X}\mathbf{C}^-\mathbf{B}'\mathbf{D}^-\mathbf{B}\mathbf{C}^-\mathbf{X}'$,

(iv)

$$\operatorname{Var}(\mathbf{X}\mathbf{P}^{\mathbf{G}}_{\mathcal{K}er(\mathbf{B})}\tilde{\tilde{\beta}} \mid \Sigma) = \operatorname{Var}(\mathbf{X}\mathbf{P}^{\mathbf{G}}_{\mathcal{K}er(\mathbf{B})}\hat{\beta} \mid \Sigma) = \mathbf{X}\mathbf{G}^-\mathbf{X}' - \mathbf{X}\mathbf{G}^-\mathbf{B}'\mathbf{H}^-\mathbf{B}\mathbf{G}^-\mathbf{X}'.$$

Further,

(v) $\operatorname{Var}(\mathbf{X}\hat{\beta} \mid \Sigma) \geq_{\mathbf{L}} \operatorname{Var}(\mathbf{X}\tilde{\tilde{\beta}} \mid \Sigma) \geq_{\mathbf{L}} \operatorname{Var}(\mathbf{X}\mathbf{P}^{\mathbf{G}}_{\mathcal{K}er(\mathbf{B})}\tilde{\tilde{\beta}} \mid \Sigma)$,

(vi) $\operatorname{Var}(\mathbf{X}\hat{\beta} \mid \Sigma) \geq_{\mathbf{L}} \operatorname{Var}(\mathbf{X}\mathbf{P}^{\mathbf{C}}_{\mathcal{K}er(\mathbf{B})}\hat{\beta} \mid \Sigma)$.

Proof. It is an elementary task to prove the unbiasedness and the statements (i), (ii), (iii), (iv) and the first inequality in (v). As far as the second inequality in (v) is concerned, obviously

$$\mathbf{I} \geq_{\mathbf{L}} \mathbf{B}(\mathbf{B}'\mathbf{B})^-\mathbf{B}' \geq_{\mathbf{L}} \mathbf{B}(\mathbf{C} + \mathbf{B}'\mathbf{B})^-\mathbf{B}' = \mathbf{H}.$$

As $\mathcal{M}(\mathbf{B}) \subset \mathcal{M}(\mathbf{H})$, the matrix $\mathbf{B}'\mathbf{H}^-\mathbf{B}$ does not depend on the g-inverse and $\mathbf{B}'\mathbf{H}^-\mathbf{B} \geq_{\mathbf{L}} \mathbf{B}'\mathbf{B}$. The last inequality is implied by the following consideration: $\mathbf{I} \geq_{\mathbf{L}} \mathbf{H} \implies \mathbf{I} \geq_{\mathbf{L}} \mathbf{R}$, where $\mathbf{R} = \operatorname{Diag}(R_{1,1}, \dots, R_{q,q})$, $\mathbf{H} = \mathbf{Q}\mathbf{R}\mathbf{Q}'$, and $\mathbf{Q}$ is a proper orthogonal matrix. Here $1 \geq R_{i,i} \geq 0$, $i = 1, \dots, q$; thus $\mathbf{R} \geq_{\mathbf{L}} \mathbf{R}^2$. As $\mathcal{M}(\mathbf{B}) \subset \mathcal{M}(\mathbf{H})$, there exists a matrix $\mathbf{S}$ such that $\mathbf{B} = \mathbf{H}\mathbf{S}$. Thus $\mathbf{B}'\mathbf{B} = \mathbf{S}'\mathbf{H}^2\mathbf{S} = \mathbf{S}'\mathbf{Q}\mathbf{Q}'\mathbf{H}^2\mathbf{Q}\mathbf{Q}'\mathbf{S} = \mathbf{S}'\mathbf{Q}\mathbf{R}^2\mathbf{Q}'\mathbf{S}$. Further $\mathbf{B}'\mathbf{H}^-\mathbf{B} = \mathbf{S}'\mathbf{H}\mathbf{S} = \mathbf{S}'\mathbf{Q}\mathbf{Q}'\mathbf{H}\mathbf{Q}\mathbf{Q}'\mathbf{S} = \mathbf{S}'\mathbf{Q}\mathbf{R}\mathbf{Q}'\mathbf{S}$, which implies $\mathbf{B}'\mathbf{B} = \mathbf{S}'\mathbf{Q}\mathbf{R}^2\mathbf{Q}'\mathbf{S} \leq_{\mathbf{L}} \mathbf{S}'\mathbf{Q}\mathbf{R}\mathbf{Q}'\mathbf{S} = \mathbf{B}'\mathbf{H}^-\mathbf{B} \implies \mathbf{X}\mathbf{G}^-\mathbf{B}'\mathbf{H}^-\mathbf{B}\mathbf{G}^-\mathbf{X}' \geq_{\mathbf{L}} \mathbf{X}\mathbf{G}^-\mathbf{B}'\mathbf{B}\mathbf{G}^-\mathbf{X}'$.

(vi) is obvious. ☐

3. Indirect observation of the incomplete first order vector parameter with constraints

For a motivation of the problem, let us consider a simple geodetic network consisting of four points, which is constructed for a measurement of the recent crustal movement (see Fig. 1).

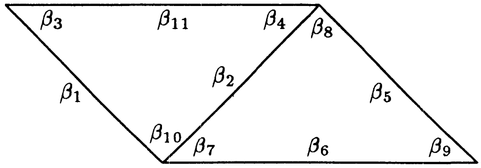

Figure 1.

The parameters $\beta_1, \dots, \beta_9$ (angles and distances) were measured directly according a design which prescribes to repeat n_i times the measurement of the ith parameter ($i = 1, \dots, 9$). There was no possibility to measure the parameter β_{10}, which is important from the point of view of the problem being solved.

The measurement of angles β_3, β_4, β_7, β_8 and β_9 is modelled by random variables Y_i, $i = 3,4,7,8,9$, characterized by a class of probability distributions such that the mean value is $E(Y_i \mid \beta_i) = \beta_i$ and the known dispersion is $\mathrm{Var}(Y_i) = \vartheta_1$. Analogously the measurement of distances β_1, β_2, β_5 and β_6 is modelled by the random variables Y_j, $j = 1,2,5,6$, such that $E(Y_j \mid \beta_j) = \beta_j$ and $\mathrm{Var}(Y_j) = \vartheta_2 + \vartheta_3 v_{jj}$, where ϑ_2 and ϑ_3 are known parameters of the second order and v_{jj} a known number. (The measurements in the network considered were performed by one theodolit and one range finder.)

The experiment is aimed to investigate the effect of recent crustal movements on the parameters β_1, β_2 and β_{10}.

Let the design prescribe n_i given in Table 1;

i	1	2	3	4	5	6	7	8	9
n_i	1	1	2	2	1	1	2	2	2

Table 1.

the measurements in the network are modelled by a 14-dimensional random variable $\boldsymbol{Y} = (Y_1, Y_2, Y_{3,1}, Y_{3,2}, Y_{4,1}, Y_{4,2}, Y_5, Y_6, Y_{7,1}, Y_{7,2}, Y_{8,1}, Y_{8,2}, Y_{9,1}, Y_{9,2})'$,

$$E(\boldsymbol{Y} \mid \boldsymbol{\beta}) = \mathbf{X}\boldsymbol{\beta}_{[1]}, \qquad \mathrm{Var}(\boldsymbol{Y} \mid \boldsymbol{\vartheta}) = \vartheta_1 \mathbf{V}_1 + \vartheta_2 \mathbf{V}_2 + \vartheta_3 \mathbf{V}_3,$$
$$\boldsymbol{b} + \mathbf{B}_1 \Delta\boldsymbol{\beta}_{[1]} + \mathbf{B}_2 \boldsymbol{\beta}_{[2]} = \boldsymbol{O},$$

where $\mathbf{X}$ is a design matrix of the form

$$\mathbf{X}_{(14,9)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

the 10-dimensional vector of the unknown parameters is split into two parts

$$\begin{aligned}\boldsymbol{\beta}_{[1]} = \boldsymbol{\beta}_{[1a]} + \boldsymbol{\Delta\beta}_{[1]} &= (\boldsymbol{\beta}_{1a}, \boldsymbol{\beta}_{2a}, \boldsymbol{\beta}_{3a}, \boldsymbol{\beta}_{4a}, \boldsymbol{\beta}_{5a}, \boldsymbol{\beta}_{6a}, \boldsymbol{\beta}_{7a}, \boldsymbol{\beta}_{8a}, \boldsymbol{\beta}_{9a})' \\ &\quad + (\Delta\beta_1, \Delta\beta_2, \Delta\beta_3, \Delta\beta_4, \Delta\beta_5, \Delta\beta_6, \Delta\beta_7, \Delta\beta_8, \Delta\beta_9)',\end{aligned}$$

where β_{ia}, $i = 1, \dots, 9$, are the approximate values of the unknown parameters, $\boldsymbol{\beta}_{[2]} = \boldsymbol{\beta}_{10}$ and $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ are diagonal matrices with diagonals

$$\begin{aligned}\operatorname{diag}\mathbf{V}_1 &= (0,0,1,1,1,1,0,0,1,1,1,1,1,1), \\ \operatorname{diag}\mathbf{V}_2 &= (1,1,0,0,0,0,1,1,0,0,0,0,0,0)\end{aligned}$$

and

$$\operatorname{diag}\mathbf{V}_3 = (v_{11}, v_{22}, 0,0,0,0, v_{55}, v_{66}, 0,0,0,0,0,0),$$

respectively.

In the problem considered the parameters $\boldsymbol{\beta}_{[1]} = \boldsymbol{\beta}_{[1a]} + \boldsymbol{\Delta\beta}_{[1]}$ and $\boldsymbol{\beta}_{[2]} = \boldsymbol{\beta}_{10}$ must satisfy the conditions

$$\boldsymbol{b} + \mathbf{B}_1 \boldsymbol{\Delta\beta}_{[1]} + \mathbf{B}_2 \boldsymbol{\beta}_{10} = \boldsymbol{O},$$

where

$$\begin{aligned}\boldsymbol{b}' = \Big(&\beta_{7a} + \beta_{8a} + \beta_{9a} - \pi,\ \beta_{3a} + \beta_{4a} - \pi, \\ &\beta_{5a} - \beta_{6a}\tfrac{\sin\beta_{7a}}{\sin\beta_{8a}},\ \beta_{2a} - \beta_{6a}\tfrac{\sin\beta_{9a}}{\sin\beta_{8a}},\ \beta_{1a} - \beta_{2a}\tfrac{\sin\beta_{4a}}{\sin\beta_{3a}}\Big)',\end{aligned}$$

$$\mathbf{B}_1 = \begin{pmatrix} 0, & 0, & 0, & 0, & 0, & 0, & 1, & 1, & 1 \\ 0, & 0, & 1, & 1, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 1, & -\frac{\sin\beta_{7a}}{\sin\beta_{8a}}, & \beta_{6a}\frac{\cos\beta_{7a}}{\sin\beta_{8a}}, & \beta_{6a}\frac{\sin\beta_{7a}\cos\beta_{8a}}{\sin^2\beta_{8a}}, & 0 \\ 0, & 1, & 0, & 0, & 0, & -\frac{\sin\beta_{9a}}{\sin\beta_{8a}}, & 0, & \beta_{6a}\frac{\sin\beta_{9a}\cos\beta_{8a}}{\sin\beta_{8a}}, & -\beta_{6a}\frac{\cos\beta_{9a}}{\sin\beta_{8a}} \\ 1, & -\frac{\sin\beta_{4a}}{\sin\beta_{3a}}, & \beta_{2a}\frac{\sin\beta_{4a}\cos\beta_{3a}}{\sin^2\beta_{3a}}, & -\beta_{2a}\frac{\cos\beta_{4a}}{\sin\beta_{3a}}, & 0, & 0, & 0, & 0, & 0 \end{pmatrix},$$

and $\mathbf{B}_2' = (0,1,0,0,0)$.

In the following, the covariance matrix $\boldsymbol{\Sigma} = \sum_{i=1}^{3} \vartheta_i \mathbf{V}_i$ is supposed to be known.

Definition 3.1. Let $\boldsymbol{Y} \sim (\mathbf{X}\boldsymbol{\beta}_1, \boldsymbol{\Sigma})$,

$$\boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix} \in \mathcal{V} = \left\{ \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} : \ \boldsymbol{u} \in \mathbb{R}^{k_1}, \ \boldsymbol{v} \in \mathbb{R}^{k_2}, \ \boldsymbol{b} + \mathbf{B}_1\boldsymbol{u} + \mathbf{B}_2\boldsymbol{v} = \boldsymbol{O} \right\},$$

where $\boldsymbol{b}$ is a q-dimensional vector and the matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ are of the type $q \times k_1$ and $q \times k_2$, respectively. It is a *model of an indirect observation of an incomplete first order vector parameter* $\boldsymbol{\beta}_1$ *with the constraints* $\mathcal{V}$. This model is regular if $r(\mathbf{X}_{n,k_1}) = k_1 < n$, $r(\mathbf{B}_1, \mathbf{B}_2) = q < k_1 + k_2$, $r(\mathbf{B}_2) = k_2 < q$, and $r(\boldsymbol{\Sigma}) = n$.

Lemma 3.2. *Let* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *be any* $q \times k_1$ *and* $q \times k_2$ *matrices, respectively. Then*

$$\mathcal{K}er(\mathbf{B}_1, \mathbf{B}_2) = \mathcal{M}\begin{pmatrix} \mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}} \\ \mathbf{M}_{\mathbf{B}_2'\mathbf{M}_{\mathbf{B}_1}} \end{pmatrix}.$$

Proof.

$$\mathcal{K}er(\mathbf{B}_1, \mathbf{B}_2) = \mathcal{M}\left(\mathbf{M}_{\begin{pmatrix} \mathbf{B}_1' \\ \mathbf{B}_2' \end{pmatrix}}\right) = \begin{pmatrix} \mathbf{I} - \mathbf{B}_1'\mathbf{R}^-\mathbf{B}_1, & -\mathbf{B}_1'\mathbf{R}^-\mathbf{B}_2 \\ -\mathbf{B}_2'\mathbf{R}^-\mathbf{B}_1, & \mathbf{I} - \mathbf{B}_2'\mathbf{R}^-\mathbf{B}_2' \end{pmatrix},$$

where $\mathbf{R} = \mathbf{B}_1\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2'$. Thus

$$\begin{aligned} &\mathcal{M}(\mathbf{I} - \mathbf{B}_1'\mathbf{R}^-\mathbf{B}_1, \ -\mathbf{B}_1'\mathbf{R}^-\mathbf{B}_2) \\ =&\mathcal{M}\big(\mathbf{I} - \mathbf{B}_1'\mathbf{R}^-\mathbf{B}_1, \ \mathbf{B}_1'\mathbf{R}^-\mathbf{B}_2(\mathbf{B}_2'\mathbf{R}^-\mathbf{B}_2)^-\mathbf{B}_2'\mathbf{R}^-\mathbf{B}_1\big) \\ =&\mathcal{M}\big(\mathbf{I} - \mathbf{B}_1'\mathbf{R}^-\mathbf{B}_1 + \mathbf{B}_1'\mathbf{R}^-\mathbf{B}_2(\mathbf{B}_2'\mathbf{R}^-\mathbf{B}_2)^-\mathbf{B}_2'\mathbf{R}^-\mathbf{B}_1\big) \end{aligned}$$

since both matrices $\mathbf{I} - \mathbf{B}_1'\mathbf{R}^-\mathbf{B}_1$ and $\mathbf{B}_1'\mathbf{R}^-\mathbf{B}_2(\mathbf{B}_2'\mathbf{R}^-\mathbf{B}_2)^-\mathbf{B}_2'\mathbf{R}^-\mathbf{B}_1$ are p.s.d. (cf. [5; p. 122]). Further

$$\begin{aligned}\mathcal{M}\big(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\big) &= \mathcal{M}\big[\mathbf{I} - \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}(\mathbf{M}_{\mathbf{B}_2}\mathbf{R}\mathbf{M}_{\mathbf{B}_2})^+\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\big]\\ &= \mathcal{M}\big[\mathbf{I} - \mathbf{B}_1'\mathbf{R}^-\mathbf{B}_1' + \mathbf{B}_1'\mathbf{R}^-\mathbf{B}_2'(\mathbf{B}_2'\mathbf{R}^-\mathbf{B}_2')^-\mathbf{B}_2'\mathbf{R}^-\mathbf{B}_1'\big].\end{aligned}$$

Analogously, $\mathcal{M}(-\mathbf{B}_2'\mathbf{R}^-\mathbf{B}_1', \mathbf{I} - \mathbf{B}_2'\mathbf{R}^-\mathbf{B}_2') = \mathcal{M}\big(\mathbf{M}_{\mathbf{B}_2'\mathbf{M}_{\mathbf{B}_1}}\big)$. □

LEMMA 3.3. *In the model from Definition* 3.1,

$$\begin{aligned}&\mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\Big[\big(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'\big)^-_{m(\boldsymbol{\Sigma})}\Big]'\\ &= \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\Big[\big(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'\big)^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\Big]'\mathbf{X}\Big[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\Big]',\end{aligned}$$

where $\mathbf{C}_4$ *is given in Lemma* 2.4.

P r o o f is analogous to that of Lemma 2.4. □

LEMMA 3.4. *In the model from Definition* 3.1, *a linear function*

$$p(\beta_1,\beta_2) = p_0 + \boldsymbol{p}_1'\beta_1 + \boldsymbol{p}_2'\beta_2\,, \qquad \binom{\beta_1}{\beta_2} \in \mathcal{V},$$

is unbiasedly estimable if and only if

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \mathcal{M}\begin{pmatrix} \mathbf{X}', & \mathbf{B}_1' \\ \mathbf{O}, & \mathbf{B}_2' \end{pmatrix}.$$

P r o o f . It is sufficient to substitute in Lemma 2.2 the matrix $(\mathbf{X},\mathbf{O})$ for $\mathbf{X}$ and the matrix $(\mathbf{B}_1,\mathbf{B}_2)$ for $\mathbf{B}$. □

THEOREM 3.5. *In the model from Definition* 3.1,

(i) *the* Σ*-LBLUE of* $\mathbf{X}\beta_1$ *is*

$$\widehat{\widehat{\mathbf{X}\beta_1}} = \mathbf{X}\mathbf{P}^{\mathbf{Q}}_{\mathcal{K}er\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\right)}\hat{\beta}_1 + \mathbf{X}\boldsymbol{u}^*\,,$$

where $\hat{\beta}_1 = \big[(\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\big]'\boldsymbol{Y}$ *and* $\boldsymbol{u}^* = -(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1)^-_{m(\mathbf{Q})}\mathbf{M}_{\mathbf{B}_2}\boldsymbol{b}$,

(ii)

$$\boldsymbol{u}^* \in \big\{\boldsymbol{u}:\ \mathbf{M}_{\mathbf{B}_2}\boldsymbol{b} + \mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\boldsymbol{u} = \boldsymbol{O}\big\}\,, \qquad \boldsymbol{u}^* \perp_{\mathbf{Q}} \mathcal{K}er\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\big)\,,$$

(iii)

$$\begin{aligned}\operatorname{Var}\big(\widehat{\mathbf{X}\beta_1} \mid \Sigma\big) &= \mathbf{X}\big\{\big[\mathbf{X}'(\Sigma + \mathbf{X}\mathbf{X}')^-\mathbf{X}\big]^+ - \mathbf{I}\big\}\mathbf{X}' = \mathbf{X}\mathbf{C}_4\mathbf{X}'\,,\\ \operatorname{Var}\Big(\widehat{\widehat{\mathbf{X}\beta_1}} \mid \Sigma\Big) &= \mathbf{X}\Big[\big(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{Q}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\big)^+ - \mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\Big]\mathbf{X}'\,,\\ \operatorname{Var}\Big(\widehat{\widehat{\mathbf{X}\beta_1}} \mid \Sigma\Big) &\leq_{\mathbf{L}} \operatorname{Var}\big(\widehat{\mathbf{X}\beta_1} \mid \Sigma\big)\,.\end{aligned}$$

Proof. If $\mathbf{K}_1$, $\mathbf{K}_2$ are $k_1 \times r$ and $k_2 \times r$ matrices with the properties $\mathcal{K}er(\mathbf{B}_1, \mathbf{B}_2) = \mathcal{M}\binom{\mathbf{K}_1}{\mathbf{K}_2}$ and $r = k_1 + k_2 - r(\mathbf{B}_1, \mathbf{B}_2)$, respectively, then the model from Definition 3.1 is equivalent to the model

$$\mathbf{Y} - \mathbf{X}\beta_{1,0} \sim (\mathbf{X}\mathbf{K}_1\gamma, \Sigma), \qquad \gamma \in \mathbb{R}^r;$$

here $\beta_{1,0}$ is any fixed solution of the equation $\boldsymbol{b} + \mathbf{B}_1\beta_{1,0} + \mathbf{B}_2\beta_{2,0} = \boldsymbol{O}$. Thus

$$\widehat{\widehat{\mathbf{X}\beta_1}} = \mathbf{X}\beta_{1,0} + \widehat{\widehat{\mathbf{X}\mathbf{K}_1\gamma}} = \mathbf{X}\beta_{1,0} + \mathbf{X}\mathbf{K}_1\left[(\mathbf{K}_1'\mathbf{X}')^-_{m(\Sigma)}\right]'(\mathbf{Y} - \mathbf{X}\beta_{1,0}).$$

By Lemmas 3.2, 1.4 and 3.3 we obtain

$$\begin{aligned}
&\mathbf{X}\mathbf{K}_1\left[(\mathbf{K}_1'\mathbf{X}')^-_{m(\Sigma)}\right]' \\
&= \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\left[(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\right]'\mathbf{X}\left[(\mathbf{X}')^-_{m(\Sigma)}\right]' \\
&= \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\left[\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'(\mathbf{X}\mathbf{C}_4\mathbf{X}' + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^-\mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\right]^- \cdot \\
&\qquad \cdot \mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'(\mathbf{X}\mathbf{C}_4\mathbf{X}' + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^-\mathbf{X}\left[(\mathbf{X}')^-_{m(\Sigma)}\right]' \\
&= \mathbf{X}\left(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{Q}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\right)^+\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}(\mathbf{Q} - \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1)\left[(\mathbf{X}')^-_{m(\Sigma)}\right]' \\
&= \mathbf{X}\mathbf{P}^{\mathbf{Q}}_{\mathcal{K}er\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\right)}\left[(\mathbf{X}')^-_{m(\Sigma)}\right]'.
\end{aligned}$$

Further we continue analogously as in the proof of Theorem 2.6 □

THEOREM 3.6. *Let $p(\beta_1, \beta_2) = \boldsymbol{p}_1'\beta_1 + \boldsymbol{p}_2'\beta_2$, $\binom{\beta_1}{\beta_2} \in \mathcal{V}$, be an unbiasedly estimable function in the model from Definition* 3.1. *Then*

(i) *the Σ-LBLUE of the function $p(\cdot)$ is*

$$\widehat{\widehat{p(\beta_1, \beta_2)}} = (\boldsymbol{p}_1' - \boldsymbol{p}_2'\mathbf{B}_2^+\mathbf{B}_1)\mathbf{P}^{\mathbf{Q}}_{\mathcal{K}er\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\right)}\left[(\mathbf{X}')^-_{m(\Sigma)}\right]'\mathbf{Y} - \boldsymbol{p}_2'\mathbf{B}_2^+\boldsymbol{b},$$

and

(ii)

$$\begin{aligned}
&\operatorname{Var}\left(\widehat{\widehat{p(\beta_1, \beta_2)}} \mid \Sigma\right) \\
&= (\boldsymbol{p}_1' - \boldsymbol{p}_2'\mathbf{B}_2^+\mathbf{B}_1)\left[\left(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{Q}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\right)^+ - \mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\right]\left[\boldsymbol{p}_1 - \mathbf{B}_1'(\mathbf{B}_2^+)'\boldsymbol{p}_2\right].
\end{aligned}$$

Proof.

(i) With respect to Lemma 3.4 the function $p(\cdot)$ is unbiasedly estimable if and only if there exist vectors $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^q$ such that $\mathbf{p}_1 = \mathbf{X}'\mathbf{u} + \mathbf{B}_1'\mathbf{v}$, $\mathbf{p}_2 = \mathbf{B}_2'\mathbf{v}$; thus $\mathbf{v} = (\mathbf{B}_2')^+\mathbf{p}_2 = (\mathbf{B}_2^+)'\mathbf{p}_2$, $\mathbf{u}'\mathbf{X} = \mathbf{p}_1' - \mathbf{p}_2'\mathbf{B}_2^+\mathbf{B}_1$ and

$$\mathbf{p}_1'\beta_1 + \mathbf{p}_2'\beta_2 = \mathbf{u}'\mathbf{X}\beta_1 + \mathbf{v}'(\mathbf{B}_1\beta_1 + \mathbf{B}_2\beta_2) = (\mathbf{p}_1' - \mathbf{p}_2'\mathbf{B}_2^+\mathbf{B}_1)\beta_1 - \mathbf{p}_2'\mathbf{B}_2^+\mathbf{b}.$$

The Σ-LBLUE of $(\mathbf{p}_1' - \mathbf{p}_2'\mathbf{B}_2^+\mathbf{B}_1)\beta_1 - \mathbf{p}_2'\mathbf{B}_2^+\mathbf{b}$, $\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \in \mathcal{V}$, is

$$(\mathbf{p}_1' - \mathbf{p}_2'\mathbf{B}_2^+\mathbf{B}_1)\mathbf{P}^{\mathbf{Q}}_{\mathcal{K}er\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\right)}\left[(\mathbf{X}')^-_{m(\Sigma)}\right]'\mathbf{Y} - \mathbf{p}_2'\mathbf{B}_2^+\mathbf{b}.$$

(ii) With respect to Lemma 3.3 and (i)

$$\operatorname{Var}\left[\widehat{p(\beta_1,\beta_2)} \mid \Sigma\right] = \mathbf{u}'\mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\left[(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\right]'\mathbf{X}\mathbf{C}_4\mathbf{X}'\mathbf{u};$$

further

$$\begin{aligned}
&\mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\left[(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^-_{m(\mathbf{X}\mathbf{C}_4\mathbf{X}')}\right]'\mathbf{X}\mathbf{C}_4\mathbf{X}'\\
&=\mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{Q}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}})^+\Big[\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'\\
&\qquad - \mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'(\mathbf{X}\mathbf{C}_4\mathbf{X}' + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^+\mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'\Big]\\
&=\mathbf{X}\left[(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{Q}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}})^+ - \mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\right]\mathbf{X}'.
\end{aligned}$$

□

THEOREM 3.7. *Let the model from Definition* 3.1 *be regular and* $\hat{\beta}_1$ *be the* Σ*-LBLUE of* β_1 *in the model* $\mathbf{Y} \sim (\mathbf{X}\beta_1, \Sigma)$, $\beta_1 \in \mathbb{R}^{k_1}$.

(i) *The* $\mathbf{D}_1$*-LBLUE of* β_2 *in the model* $\mathbf{B}\hat{\beta}_1 + \mathbf{b} \sim (-\mathbf{B}_2\beta_2, \mathbf{D}_1)$, *is* $\tilde{\hat{\beta}}_2 = -\mathbf{F}^{-1}\mathbf{B}_2'\mathbf{E}^{-1}(\mathbf{B}_1\hat{\beta}_1 + \mathbf{b})$;

(ii) *the* Σ*-LBLUE of* β_2 *in the model from Definition* 3.1 *is* $\hat{\hat{\beta}}_2 = \tilde{\hat{\beta}}_2$;

(iii) *the* Σ*-LBLUE of* β_1 *in the model from Definition* 3.1 *is*

$$\begin{aligned}
\hat{\hat{\beta}}_1 &= \mathbf{P}^{\mathbf{C}}_{\mathcal{K}er\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\right)}\hat{\beta}_1 - (\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1)^-_{m(\mathbf{C})}\mathbf{b}\\
&= (\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{E}^{-1}\mathbf{B}_1 + \mathbf{C}^{-1}\mathbf{B}_1'\mathbf{E}^{-1}\mathbf{B}_2\mathbf{F}^{-1}\mathbf{B}_2'\mathbf{E}^{-1}\mathbf{B}_1)\hat{\beta}_1\\
&\qquad - (\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{E}^{-1} - \mathbf{C}^{-1}\mathbf{B}_1'\mathbf{E}^{-1}\mathbf{B}_2\mathbf{F}^{-1}\mathbf{B}_2'\mathbf{E}^{-1})\mathbf{b}\\
&= \left(\mathbf{I} - \mathbf{C}^{-1}\mathbf{B}_1'\mathbf{E}^{-1}\mathbf{M}^{\mathbf{E}^{-1}}_{\mathbf{B}_2}\mathbf{B}_1\right)\hat{\beta}_1 - \mathbf{C}^{-1}\mathbf{B}_1'\mathbf{E}^{-1}\mathbf{M}^{\mathbf{E}^{-1}}_{\mathbf{B}_2}\mathbf{b};
\end{aligned}$$

(iv) *the $\boldsymbol{\Sigma}$-LBLUE of* $\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{pmatrix}$ *in the model from Definition* 3.1 *is*

$$\begin{pmatrix}\hat{\hat{\boldsymbol{\beta}}}_1\\ \hat{\hat{\boldsymbol{\beta}}}_2\end{pmatrix} = \mathbf{P}\begin{pmatrix}\hat{\boldsymbol{\beta}}_1\\ \tilde{\tilde{\boldsymbol{\beta}}}_2\end{pmatrix} + \begin{pmatrix}\boldsymbol{u}^*\\ \boldsymbol{v}^*\end{pmatrix} = \mathbf{P}\begin{pmatrix}\hat{\boldsymbol{\beta}}_1 - \boldsymbol{u}^*\\ \tilde{\boldsymbol{\beta}}_2 - \boldsymbol{v}^*\end{pmatrix} + \begin{pmatrix}\boldsymbol{u}^*\\ \boldsymbol{v}^*\end{pmatrix},$$

where $\mathbf{P} = \mathbf{P}^{(*)}_{\mathcal{K}er(\mathbf{B}_1,\mathbf{B}_2)}$, $(*) = \begin{pmatrix}\mathbf{C}, & \mathbf{O}\\ \mathbf{O}, & \mathbf{I}\end{pmatrix}$, $\boldsymbol{u}^* = -\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{E}^{-1}\boldsymbol{b}$, $\boldsymbol{v}^* = -\mathbf{B}_2'\mathbf{E}^{-1}\boldsymbol{b}$;

(v)

$$\begin{pmatrix}\boldsymbol{u}^*\\ \boldsymbol{v}^*\end{pmatrix} \in \mathcal{V} = \left\{\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{pmatrix} :\ \boldsymbol{b} + \mathbf{B}_1\boldsymbol{u} + \mathbf{B}_2\boldsymbol{v} = \boldsymbol{O}\right\}, \qquad \begin{pmatrix}\boldsymbol{u}^*\\ \boldsymbol{v}^*\end{pmatrix} \perp_{(*)} \mathcal{K}er(\mathbf{B}_1,\mathbf{B}_2);$$

(vi)

$$\operatorname{Var}\left(\begin{pmatrix}\hat{\hat{\boldsymbol{\beta}}}_1\\ \hat{\hat{\boldsymbol{\beta}}}_2\end{pmatrix} \middle| \boldsymbol{\Sigma}\right) = \begin{pmatrix}\mathbf{V}_{1,1}, & \mathbf{V}_{1,2}\\ \mathbf{V}_{2,1}, & \mathbf{V}_{2,2}\end{pmatrix},$$

where

$$\begin{aligned}\mathbf{V}_{1,1} &= \mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{B}_1'(\mathbf{E}^{-1} - \mathbf{E}^{-1}\mathbf{B}_2\mathbf{F}^{-1}\mathbf{B}_2'\mathbf{E}^{-1})\mathbf{B}_1\mathbf{C}^{-1}\\ &= \mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{B}_1'(\mathbf{M}_{\mathbf{B}_2}\mathbf{E}\mathbf{M}_{\mathbf{B}_2})^+\mathbf{B}_1\mathbf{C}^{-1},\\ \mathbf{V}_{1,2} &= -\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{E}^{-1}\mathbf{B}_2\mathbf{F}^{-1} = \mathbf{V}_{2,1}',\\ \mathbf{V}_{2,2} &= \mathbf{F}^{-1} - \mathbf{I}.\end{aligned}$$

Proof. With respect to [4] or [1; Section 5.3], the $\boldsymbol{\Sigma}$-LBLUE of $\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{pmatrix}$ can be obtained as a solution of the problem to minimize the function $\phi(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{Y} - \mathbf{X}\boldsymbol{u})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{Y} - \mathbf{X}\boldsymbol{u})$, $\boldsymbol{u} \in \mathbb{R}^{k_1}$, under the side condition $\boldsymbol{b} + \mathbf{B}_1\boldsymbol{u} + \mathbf{B}_2\boldsymbol{v} = \boldsymbol{O}$. The auxiliary function of the Lagrange procedure is $\Phi(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{Y}-\mathbf{X}\boldsymbol{u})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{Y}-\mathbf{X}\boldsymbol{u}) - 2\boldsymbol{\lambda}'(\boldsymbol{b}+\mathbf{B}_1\boldsymbol{u}+\mathbf{B}_2\boldsymbol{v})$ (the vector of the indefinite Lagrange multipliers is $\boldsymbol{\lambda}$). By a standard procedure we obtain: $\mathbf{C}\hat{\hat{\boldsymbol{\beta}}}_1 - \mathbf{X}\boldsymbol{Y}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y} - \mathbf{B}_1'\boldsymbol{\lambda} = \boldsymbol{O}$, $\mathbf{B}_2'\boldsymbol{\lambda} = \boldsymbol{O}$ and $\boldsymbol{b} + \mathbf{B}_1\hat{\hat{\boldsymbol{\beta}}}_1 + \mathbf{B}_2\hat{\hat{\boldsymbol{\beta}}}_2 = \boldsymbol{O}$. Thus

$$\hat{\hat{\boldsymbol{\beta}}}_1 = \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y} + \mathbf{C}^{-1}\mathbf{B}_1'\boldsymbol{\lambda}$$

and

$$\begin{pmatrix}\mathbf{D}_1, & \mathbf{B}_2\\ \mathbf{B}_2', & \mathbf{O}\end{pmatrix}\begin{pmatrix}\boldsymbol{\lambda}\\ \hat{\hat{\boldsymbol{\beta}}}_2\end{pmatrix} = \begin{pmatrix}-\boldsymbol{b} - \mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}\\ \mathbf{O}\end{pmatrix}.$$

The matrix

$$\begin{pmatrix}\mathbf{D}_1, & \mathbf{B}_2\\ \mathbf{B}_2', & \mathbf{O}\end{pmatrix}$$

is regular in a consequence of the assumption on the regularity of the model. With respect to the theory of the Pandora-Box matrix (cf. [4] and [1; p. 176])

$$\begin{pmatrix} \mathbf{D}_1, & \mathbf{B}_2 \\ \mathbf{B}_2', & \mathbf{O} \end{pmatrix}^- = \begin{pmatrix} (\mathbf{M}_{\mathbf{B}_2}\mathbf{D}_1\mathbf{M}_{\mathbf{B}_2})^+, & (\mathbf{B}_2')^-_{m(\mathbf{D}_1)} \\ ((\mathbf{B}_2')^-_{m(\mathbf{D}_1)})', & -[(\mathbf{B}_2')^-_{m(\mathbf{D}_1)}]'\mathbf{D}_1(\mathbf{B}_2')^-_{m(\mathbf{D}_1)} \end{pmatrix}.$$

As $\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_2 = \boldsymbol{O}$, the matrix $(\mathbf{M}_{\mathbf{B}_2}\mathbf{D}_1\mathbf{M}_{\mathbf{B}_2})^+$ can be rewritten as $(\mathbf{M}_{\mathbf{B}_2}\mathbf{E}\mathbf{M}_{\mathbf{B}_2})^+$. Because of $\mathcal{M}(\mathbf{B}_2) \subset \mathcal{M}(\mathbf{E})$, we can write (cf. Lemma 1.2) $(\mathbf{M}_{\mathbf{B}_2}\mathbf{D}_1\mathbf{M}_{\mathbf{B}_2})^+ = \mathbf{E}^{-1} - \mathbf{E}^{-1}\mathbf{B}_2\mathbf{F}^{-1}\mathbf{B}_2'\mathbf{E}^{-1}$. In the last relationship all the inverses exist because of the regularity of the model. Further, $(\mathbf{B}_2')^-_{m(\mathbf{D}_1)} = (\mathbf{B}_2')^-_{m(\mathbf{E})}$, and thus $(\mathbf{B}_2')^-_{m(\mathbf{D}_1)} = \mathbf{E}^{-1}\mathbf{B}_2\mathbf{F}^{-1}$. Finally

$$[(\mathbf{B}_2')^-_{m(\mathbf{E})}]'\mathbf{D}_1(\mathbf{B}_2')^-_{m(\mathbf{E})} = \mathbf{F}^{-1}\mathbf{B}_2'\mathbf{E}^{-1}(\mathbf{E} - \mathbf{B}_2\mathbf{B}_2')\mathbf{E}^{-1}\mathbf{B}_2\mathbf{F}^{-1} = \mathbf{F}^{-1} - \mathbf{I}.$$

Thus $\boldsymbol{\lambda} = -(\mathbf{M}_{\mathbf{B}_2}\mathbf{D}_1\mathbf{M}_{\mathbf{B}_2})^+(\boldsymbol{b} + \mathbf{B}_1\hat{\boldsymbol{\beta}}_1)$ and

$$\hat{\hat{\boldsymbol{\beta}}}_1 = \hat{\boldsymbol{\beta}}_1 - \mathbf{C}^{-1}\mathbf{B}_1'(\mathbf{M}_{\mathbf{B}_2}\mathbf{D}_1\mathbf{M}_{\mathbf{B}_2})^+\mathbf{B}_1\hat{\boldsymbol{\beta}}_1 - \mathbf{C}^{-1}\mathbf{B}_1'(\mathbf{M}_{\mathbf{B}_2}\mathbf{D}_1\mathbf{M}_{\mathbf{B}_2})^+\boldsymbol{b}.$$

As

$$\mathbf{C}^{-1}\mathbf{B}_1'(\mathbf{M}_{\mathbf{B}_2}\mathbf{D}_1\mathbf{M}_{\mathbf{B}_2})^+\mathbf{B}_1 = \big(\mathbf{M}^{\mathbf{C}^{-1}}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\big)' = \mathbf{P}^{\mathbf{C}}_{\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}} = \mathbf{I} - \mathbf{P}^{\mathbf{C}}_{\mathcal{K}er\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\right)}$$

and

$$\mathbf{C}^{-1}\mathbf{B}_1'(\mathbf{M}_{\mathbf{B}_2}\mathbf{D}_1\mathbf{M}_{\mathbf{B}_2})^+ = (\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1)^-_{m(\mathbf{C})} = \mathbf{C}^{-1}\mathbf{B}_1'\mathbf{E}^{-1}\mathbf{M}^{\mathbf{E}^{-1}}_{\mathbf{B}_2},$$

we have proved (iv).

Further

$$\hat{\hat{\boldsymbol{\beta}}}_1 = -\mathbf{F}^{-1}\mathbf{B}_2'\mathbf{E}^{-1}\mathbf{B}_1\hat{\boldsymbol{\beta}}_1 = \tilde{\tilde{\boldsymbol{\beta}}}_2$$

(the $\boldsymbol{\Sigma}$-LBLUE in the model $\mathbf{B}_1\tilde{\boldsymbol{\beta}}_1 + \boldsymbol{b} \sim (-\mathbf{B}_2\boldsymbol{\beta}_2, \mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' = \mathbf{D}_1)$).

As the matrix $\mathbf{B}_2$ is of the full rank in columns, the parameter $\boldsymbol{\beta}_2$ in the model $\mathbf{B}_1\hat{\boldsymbol{\beta}}_1 + \boldsymbol{b} \sim (-\mathbf{B}_2\boldsymbol{\beta}_2, \mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1')$ is unbiasedly estimable and the $\mathbf{D}_1$-LBLUE is $\tilde{\tilde{\boldsymbol{\beta}}}_2$. Here it is necessary to remark that the matrix $\mathbf{D}_1$ need not be regular, however the matrix $\mathbf{E} = \mathbf{D}_1 + \mathbf{B}_2\mathbf{B}_2'$ is regular in consequence of the assumption on the regularity of the model. The relationship $(\mathbf{B}_2')^-_{m(\mathbf{D}_1)} = (\mathbf{B}_2')^-_{m(\mathbf{E})}$ is utilized as well. By this, (iii) is proved.

With respect to Lemma 1.1,

$$\mathbf{P}^{(*)}_{\mathcal{K}er(\mathbf{B}_1,\mathbf{B}_2)} = \begin{pmatrix} \mathbf{I} - \mathbf{C}^{-1}\mathbf{B}_1'\mathbf{E}^{-1}\mathbf{B}_1, & -\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{E}^{-1}\mathbf{B}_2 \\ -\mathbf{B}_2'\mathbf{E}^{-1}\mathbf{B}_1, & \mathbf{I} - \mathbf{F} \end{pmatrix},$$

where $(*) = \begin{pmatrix} \mathbf{C}, & \mathbf{O} \\ \mathbf{O}, & \mathbf{I} \end{pmatrix}$. If we compare expressions for $\hat{\hat{\boldsymbol{\beta}}}_1$, $\hat{\hat{\boldsymbol{\beta}}}_2$ and $\tilde{\tilde{\boldsymbol{\beta}}}_2$, we obtain the statements in (v).

Now it is easy to prove the other statements and therefore their proofs are omitted. □

COROLLARY 3.8. *If in the regular model from Theorem* 3.7, $\mathbf{X} = \mathbf{I}$, *i.e., if a model of a direct observation of an incomplete vector parameter with constraints is under consideration* (*cf.* [3]), *then*

$$\hat{\beta}_1 = \boldsymbol{Y}\,, \qquad \tilde{\tilde{\beta}}_2 = -\mathbf{F}_1^{-1}\mathbf{B}_2'\mathbf{E}_1^{-1}(\mathbf{B}\hat{\beta}_1 + \boldsymbol{b}) = \hat{\hat{\beta}}_2\,,$$

where

$$\begin{aligned}\hat{\hat{\beta}}_1 &= \left[\mathbf{I} - \mathbf{C}^{-1}\mathbf{B}_2'(\mathbf{I} - \mathbf{E}_1^{-1}\mathbf{B}_2\mathbf{F}_1^{-1}\mathbf{B}_2')\mathbf{E}_1^{-1}\mathbf{B}_1\right]\hat{\beta}_1 - \mathbf{C}^{-1}\mathbf{B}_1'\mathbf{E}_1^{-1}(\mathbf{I} - \mathbf{B}_2\mathbf{F}_1^{-1}\mathbf{B}_2'\mathbf{E}^{-1})\boldsymbol{b} \\ &= \mathbf{P}^{\Sigma^{-1}}_{\mathcal{K}er\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\right)}\hat{\beta}_1 - (\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1)^-_{m(\Sigma^{-1})}\boldsymbol{b}\,,\end{aligned}$$

and $\mathbf{E}_1 = \mathbf{B}_1\Sigma\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2'$, $\mathbf{F}_1 = \mathbf{B}_2'\mathbf{E}_1^{-1}\mathbf{B}_2$, *etc.*

Remark 3.9. If the vector parameter β_1 in Theorem 3.7 is a nuisance one, then, with respect to (ii), $\hat{\hat{\beta}}_2$ can be obtained in a simpler way saving simultaneously the numerical labour. Further, the statement (iv) from Theorem 3.7 is of great importance for obtaining the equivalent algorithms necessary for checking the numerical reliability of the estimation.

COROLLARY 3.10. *Let in the model from Definition* 3.1 $\mathcal{M}(\mathbf{B}_1') \subset \mathcal{M}(\mathbf{X}') \subset \mathcal{M}(\Sigma)$. *Then*

(i)

$$\widehat{\widehat{\mathbf{X}\beta_1}} = \mathbf{X}\mathbf{P}^{\mathbf{C}}_{\mathcal{K}er\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\right)}\hat{\beta}_1 + \mathbf{X}\boldsymbol{u}^*\,,$$

where

$$\begin{aligned}\mathbf{P}^{\mathbf{C}}_{\mathcal{K}er\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\right)} &= \mathbf{I} - \mathbf{C}^-\mathbf{B}_1'(\mathbf{M}_{\mathbf{B}_2}\mathbf{E}\mathbf{M}_{\mathbf{B}_2})^-\mathbf{B}_1\,, \\ \boldsymbol{u}^* &= -\mathbf{C}^-\mathbf{B}_1'(\mathbf{M}_{\mathbf{B}_2}\mathbf{E}\mathbf{M}_{\mathbf{B}_2})^-\boldsymbol{b}\,;\end{aligned}$$

(ii)

$$\boldsymbol{u}^* \perp_{\mathbf{C}} \mathcal{K}er\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\right)\,, \qquad \boldsymbol{u}^* \in \{\boldsymbol{u}:\ \mathbf{M}_{\mathbf{B}_2}\boldsymbol{b} + \mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\boldsymbol{u} = \boldsymbol{O}\}\,;$$

(iii) *the* $\mathbf{B}_1\mathbf{C}^-\mathbf{B}_1'$*-LBLUE of* β_2 *in the model*

$$\widehat{\mathbf{B}_1\beta_1} + \boldsymbol{b} \sim (-\mathbf{B}_2\beta_2,\ \mathbf{D}_1 = \mathbf{B}_1\mathbf{C}^-\mathbf{B}_1')$$

is

$$\tilde{\tilde{\beta}}_2 = -(\mathbf{B}_2'\mathbf{E}^-\mathbf{B}_2)^-\mathbf{B}_2'\mathbf{E}^-\left(\widehat{\mathbf{B}_1\beta_1} + \boldsymbol{b}\right)\,;$$

(iv) *one version of* $\hat{\hat{\beta}}_2$ *is* $\tilde{\tilde{\beta}}_2$,

(v)

$$\begin{pmatrix}\hat{\hat{\beta}}_1 \\ \hat{\hat{\beta}}_2\end{pmatrix} = \mathbf{P}^{(*)}_{\mathcal{K}er(\mathbf{B}_1,\mathbf{B}_2)}\begin{pmatrix}\hat{\beta}_1 \\ \tilde{\tilde{\beta}}_2\end{pmatrix} + \begin{pmatrix}\boldsymbol{u}^* \\ \boldsymbol{v}^*\end{pmatrix}\,,$$

where

$$(*) = \begin{pmatrix} \mathbf{C}, & \mathbf{O} \\ \mathbf{O}, & \mathbf{I} \end{pmatrix}, \qquad \mathbf{P}^{(*)}_{\mathcal{K}er(\mathbf{B}_1,\mathbf{B}_2)} = \begin{pmatrix} \mathbf{I} - \mathbf{C}^-\mathbf{B}_1'\mathbf{E}^-\mathbf{B}_1, & -\mathbf{C}^-\mathbf{B}_1'\mathbf{E}^-\mathbf{B}_2 \\ -\mathbf{B}_2'\mathbf{E}^-\mathbf{B}_1, & \mathbf{I} - \mathbf{B}_2'\mathbf{E}^-\mathbf{B}_2 \end{pmatrix},$$

$$\boldsymbol{u}^{(*)} = -\mathbf{C}^-\mathbf{B}_1'\mathbf{E}^-\boldsymbol{b}, \qquad \boldsymbol{v}^{(*)} = -\mathbf{B}_2'\mathbf{E}^-\boldsymbol{b};$$

(vi)

$$\begin{pmatrix} \boldsymbol{u}^* \\ \boldsymbol{v}^* \end{pmatrix} \perp_{(*)} \mathcal{K}er(\mathbf{B}_1, \mathbf{B}_2), \qquad \begin{pmatrix} \boldsymbol{u}^* \\ \boldsymbol{v}^* \end{pmatrix} \in \mathcal{V} = \left\{ \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} : \ \boldsymbol{b} + \mathbf{B}_1\boldsymbol{u} + \mathbf{B}_2\boldsymbol{v} = \boldsymbol{O} \right\}.$$

The proof can be performed analogously as in Theorem 3.7.

COROLLARY 3.11. *Let* $\begin{pmatrix} \hat{\hat{\beta}}_1 \\ \hat{\hat{\beta}}_2 \end{pmatrix}$ *be the estimator from Corollary* 3.10. *Then*

$$\operatorname{Var}\left(\begin{pmatrix} \hat{\hat{\beta}}_1 \\ \hat{\hat{\beta}}_2 \end{pmatrix} \Big| \boldsymbol{\Sigma} \right) = \begin{pmatrix} (\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{C}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}})^+, & -\mathbf{C}^+\mathbf{B}_1'\mathbf{E}^-\mathbf{B}_2\mathbf{F}^+ \\ -\mathbf{F}^+\mathbf{B}_2'\mathbf{E}^-\mathbf{B}_1\mathbf{C}^+, & \mathbf{F}^+ - \mathbf{F}^+\mathbf{F} \end{pmatrix}.$$

P r o o f. With respect to (i) and (iv) from Corollary 3.10,

$$\operatorname{Var}\left(\begin{pmatrix} \hat{\hat{\beta}}_1 \\ \hat{\hat{\beta}}_2 \end{pmatrix} \Big| \boldsymbol{\Sigma} \right) = \operatorname{Var}\left[\begin{pmatrix} (\mathbf{M}^{\mathbf{C}^+}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}})'\mathbf{C}^-\mathbf{X}'\boldsymbol{\Sigma}^- \\ -\mathbf{F}^+\mathbf{B}_2'\mathbf{E}^-\mathbf{B}_1\mathbf{C}^-\mathbf{X}'\boldsymbol{\Sigma}^- \end{pmatrix} \boldsymbol{Y} \Big| \boldsymbol{\Sigma} \right].$$

As

$$(\mathbf{M}^{\mathbf{C}^+}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}})'\mathbf{C}^+\mathbf{C}\mathbf{C}^+\mathbf{M}^{\mathbf{C}^+}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}} = \mathbf{C}^+\mathbf{M}^{\mathbf{C}^+}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}} = (\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{C}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}})^+$$

and

$$\begin{aligned} &- \mathbf{F}^+\mathbf{B}_2'\mathbf{E}^-\mathbf{B}_1\mathbf{C}^+\mathbf{C}\mathbf{C}^+\mathbf{M}^{\mathbf{C}^+}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}} \\ &= - \mathbf{F}^+\mathbf{B}_2'\mathbf{E}^-\mathbf{B}_1\mathbf{C}^+\left[\mathbf{I} - \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{C}^-\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2})^+\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{C}^+\right] \\ &= - \mathbf{F}^+\mathbf{B}_2'\mathbf{E}^-\mathbf{B}_1\mathbf{C}^+, \end{aligned}$$

we can finish the proof in the obvious way.

Here the facts $\mathcal{M}(\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}) \subset \mathcal{M}(\mathbf{X}') = \mathcal{M}(\mathbf{C})$, $\mathbf{P}^{\mathbf{C}}_{\mathcal{K}er(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1)} = (\mathbf{M}^{\mathbf{C}^+}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}})'$ and $\mathbf{F}\mathbf{F}^+ = \mathbf{F}^+\mathbf{F} \implies \mathbf{F}^+\mathbf{F}\mathbf{F}\mathbf{F}^+ = \mathbf{F}^+\mathbf{F}$ were utilized. □

COROLLARY 3.12. *Let in the model from Definition* 3.1, $\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\boldsymbol{\Sigma})$. *Then*

(i) *the* $\boldsymbol{\Sigma}$*-LBLUE of* $\mathbf{X}\beta_1$ *is* $\widehat{\widehat{\mathbf{X}\beta_1}} = \mathbf{X}\mathbf{P}^{\mathbf{K}}_{\mathcal{K}er(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1)}\tilde{\tilde{\beta}}_1 + \mathbf{X}\boldsymbol{u}^*$, *where*

$$\mathbf{P}^{\mathbf{K}}_{\mathcal{K}er(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1)} = \mathbf{I} - \mathbf{K}^-\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{K}^-\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2})^+\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1,$$

$$\tilde{\tilde{\beta}}_1 = \mathbf{K}^-\mathbf{X}'\boldsymbol{\Sigma}^-\boldsymbol{Y} - \mathbf{K}^-\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\boldsymbol{b},$$

$$\forall \left\{ \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix} \in \left\{ \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} : \ \boldsymbol{b} + \mathbf{B}_1\boldsymbol{u} + \mathbf{B}_2\boldsymbol{v} = \boldsymbol{O} \right\} \right\}$$

$$\left[E\big(\mathbf{X}\tilde{\tilde{\boldsymbol{\beta}}}_1 \mid \boldsymbol{\beta}_1, \boldsymbol{\beta}_2\big) = \mathbf{X}\boldsymbol{\beta}_1 \ \ and \ \ E\big(\mathbf{B}_1\tilde{\tilde{\boldsymbol{\beta}}}_1 \mid \boldsymbol{\beta}_1, \boldsymbol{\beta}_2\big) = \mathbf{B}_1\boldsymbol{\beta}_1 \right],$$

$$\boldsymbol{u}^* = -\mathbf{K}^-\mathbf{B}_1'(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{K}^-\mathbf{B}_1\mathbf{M}_{\mathbf{B}_2})^+\boldsymbol{b};$$

(ii) $\boldsymbol{u}^* \perp_{\mathbf{K}} \mathcal{K}er\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\big)$, $\boldsymbol{u}^* \in \big\{\boldsymbol{u} : \ \mathbf{M}_{\mathbf{B}_2}\boldsymbol{b} + \mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\boldsymbol{u} = \boldsymbol{O}\big\}$,

(iii) *the* $\boldsymbol{\Sigma}$*-LBLUE of any unbiasedly estimable function* $p(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) = p_0 + \boldsymbol{p}_1'\boldsymbol{\beta}_1 + \boldsymbol{p}_2'\boldsymbol{\beta}_2$, $\begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix} \in \mathcal{V}$, *is*

$$\widehat{p(\widehat{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2})} = p_0 + (\boldsymbol{p}_1' - \boldsymbol{p}_2'\mathbf{B}_2^-\mathbf{B}_1)\hat{\hat{\boldsymbol{\beta}}}_1 - (\boldsymbol{p}_1' - \boldsymbol{p}_2'\mathbf{B}_2^-\mathbf{B}_1)\mathbf{K}^-\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\boldsymbol{b} - \boldsymbol{p}_2'\mathbf{B}_2^-\boldsymbol{b}.$$

The proof can be established analogously as for Theorems 3.7 and 3.5.

REFERENCES

[1] KUBÁČEK, L.: *Foundations of Estimation Theory*, Elsevier, Amsterdam-Oxford-New York-Tokyo, 1988.

[2] KUBÁČEK, L.: *Equivalent algorithms for estimation in linear model with condition*, Math. Slovaca **41** (1991), 401–421.

[3] KUBÁČKOVÁ, L.: *Foundations of Experimental Data Analysis*, CRC Press, Boca Raton-Ann Arbor-London-Tokyo, 1992.

[4] RAO, C. R.: *Unified theory of linear estimation*, Sankhyā Ser. A **33** (1971), 370–396.

[5] RAO, C. R.—MITRA, S. K.: *Generalized Inverse of Matrices and Its Applications*, J. Wiley, New York, 1971.

Received March 9, 1993
Revised June 24, 1993

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK - 814 73 Bratislava
SLOVAKIA

Present address: Department of Mathematics
Faculty of Science
Tomkova 40
CZ - 779 00 Olomouc
CZECH REPUBLIC

E-mail: kubacekl@risc.upol.cz