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GEOMETRIC PROOF OF THE EASY PART OF THE HOPF INVARIANT ONE THEOREM

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ABSTRACT. This paper gives a geometric proof of Adem's result that if there is a map $f\colon S^{2n-1} \to S^n$ with Hopf invariant 1, then n is a power of 2.

In 1960 Adams proved the following famous theorem (see [Ada]).

THEOREM. (Adams) *There is a map $f\colon S^{2n-1} \to S^n$ with Hopf invariant* 1 *if and only if $n = 2, 4, 8$.*

A few years earlier Adem proved the following weaker result (see [Ade]).

THEOREM. (Adem) *If there is a map $f\colon S^{2n-1} \to S^n$ with Hopf invariant* 1, *then n is a power of* 2.

Adem's theorem was proved using a description of the generators of the Steenrod algebra. Adams' theorem was first proved by using secondary cohomology operations then by using Adams' operations in K-theory. Here we shall give a geometric proof of the result of Adem.

We first recall several definitions and facts.

DEFINITION 1. Given a smooth map $f\colon S^{2n-1} \to S^n$, the *Hopf invariant* $H(f) \in \mathbb{Z}$ of f can be defined as follows. Pick two regular values of f, let us denote them by p and q, and consider their preimages $L_p = f^{-1}(p)$ and $L_q = f^{-1}(q)$. These are $n-1$ dimensional, closed, oriented submanifolds in S^{2n-1}, therefore their linking number $\operatorname{lk}(L_p, L_q)$ is well defined. Put $H(f) = \operatorname{lk}(L_p, L_q)$.

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Equivalent definition. Consider the preimage of a regular value p of f, let us denote it by L_p. This is a framed submanifold in $\mathbb{R}^{2n-1} \subset S^{2n-1}$. By Hirsch theory, there is a regular homotopy deforming the embedding $i: L_p \subset \mathbb{R}^{2n-1}$ into a selftransverse immersion $g: L_p \looparrowright \mathbb{R}^{2n-2} \subset \mathbb{R}^{2n-1}$. Then $H(f)$ is the algebraic number of the double points of g, where the double points are signed as follows. By Hirsch theory, there is actually an embedding $i': L_p \subset \mathbb{R}^{2n-1}$ isotopic to i such that its composition with the vertical projection $\mathbb{R}^{2n-1} \to \mathbb{R}^{2n-2}$ gives the immersion g. Because of this, there is an ordering on the pairs of double points of g having the same image. Namely, if $g(P) = g(Q)$, then $P < Q$ if the omitted (vertical) coordinate of P was smaller than that of Q. Therefore there is an ordering of the two branches at a double point, and this gives a sign.

The Hopf invariant depends only on the homotopy class of f and gives a homomorphism $H: \pi_{2n-1}(S^n) \to \mathbb{Z}$.

Definition 3. The *stable Hopf invariant* $H^s: \pi^s(n-1) \to \mathbb{Z}_2$ is roughly defined as the mod 2 reduction of H. More precisely, given an element $\alpha \in \pi^s(n-1)$ of the stable homotopy group of spheres, its stable Hopf invariant is defined as the mod 2 reduction of $H(\beta)$, where β is any element of $\pi_{2n-1}(S^n)$ mapped to α by the suspension homomorphism.

Definition 3 is correct, i.e. it does not depend on the choice of β. The usual argument for showing this is the following. The kernel of the suspension homomorphism $S: \pi_{2n-1}(S^n) \to \pi^s(n-1)$ is generated by the Whitehead product $[1_n, 1_n]$ of the identity map $1_n: S^n \to S^n$ and the Hopf invariant of this Whitehead product is even. (It is 2 if n is even and zero if n is odd.)

Geometrically, the correctness of the definition of H^s is even more trivial. The stable homotopy group $\pi^s(n-1)$ can be identified with the cobordism group of framed immersions of $n-1$ dimensional manifolds in $\mathbb{R}^{2n-2}$. Given a cobordism class $\alpha \in \pi^s(n-1)$, the stable Hopf invariant $H^s(\alpha)$ is the parity of the double points of any immersion from the class α. But the parity of double points is obviously the same for any two cobordant immersions. (The double lines of a joining cobordism end in the double points of the two immersions.)

By what we said above, Adem's theorem is equivalent to the following.

Proposition. *If there is a closed $n-1$ dimensional manifold M^{n-1} having an immersion $g: M^{n-1} \looparrowright \mathbb{R}^{2n-2}$ with trivial normal bundle and odd number of double points, then n is a power of 2.*

Proof of Proposition. Let $g: M^{n-1} \looparrowright \mathbb{R}^{2n-2}$ be a framed immersion with an odd number of double points. Suppose that n is not a power of 2, i.e. $n = (2r+1)2^s$ where $r \geq 1$. By Hirsch theory, g can be deformed by a regular homotopy into a selftransverse immersion g' mapping M^{n-1} into

$\mathbb{R}^{2n-2-2^s} \subset \mathbb{R}^{2n-2}$. Let $\Delta_2 \subset \mathbb{R}^{2n-2-2^s}$ be the 2^s dimensional double points manifold of the map g', let $\tilde{\Delta}_2$ be its preimage and let $l \to \Delta_2$ be the line bundle associated with the double cover $\tilde{\Delta}_2 \to \Delta_2$. The normal bundle of Δ_2 is isomorphic to $k(\varepsilon^1 \oplus l)$, where k is the codimension of the immersion g', i.e. $k = (n-1) - 2^s = r2^{s+1} - 1$. The total normal Stiefel-Whitney class of Δ_2 is $\bar{w}(\Delta_2) = \big(1 + w_1(l)\big)^k$. In particular, $\bar{w}_{2^s}(\Delta_2) = \binom{k}{2^s} w_1(l)^{2^s} = w_1(l)^{2^s}$. But $\bar{w}_{2^s}(\Delta_2) = 0$, since $\dim(\Delta_2) = 2^s$. On the other hand, by a theorem of Miller [M], $w_1(l)^{2^s}$ gives the parity of double points. □

For reader's convenience we give here a short proof of Miller's theorem. First the formulation.

THEOREM. (Miller) *Let $f\colon M^m \looparrowright \mathbb{R}^{2m-p}$ be a selftransverse immersion, $p < m/2$ with double points manifold $\Delta_2(f)$, preimage $\tilde{\Delta}_2(f) = f^{-1}(\Delta_2(f))$, and line bundle l associated with the double cover $\tilde{\Delta}_2(f) \to \Delta_2(f)$. Let $g\colon M^m \looparrowright \mathbb{R}^{2m-p+t}$ be a selftransverse immersion of the form $g(x) = \big(f(x), \phi(x)\big)$ where $\phi\colon M^m \to \mathbb{R}^t$ is a smooth map.*

Then $\Delta_2(g) \subset \Delta_2(f)$ represents the $\mathbb{Z}_2$-homology class dual to $w_1(l)^t$, i.e. $[\Delta_2(g)] = \mathcal{D}w_1^t(l) \in H_{p-t}(\Delta_2(f); \mathbb{Z}_2)$.

Proof. The double cover $\tilde{\Delta}_2(f) \to \Delta_2(f)$ can be identified with the S^0-bundle associated with the line bundle l, i.e. $\tilde{\Delta}_2(f) = S(l)$. If $x \in \tilde{\Delta}_2(f)$, then $-x$ is the other point having the same image as x under the map f. If $\tilde{\Delta}_2(g)$ is the preimage of $\Delta_2(g) \subset \Delta_2(f)$ at the projection $S(l) \to \Delta_2(f)$, then $x \in \tilde{\Delta}_2(f)$ belongs to $\tilde{\Delta}_2(g)$ if and only if $\phi(x) = \phi(-x)$.

Let us define a map $h\colon l \to \mathbb{R}^t$ from the total space of l as follows. For $x \in S(l)$ and $\alpha \in \mathbb{R}^1$ put $h(\alpha \cdot x) = \alpha\big(\phi(x) - \phi(-x)\big)$. Then $\Delta_2(g) = h^{-1}(0)$. The function h can be considered as a section of the bundle $\mathrm{Hom}(l, \varepsilon^t) \approx l^* \oplus \cdots \oplus l^* = t \cdot l^*$. But $l^* \approx l$. Therefore $\Delta_2(g)$ is the zero set of a generic section of $l \oplus \cdots \oplus l = t \cdot l$, and so it represents the homology class dual to $w_t(t \cdot l) = w_1(l)^t$. Miller's theorem and thus Adem's theorem are proved. □

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