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COMPARISON AND OSCILLATION THEOREMS FOR SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. Our aim in this paper is to present comparison theorems for nonlinear differential equations of the form

$$(r(t)u'(t))' + p(t)f(u(g(t))) = 0 . \tag{$*$}$$

We present sufficient conditions for $(*)$ to be oscillatory.

We consider the second order functional differential equation with deviating argument

$$\big(r(t)u'(t)\big)' + p(t)f\big(u(g(t))\big) = 0 , \tag{1}$$

where $r, p, g \in C\big([t_0, \infty)\big)$ are positive, $f \in C(\mathbb{R})$, $f(x)x > 0$ for $x \neq 0$, and $g(t) \to \infty$ as $t \to \infty$.

Our attention is restricted to those solutions of (1) that satisfy $\sup\big\{|u(t)| : t \geq T\big\} > 0$. We make a standing hypothesis that (1) does possess such solutions. A solution of (1) is called *oscillatory* if the set of its zeros is unbounded. Otherwise, it is said to be *nonoscillatory*. An equation itself is called *oscillatory* if all its solutions are oscillatory.

In this paper, we have been motivated by the observation that there are many papers that reduce the problem of oscillation of higher order differential equations to the oscillation of a set of second order differential equations (see, e.g., [2], [6] and [9]). Thus it is desirable to have criteria for oscillation of the second order equations and the comparison method is one of the important methods in oscillation theory of second order equations.

We are interested in comparing the oscillatory behavior of (1) with that of the equation

$$\big(l(t)u'(t)\big)' + z(t)h\big(u(w(t))\big) = 0 , \tag{2}$$

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where l, z, $w \in C([t_0,\infty))$ are positive, $h \in C(\mathbb{R})$, $h(x)x > 0$ for $x \neq 0$ and $w(t) \to \infty$ as $t \to \infty$.

We say that (1) is *in a canonical form* if

$$\int^{\infty} \frac{\mathrm{d}s}{r(s)} = \infty .$$

On the other hand, if

$$\int^{\infty} \frac{\mathrm{d}s}{r(s)} < \infty ,$$

then (1) is said to be *in a noncanonical form.*

We make use of the following functions in the remainder of this paper:

$$R(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{r(s)} , \quad \text{and} \quad L(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{l(s)} , \qquad t \geq t_0 ,$$

for the canonical case of (1) and (2), and

$$\rho(t) = \int_{t}^{\infty} \frac{\mathrm{d}s}{r(s)} , \quad \text{and} \quad \lambda(t) = \int_{t}^{\infty} \frac{\mathrm{d}s}{l(s)} , \qquad t \geq t_0 ,$$

for the noncanonical case of (1) and (2). Let R^{-1}, L^{-1}, ρ^{-1}, and λ^{-1} be the inverse functions to R, L, ρ and λ, respectively.

THEOREM 1. *A function $u(t)$ is a solution of the noncanonical equation* (1) *on $[t_0,\infty)$ if and only if the function $y(s) = su(\rho^{-1}(1/s))$ is a solution of the canonical equation*

$$y''(s) + p_1(s) f\left(\frac{y(g_1(s))}{g_1(s)}\right) = 0 , \qquad s \geq s_0 = 1/\rho(t_0) , \tag{3}$$

where

$$p_1(s) = \frac{p(\rho^{-1}(1/s)) r(\rho^{-1}(1/s))}{s^3} \quad \text{and} \quad g_1(s) = \frac{1}{\rho(g(\rho^{-1}(1/s)))} .$$

Proof. Differentiating the relation $u(t) = \rho(t) y(1/\rho(t))$ and considering $\rho'(t) = -1/r(t)$, we conclude that

$$r(t)u'(t) = -y(1/\rho(t)) + \frac{1}{\rho(t)} y'(1/\rho(t)) . \tag{4}$$

Differentiating again, we obtain

$$r(t)\big(r(t)u'(t)\big)' = \frac{1}{\rho^3(t)} y''\big(1/\rho(t)\big) . \tag{5}$$

From (5) and the transformation $t = \rho^{-1}(1/s)$, it follows that

$$r(t)\Big\{\big(r(t)u'(t)\big)' + p(t)f\big(u(g(t))\big)\Big\} = s^3\Big\{y''(s) + p_1(s)f\Big(\frac{y(g_1(s))}{g_1(s)}\Big)\Big\} .$$

Now we see that $u(t)$ is a solution of (1) on $[t_0,\infty)$ if and only if $y(s) = su\big(\rho^{-1}(1/s)\big)$ is a solution of (3) on $[s_0,\infty)$. The proof is complete. □

Note that Theorem 1 generalizes and extends Theorem 1 in [4].

COROLLARY 1. *The noncanonical equation* (1) *is oscillatory if and only if* (3) *is oscillatory.*

For the canonical form of (1) we have the following result, which is due to Ohriska [8].

THEOREM 2. *The canonical equation* (1) *is oscillatory if and only if the equation*

$$y''(t) + p_2(t)f\big(y(g_2(t))\big) = 0 \tag{6}$$

is oscillatory, where

$$p_2(t) = r\big(R^{-1}(t)\big)p\big(R^{-1}(t)\big) , \qquad \text{and} \qquad g_2(t) = R\big(g(R^{-1}(t))\big) .$$

Now we turn to (2) and its corresponding equations

$$y''(t) + z_1(t)h\Big(\frac{y(w_1(t))}{w_1(t)}\Big) = 0 \qquad \text{and} \tag{7}$$

$$y''(t) + z_2(t)h\big(w_2(t)\big) = 0 , \tag{8}$$

where

$$z_1(t) = \frac{z\big(\lambda^{-1}(1/t)\big)l\big(\lambda^{-1}(1/t)\big)}{t^3} \qquad \text{and} \qquad w_1(t) = \frac{1}{\lambda\big(w(\lambda^{-1}(1/t))\big)} ,$$

$$z_2(t) = l\big(L^{-1}(t)\big)z\big(L^{-1}(t)\big) \qquad \text{and} \qquad w_2(t) = L\big(w(L^{-1}(t))\big) .$$

For (2), (7) and (8), the results are analogous to those presented in Theorems 1 and 2.

The following comparison theorem enables us to transfer oscillation properties from (2) to (1), in case that both equations are in the canonical forms.

THEOREM 3. *Let* (1) *and* (2) *be in their canonical forms. Suppose that for all large* t *and* $x \in \mathbb{R}$

$$f(x)\operatorname{sgn} x \ge h(x)\operatorname{sgn} x\,, \tag{9}$$

$$g_2(t) \ge w_2(t)\,, \tag{10}$$

$$p_2(t) \ge z_2(t)\,, \tag{11}$$

$$h \text{ is nondecreasing}. \tag{12}$$

Then (1) *is oscillatory if* (2) *is oscillatory.*

P r o o f. In view of Theorem 2, it is sufficient to show that (6) is oscillatory. To obtain a contradiction, assume that $y(t)$ is a nonoscillatory solution of (6). Without loss of generality, we may assume that $y(t)$ is positive on $[t_0\infty)$. Since $y''(t) < 0$, a generalization of a lemma of K i g u r a d z e [5] implies that $y'(t) > 0$ for all large t, say $t \ge t_1$. Therefore, integrating (6) from t $(\ge t_1)$ to ∞, we have, in view of (9) and (11),

$$y'(t) \ge \int_t^\infty p_2(s) f\big(y(g_2(s))\big)\,\mathrm{d}s \ge \int_t^\infty z_2(s) h\big(y(g_2(s))\big)\,\mathrm{d}s\,. \tag{13}$$

Consequently, noting that $y(t)$ is increasing, (10) and (12) imply

$$y'(t) \ge \int_t^\infty z_2(s) h\big(y(w_2(s))\big)\,\mathrm{d}s\,, \qquad t \ge t_1\,.$$

Integrating the last inequality from t_1 to t, we have

$$y(t) \ge y(t_1) + \int_{t_1}^{t} \left(\int_{s_1}^{\infty} z_2(s) h\big(y(w_2(s))\big)\,\mathrm{d}s \right) \mathrm{d}s_1\,. \tag{14}$$

Let us denote the right hand side of (14) by $u(t)$. Repeated differentiation of $u(t)$ leads to

$$u''(t) + z_2(t) h\big(y(w_2(t))\big) = 0\,. \tag{15}$$

Since $y\big(w_2(t)\big) \ge u\big(w_2(t)\big)$ for all large t, say $t \ge t_2$, we see from (12) that $u(t)$ is a positive solution of the differential inequality

$$\Big\{u''(t) + z_2(t) h\big(u(w_2(t))\big)\Big\} \operatorname{sgn} u\big(w_2(t)\big) \le 0\,, \qquad t \ge t_1\,.$$

It follows from Corollary 1 of K u s a n o and N a i t o [7] that equation (8) also has a positive solution, and so, by Theorem 2, equation (2) is not oscillatory. This is a contradiction, and the proof is complete. □

We can compare Theorem 3 with the following similar comparison result, which is due to K u s a n o and N a i t o [7], and C h a n t u r i a [1].

THEOREM A. *Let* (9) *and* (12) *be satisfied. Further assume that*

$$\begin{aligned} r(t) &\le l(t), &\quad t &\ge t_0, \\ g(t) &\ge w(t), &\quad t &\ge t_0, \\ p(t) &\ge z(t), &\quad t &\ge t_0. \end{aligned} \tag{16}$$

Then (1) *is oscillatory if* (2) *is oscillatory.*

Note that, by Theorem 3, equation (1) can inherit oscillatory behavior from (2) even if (16) is violated.

Now we turn to noncanonical equations. As Kusano and Naito have emphasized in [7], comparison theorems for canonical functional equations do not work for noncanonical functional equations. We attempt to give a comparison result for noncanonical equations (1) and (2) for which $w(t) = g(t)$.

THEOREM 4. *Let* (1) *and* (2) *be noncanonical equations. Assume that* (9) *and* (12) *hold. Further assume that for all large* t

$$p_1(t) \ge z_1(t), \tag{17}$$

$$w(t) = g(t) = t. \tag{18}$$

Then (1) *is oscillatory if* (2) *is oscillatory.*

Proof. By Corollary 1, it is enough to show that (3) with $g_1(t) = t$ is oscillatory. Let $y(t)$ be an eventually positive solution of (3). Then $y''(t) < 0$, and according to a generalization of a lemma of Kiguradze [5], $y'(t) > 0$ for all large t, say $t \ge t_1$. Therefore, integrating (3) from t ($\ge t_1$) to ∞ and applying (9), (17) and (18), we have

$$y'(t) \ge \int_t^\infty p_1(s) f\left(\frac{y(s)}{s}\right) \mathrm{d}s \ge \int_t^\infty z_1(s) h\left(\frac{y(s)}{s}\right) \mathrm{d}s .$$

Integrating the last inequality from t_1 to t, we have

$$y(t) \ge y(t_1) + \int_{t_1}^{t} \left(\int_{s_1}^{\infty} z_1(s) h\left(\frac{y(s)}{s}\right) \mathrm{d}s \right) \mathrm{d}s_1 . \tag{19}$$

If we denote the right hand side of (19) by $u(t)$, then

$$u''(t) + z_1(t) h\left(\frac{y(t)}{t}\right) = 0 . \tag{20}$$

Since $y(t) \ge u(t)$ for all $t \ge t_1$, from (12), we see that $u(t)$ is a positive solution of the differential inequality

$$\left\{ u''(t) + z_1(t) h\left(\frac{u(t)}{t}\right) \right\} \operatorname{sgn} u(t) \le 0 .$$

It follows from Corollary 1 of Kusano and Naito [7] that (7) has also a positive solution, and so, by Corollary 1, equation (2) is not oscillatory. This is a contradiction, and the proof is complete. □

Next, we relax condition (18) in Theorem 4 and provide a comparison theorem between noncanonical equations (1) and (2) with different deviating arguments. However, the further assumptions on the function $h(x)$ are needed.

THEOREM 5. *Let* (1) *and* (2) *be noncanonical equations. Assume that* (9) *and* (12) *hold. Further assume that for all large* t

$$g_1(t) \geq w_1(t)\,, \tag{21}$$

$$Np_1(t)h\left(\frac{w_1(t)}{g_1(t)}\right) \geq z_1(t)\,, \qquad N \text{ is a positive constant}, \tag{22}$$

$$-h(-xy) \geq h(xy) \geq Nh(x)h(y) \qquad \text{for} \quad x > 0 \text{ and } y > 0\,. \tag{23}$$

Then (1) *is oscillatory if* (2) *is oscillatory.*

Proof. The proof runs similarly as proofs of Theorems 3 and 4. To obtain a contradiction, assume that $y(t)$ is an eventually positive solution of (3). Then again $y'(t) > 0$ and

$$y'(t) \geq \int\limits_t^\infty p_1(s)f\left(\frac{y(g_1(s))}{g_1(s)}\right)\,\mathrm{d}s \geq \int\limits_t^\infty p_1(s)h\left(\frac{y(g_1(s))}{g_1(s)}\right)\,\mathrm{d}s\,. \tag{24}$$

Since $y(t)$ is increasing, we have, in view of (12), (21) and (23),

$$\begin{aligned} h\left(\frac{y(g_1(t))}{g_1(t)}\right) &\geq h\left(\frac{y(w_1(t))}{g_1(t)}\right) = h\left(\frac{w_1(t)}{g_1(t)}\,\frac{y(w_1(t))}{w_1(t)}\right) \\ &\geq Nh\left(\frac{w_1(t)}{g_1(t)}\right)h\left(\frac{y(w_1(t))}{w_1(t)}\right). \end{aligned}$$

Combining the last inequality with (24) and (23) we obtain

$$y'(t) \geq \int\limits_t^\infty z_1(s)h\left(\frac{y(w_1(s))}{w_1(s)}\right)\,\mathrm{d}s\,. \tag{25}$$

Integrating the last inequality from t_1 to t, we have

$$y(t) \geq y(t_1) + \int\limits_{t_1}^t \left(\int\limits_{s_1}^\infty z_1(s)h\left(\frac{y(w_1(s))}{w_1(s)}\right)\,\mathrm{d}s\right)\,\mathrm{d}s_1. \tag{26}$$

If we denote the right hand side of (26) by $u(t)$, then

$$u''(t) + z_1(t)h\left(\frac{y(w_1(t))}{w_1(t)}\right) = 0\,. \tag{27}$$

Since $y(w_1(t)) \ge u(w_1(t))$ for all large t, say $t \ge t_2$, we see from (12) that $u(t)$ is a positive solution of the differential inequality

$$\left\{u''(t) + z_1(t)h\left(\frac{y(w_1(t))}{w_1(t)}\right)\right\} \operatorname{sgn} u(w_1(t)) \le 0\,.$$

Using the same arguments as those used in the proof of Theorem 4, we can see that (2) is not oscillatory. This contradiction completes the proof of the theorem. □

The following two theorems are intended to show that a canonical equation can inherit oscillatory behavior from that of a noncanonical equation and conversely.

THEOREM 6. *Let* (1) *be a canonical equation and* (2) *be a noncanonical equation. Assume that* (9), (12) *and* (23) *hold. Further assume that for all large* t

$$g_2(t) \ge w_1(t)\,, \tag{28}$$

$$N p_2(t) h(w_1(t)) \ge z_1(t)\,, \qquad N \text{ is a positive constant.} \tag{29}$$

Then (1) *is oscillatory if* (2) *is oscillatory.*

P r o o f . It is sufficient to show that (6) is oscillatory. Assume that $y(t)$ is an eventually positive solution of (6). Then again $y'(t) > 0$ on $[t_1, \infty)$, $t_1 \ge t_0$, and

$$y'(t) \ge \int_t^\infty p_2(s) f(y(g_2(s)))\, ds \ge \int_t^\infty p_2(s) h(y(g_2(s)))\, ds. \tag{30}$$

Since $y(t)$ is increasing, (12), (28) and (23) imply

$$\begin{aligned} h(y(g_2(t))) \ge h(y(w_1(t))) &= h\left(w_1(t)\frac{y(w_1(t))}{w_1(t)}\right) \\ &\ge N h(w_1(t)) h\left(\frac{y(w_1(t))}{w_1(t)}\right). \end{aligned}$$

Combining the last inequality with (30) and (29) we obtain (27). Then repeating the same arguments as in the proof of Theorem 5, we can see that (2) is not oscillatory, and this completes the proof of the theorem. □

THEOREM 7. *Let* (1) *be a noncanonical equation and* (2) *be a canonical equation. Assume that* (9), (12) *and* (25) *hold. Further assume that for all large* t

$$g_1(t) \geq w_2(t)\,, \tag{31}$$

$$Np_1(t)h\left(\frac{1}{g_1(t)}\right) \geq z_2(t)\,, \qquad N \text{ is a positive constant.} \tag{32}$$

Then (1) *is oscillatory if* (2) *is oscillatory.*

Proof. The proof runs exactly as the proof of Theorem 5, and so details are left to the reader. The main tool in proving this theorem is the following inequality which holds for a positive increasing solution of (3)

$$h\big(y(g_1(t))\big) \geq h\left(\frac{1}{g_1(t)}y(w_2(t))\right) \geq Nh\left(\frac{1}{g_1(t)}\right)h\big(y(w_2(t))\big)\,.$$

□

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