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GEOMETRIC STRUCTURES ON TOROIDAL MAPS AND ELLIPTIC CURVES

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ABSTRACT. From the work [JONES, G. A.—SINGERMAN, D.: *Theory of maps on orientable surfaces*, Proc. London Math. Soc. (3) **37** (1978), 273–307] or [GROTHENDIECK, A.: *Esquisse d'un programme*. In: Geometric Galois Actions 1 (L. Schneps, P. Lochakeds, eds.). London Math. Soc. Lecture Note Ser. 242, Cambridge University Press, Cambridge, 1997] there is associated with every map on a surface, a geometric structure on the surface, which is either spherical, Euclidean, or hyperbolic. A surface of genus $g > 1$ necessarily has a hyperbolic structure, but the torus can have either a Euclidean or hyperbolic structure. We study the genus 1 maps which have a Euclidean structure, both from the viewpoint of graph embeddings and of elliptic curves. We also find an embedding of the complete graph K_6 which necessarily has a hyperbolic structure and where the edges are hyperbolic geodesics.

0. Introduction

When considering graph embeddings in the torus it is usual to regard the torus as a purely topological object. Sometimes, we also consider the Euclidean structure on the torus. For example in the standard embeddings of the complete graphs K_5 and K_7 drawn in Figure 8 of this paper, the tori are obtained by identifying the opposite sides of Euclidean parallelograms and as such have induced Euclidean structures. However, the more generic situation is for a graph embedding on the torus to relate to a hyperbolic structure with singularities corresponding to branch points. For example, in Figure 10 of this paper we show an embedding of K_6 on the torus where the edges are hyperbolic geodesics.

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Key words: graph embedding, geometric structure on a map, toroidal map, algebraic curve, elliptic curve, Riemann surface, Euclidean structure, hyperbolic structure.

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Another situation where this occurs is in the Grothendieck theory of dessins d'enfants ([Gro]). Here, a map on a surface of genus g corresponds to some algebraic curve of genus g defined over a number field. In the case of genus 1, the surface is a torus and the curve is an elliptic curve. Rational elliptic curves have proved very important in the solution of various diophantine problems such as Fermat's Last Theorem, and in [SSy] we showed that only 5 such curves correspond to Euclidean toroidal maps; the rest correspond to maps with a hyperbolic structure.

The purpose of this work is to briefly describe these ideas to those whose primary interest is in the theory of graph embeddings. In §1 we introduce the basic ideas involving maps and show how they relate to Riemann surfaces and algebraic curves. In §2 we introduce the idea of a geometric structure on a map. In §3 we give a brief introduction to elliptic curves. In §4 we describe the *uniform* (sometimes called combinatorially regular) maps on the torus. These are the only ones with a Euclidean structure and include the well-known regular maps on the torus. In §5 we discuss the embeddings of the complete graphs K_5, K_6 and K_7 on the torus. Now K_5 and K_7 correspond to regular maps and hence have a Euclidean structure, while for K_6 we definitely need a hyperbolic structure; an example is given where there is one hexagonal face and 8 triangular faces. In §6 we outline our proof that there are only 5 rational elliptic curves whose associated maps have a Euclidean structure.

1. Maps and Riemann surfaces

A map is informally an embedding of a connected graph $\mathcal{G}$ into an orientable surface $\mathcal{S}$ such that the connected components of $\mathcal{S} \setminus \mathcal{G}$ (called the *faces* of the map) are simply connected (see [JS1]). We allow the graph $\mathcal{G}$ to contain loops and free-edges. (A *free-edge* has only one end incident with a vertex, as shown in Figure 1. Free-edges are sometimes called semi-edges or half-edges.)

FIGURE 1. A free-edge.

Plane trees embedded in the plane and the Platonic solids embedded in the sphere are all examples of maps. Figure 2 shows a map with one vertex, one face and three edges (one of which is free) embedded in a torus. The torus is obtained by identifying opposite sides of the square.

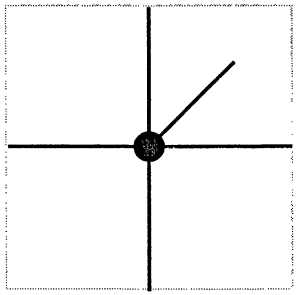

FIGURE 2. A map of genus 1.

One can associate an algebraic structure to a map $\mathcal{M}$ as follows: whenever an edge intersects a vertex we put an arrow on the edge facing that vertex; every such vertex-edge pair is called a *dart* of $\mathcal{M}$. Letting Ω denote the set of darts of $\mathcal{M}$, we define two permutations of Ω: r_0 consists of cycles formed by going round each vertex in an anticlockwise direction, while r_1 is the permutation consisting of cycles which interchange the darts on an edge or loop, and fix the dart on a free edge. The product $r_2 = (r_0 r_1)^{-1}$ consists of cycles which define anticlockwise rotations about each face of $\mathcal{M}$, where the composition is taken from left to right. We let $G = \mathrm{gp}\langle r_0, r_1\rangle \leq S^{\Omega}$ be the group generated by r_0 and r_1. Note that G is a transitive permutation group because the graph underlying $\mathcal{M}$ is connected. If r_0 and r_2 have orders m and n respectively, we say that the map $\mathcal{M}$ has *type* (m, n).

For positive integers l_0, l_1, l_2, the extended triangle group $\Gamma^*(l_0, l_1, l_2)$ is the group generated by reflections in the sides of a triangle with angles π/l_0, π/l_1, π/l_2. Note that the triangle will lie on the sphere Σ, the Euclidean plane $\mathbb{C}$ or the hyperbolic plane $\mathbb{H}$ depending on whether $1/l_0 + 1/l_1 + 1/l_2 > 1$, $= 1$ or < 1 respectively. The triangle group $\Gamma(l_0, l_1, l_2)$ is the index 2 subgroup of $\Gamma^*(l_0, l_1, l_2)$ consisting of all the orientation-preserving transformations. It is known (see [Mag]) that $\Gamma(l_0, l_1, l_2)$ has a presentation of the form

$$\Gamma(l_0, l_1, l_2) \cong \mathrm{gp}\langle x_0, x_1, x_2 \mid x_0^{l_0} = x_1^{l_1} = x_2^{l_2} = x_0 x_1 x_2 = 1\rangle .$$

These groups are the cocompact triangle groups; the quotient space is the sphere.

Given a map $\mathcal{M}$ of type (m, n) with $G = \mathrm{gp}\langle r_0, r_1\rangle$ as defined above, there is an epimorphism $\theta\colon \Gamma(m, 2, n) \to G$ given by $x_0 \mapsto r_0$, $x_1 \mapsto r_1$ and $x_2 \mapsto r_2$. If we set $G_\alpha = \{g \in G \mid \alpha g = \alpha\}$ for any $\alpha \in \Omega$, then $M = \theta^{-1}(G_\alpha)$ is called a *canonical map subgroup* for $\mathcal{M}$. The quotient space $X = \mathcal{U}/M$ (where $\mathcal{U} = \Sigma$, $\mathbb{C}$ or $\mathbb{H}$) is a Riemann surface, and furthermore $\mathcal{M}$ can be embedded naturally into X so that the edges of $\mathcal{M}$ are geodesics in X. Conversely, if $M \leq \Gamma(m, 2, n)$ is any finite index inclusion, then the Riemann surface $X = \mathcal{U}/M$ contains an embedding of some map $\mathcal{M}$ (for more details see [JS1]). (Dealing with a general triangle group $\Gamma(l_0, l_1, l_2)$ the combinatorial structures we obtain are hypermaps, not maps. However, associated with every hypermap there is a uniquely defined map (see [JS3]). Hence in the general theory it is sufficient to restrict our attention to maps.)

We now briefly describe the connection between maps and algebraic curves. If $T(x,y,z) \in \mathbb{C}[x,y,z]$ is an irreducible homogeneous polynomial with complex coefficients, then the algebraic curve

$$C_T = \{[x,y,z] \in P^2(\mathbb{C}) \mid T(x,y,z) = 0\}$$

can be normalized to obtain a compact Riemann surface X_T; conversely, given a compact Riemann surface X there exists an irreducible homogeneous polynomial $T(x,y,z) \in \mathbb{C}[x,y,z]$ such that X_T, the normalization of C_T, is isomorphic to X (see [Gri] for example). We say that a Riemann surface X is defined over a subfield $F \subseteq \mathbb{C}$ if $X \cong X_T$ for some polynomial $T(x,y,z) \in F[x,y,z]$. A complex number $\beta \in \mathbb{C}$ is an *algebraic number* if $f(\beta) = 0$ for some non-zero polynomial $f(x) \in \mathbb{Z}[x]$, and β is said to be an *algebraic integer* if $f(x)$ is a monic polynomial. The set of all algebraic numbers forms a field, denoted $\overline{\mathbb{Q}}$. Using the theorems of Belyĭ, Weil and Wolfart (see [Bel], [CIW], [Wo] or [JS3]) we can characterize those Riemann surfaces that are defined over the algebraic numbers:

THEOREM 1. *A compact Riemann surface X is defined over $\overline{\mathbb{Q}}$ if and only if $X \cong \mathcal{U}/\Lambda$, where Λ is a finite index subgroup of a cocompact triangle group and $\mathcal{U} = \Sigma$, $\mathbb{C}$ or $\mathbb{H}$.*

From the above discussion we see that a Riemann surface X defined over the algebraic numbers carries a map $\mathcal{M}$ with map subgroup Λ (more generally we may have a hypermap, but as noted above we can replace this with a map). Often we say that X *carries the map* $\mathcal{M}$. To understand this idea more clearly we introduce geometric structures on maps.

2. Geometric structures on maps

We first recall the standard theory concerning geometric structures on surfaces. There are 3 simply-connected surfaces with constant curvature, namely the sphere Σ with the spherical metric (constant positive curvature), the Euclidean plane $\mathbb{C}$ with the standard Euclidean metric (zero curvature), and the upper half-plane $\mathbb{H}$ with the Poincaré metric (constant negative curvature). The latter surface gives a model of the hyperbolic plane; the geodesics are the semicircles and straight lines orthogonal to the real axis. We often map $\mathbb{H}$ conformally onto the open unit disc $\boldsymbol{D}$ and then find the model in which the underlying space is $\boldsymbol{D}$, and the geodesics are circular arcs and straight lines orthogonal to the boundary of the disc. This model is familiar in the works of Escher.

A standard way of representing a torus is as a quotient space $\mathbb{C}/\Lambda$, where Λ is a group generated by two independent translations. This has a parallelogram, with sides spanned by the generating vectors of Λ, as its fundamental region (see §3). This means that the torus has an induced metric of zero curvature (flat metric) and each geodesic can be thought of locally as a Euclidean straight line. The parallelogram and its images under Λ then form a tiling of the plane. The standard way of obtaining a surface of genus 2 is by identifying the sides of an octagon. However, as there is no tiling of the plane by octagons, we cannot find a flat metric on a surface of genus 2. On the other hand we can find a tiling of the hyperbolic plane by octagons and from this we get a hyperbolic metric on a surface of genus 2. By analogy with the torus it is convenient to represent a surface of genus 2 as a quotient of $\boldsymbol{D}$ (with the hyperbolic metric) by a discontinuous group Λ of hyperbolic isometries acting freely on $\boldsymbol{D}$. The surface of genus 2 may now be represented as $\boldsymbol{D}/\Lambda$, and as Λ acts freely and discontinuously on $\boldsymbol{D}$, the natural projection $\pi\colon \boldsymbol{D} \to \boldsymbol{D}/\Lambda$ is a smooth covering map and induces a hyperbolic structure on the surface of genus 2. In a similar way we can obtain hyperbolic structures on all surfaces of genus $g > 2$.

Now suppose that the group Λ acts properly discontinuously on $\mathcal{U}$ (where $\mathcal{U} = \Sigma$, $\mathbb{C}$ or $\mathbb{H}$), still pairing the sides of some fundamental region, but possibly having isolated fixed points. We can still form the quotient space $\mathcal{U}/\Lambda$; this now has the structure of an *orbifold* and has *cone points* at the projections of the fixed points (which are the branch points of the cover). In 2 dimensions these orbifolds are homeomorphic to 2-dimensional manifolds, i.e. surfaces. The real difference is that at a branch point z the covering projection is locally p-to-1 for some $p > 1$. This means that if we have pq geodesic segments emanating from z at equal angles, then on the quotient surface we have q geodesic segments emanating from the projection of z.

Now, for a geometric structure on a map we want a metric on the underlying surface in which every non free-edge has the same length, every free-edge has half this length, and at every vertex of valency m each sector formed by two consecutive edges has angle $2\pi/m$. To show such a metric exists we first consider the universal maps of type (m,n). (Our notation means that every vertex has valency m and every face has valency n.) We let $\mathcal{U}$ denote one of the three simply-connected spaces of constant curvature described above. Then for each pair of integers $m, n \geq 2$, $\mathcal{U}$ carries a regular map of type (m,n), where $\mathcal{U} = \Sigma$ if $1/m + 1/n > 1/2$, $\mathcal{U} = \mathbb{C}$ if $1/m + 1/n = 1/2$, and $\mathcal{U} = \mathbb{H}$ if $1/m + 1/n < 1/2$. (By a *regular* map we mean one in which the automorphism group acts transitively on the darts.) We denote the universal map of type (m,n) by

$\hat{\mathcal{M}}(m,n)$. Some of these universal maps are very familiar. For example, in the Euclidean case $\hat{\mathcal{M}}(4,4)$ is the Gaussian integer lattice $\mathbb{Z}[\mathrm{i}]$, and $\hat{\mathcal{M}}(3,6)$ is the hexagonal lattice $\mathbb{Z}[\rho]$, where $\rho = \left(1+\sqrt{-3}\,\right)/2$.

It was shown in [JS1] that given *any* map $\mathcal{M}$ of type (m,n) on an orientable surface S (where m is the l.c.m. of the vertex valencies and n is the l.c.m. of the face sizes) there is a (possibly branched) covering map $\pi\colon \mathcal{U} \to S$ such that $\pi(\hat{\mathcal{M}}(m,n)) = \mathcal{M}$ and the branch points (if any) occur at the vertices, face-centres and edge-centres. For example, if there is a map of type $(8,12)$ with a vertex of valency 4, then there would be a branch point at a vertex of the universal map in order that a vertex of valency 8 projects to a vertex of valency 4.

Definition 2. A *geometric structure* on a map $\mathcal{M}$ of type (m,n) lying on a surface S is a pair $(\pi, \hat{\mathcal{M}}(m,n))$, where $\pi\colon \mathcal{U} \to S$ is a (possibly branched) covering, $\pi(\hat{\mathcal{M}}(m,n)) = \mathcal{M}$, and any branching of π occurs at the vertices, face-centres, and edge-centres of $\hat{\mathcal{M}}(m,n)$.

By [JS1] we then find there is a subgroup $M \le \Gamma(m,2,n)$ such that $\hat{\mathcal{M}}(m,n)/M \cong \mathcal{M}$ with S underlying the Riemann surface $X = \mathcal{U}/M$. We say that a map $\mathcal{M}$ of type (m,n) has a spherical, Euclidean, or hyperbolic structure if $\hat{\mathcal{M}}(m,n)$ lies on Σ, $\mathbb{C}$ or $\mathbb{H}$ respectively. This just depends on $1/m + 1/n$ as indicated above. (In [JS1] the groups of covering transformations are called *map subgroups*. In retrospect, these could have been called *fundamental groups of the map* since there is a close analogy with fundamental groups of orbifolds).

On each of the 3 simply-connected surfaces there is a notion of angle, so that the darts incident with a vertex v of $\hat{\mathcal{M}}(m,n)$ divide the neighbourhood of v into m sectors of angle $2\pi/m$. Every such vertex v projects to a vertex $\pi(v)$ of the map $\mathcal{M}$; if v is a branch point of order k, then the darts of $\mathcal{M}$ divide the neighbourhood of $\pi(v)$ into m/k sectors of angle $2\pi k/m$.

Definition 3. A map $\mathcal{M}$ of type (m,n) is *uniform* if every vertex of $\mathcal{M}$ has valency m, every face has size n, and it either has no free-edges or all of its edges are free.

We then have:

(i) Every uniform map on the sphere is regular.
(ii) Every Euclidean map on the torus is uniform, and conversely every uniform map on the torus has a Euclidean structure.
(iii) Every map on a surface of genus $g > 1$ has a hyperbolic structure.

Now (i) is proved as [JS1; Corollary 6.4] and (iii) follows since every surface of genus $g > 1$ has a hyperbolic structure, and never a spherical or Euclidean structure [JS2; Chapter 4]. To prove (ii) we observe that a uniform map on the torus T corresponds to an unbranched cover $\pi\colon \mathcal{U} \to T$. It is known that every cover $\pi\colon \mathbb{C} \to T$ is unbranched, and further that if $\pi\colon \mathcal{U} \to T$ is an unbranched covering by a simply-connected surface $\mathcal{U}$ then we must have $\mathcal{U} = \mathbb{C}$. Note that it is possible to have Euclidean or hyperbolic maps on the sphere, and also hyperbolic maps on the torus.

3. Elliptic curves

An elliptic curve is a cubic curve of the form

$$y^2 = ax^3 + bx^2 + cx + d\,,$$

where the polynomial on the right-hand side has distinct roots. (More generally, an elliptic curve is defined to be a non-singular curve of genus 1. However, it can be shown that such curves are birationally equivalent to curves of the above form, see Knapp [Kn].) By some easy transformations, we may normalize the equation to be in so called *Weierstrass Normal Form*

$$y^2 = 4x^3 - px - q \tag{1}$$

with

$$p^3 - 27q^2 \neq 0\,. \tag{2}$$

Now, (2) just says that the discriminant of $4x^3 - px - q$ is non-zero, which is equivalent to the roots of the cubic being distinct.

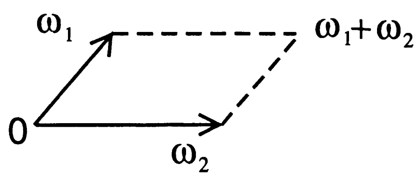

FIGURE 3.

We first wish to explain the classical connection between elliptic curves and the torus. One of the common ways of obtaining a torus is to identify the opposite edges of a rectangle. For the purposes of graph embeddings this is usually enough, but here we are also interested in conformal structures, so that we now regard the torus as being obtained by identifying the opposite edges of a parallelogram in the complex plane with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$, where $\frac{\omega_2}{\omega_1} \notin \mathbb{R}$ (see Figure 3). From these points we may construct a tessellation

$$\Lambda = \Lambda(\omega_1, \omega_2) = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$$

of the complex plane, part of which is shown in Figure 4.

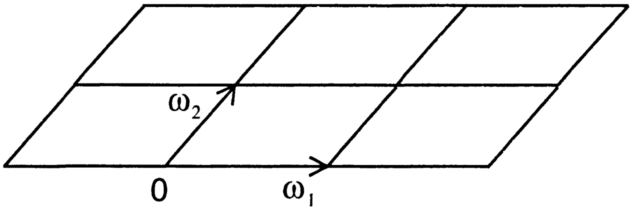

FIGURE 4.

The set Λ is called a *lattice*. Algebraically, it is a free abelian group of rank 2; geometrically, it is a discontinuous group of translations of the complex plane. It is always interesting to study functions that are invariant under the action of some group; for example trigonometric functions are invariant under the action of an infinite cyclic group. Now a meromorphic function (that is a function that is analytic except for poles) $f\colon \mathbb{C} \to \mathbb{C}$ is called *elliptic* or more strictly Λ-*elliptic* if $f(z+\omega) = f(z)$, for all $\omega \in \Lambda$. Perhaps the simplest example of an elliptic function is the Weierstrass pe-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

It can be shown that $\wp(z)$ and its derivative $\wp'(z)$ generate the field of all Λ-elliptic functions and that we have the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3\,, \tag{3}$$

where g_2, g_3 are functions of the lattice. In fact

$$g_2 = 60 \sum_{\omega \in \Lambda - \{0\}} \frac{1}{w^4}\,,$$

$$g_3 = 140 \sum_{\omega \in \Lambda - \{0\}} \frac{1}{w^6}$$

(such series are called Eisenstein series). For all this see [JS2; Chapter 3].

From equation (3), we see that we have a uniformization (or parametrization) of the elliptic curve (1) with $p = g_2$ and $q = g_3$. Moreover, it can be shown that given any p, q with $p^3 - 27q^2 \neq 0$, we can find a lattice with $p = g_2$ and $q = g_3$ (see [JS2; §6.5]). Now, the quotient space $\mathbb{C}/\Lambda$ is the torus obtained by identifying opposite sides of one of the parallelograms of the tessellation. If we let $[z]_\Lambda$ denote the Λ-orbit of z then the mapping $(\wp(z), \wp'(z)) \mapsto [z]_\Lambda$ sets up a homeomorphism (even a complex-analytic homeomorphism) from the elliptic curve (1) to the torus $\mathbb{C}/\Lambda$, where $p = g_2$ and $q = g_3$. It is in this sense that we may regard an elliptic curve to be a torus.

We now need to decide when two tori are to be regarded as equivalent. In topology, or for the purposes of graph embeddings, we use homeomorphism as the equivalence relation. For us the underlying conformal structure is important, and so our equivalence relation is conformal homeomorphism. It is the ratio

$$\tau = \omega_2/\omega_1$$

that determines the shape of the parallelogram. By interchanging ω_2 and ω_1 if necessary, we may assume that τ has positive imaginary part. Thus letting $\mathbb{H}$ denote the upper half of the complex plane, we see that each $\tau \in \mathbb{H}$ determines a torus. The lattice determining the torus is the one generated by 1 and τ. However, it is possible for different τ to determine the same torus. This is because we can change the basis of a lattice. One can easily show that $\Lambda(\omega_1, \omega_2) = \Lambda(\omega_1', \omega_2')$ if and only if there exist integers $a, b, c, d \in \mathbb{Z}$ with $ad - bc = \pm 1$ such that

$$\begin{aligned} \omega_2' &= a\omega_2 + b\omega_1 , \\ \omega_1' &= c\omega_2 + d\omega_1 \end{aligned}$$

(see [JS2; 3.4.2]). If we let $\tau = \omega_2/\omega_1$, $\tau' = \omega_2'/\omega_1'$ and choose $\tau, \tau' \in \mathbb{H}$, then the condition for τ, τ' to define the same torus is that

$$\tau' = \frac{a\tau + b}{c\tau + d} \qquad a, b, c, d \in \mathbb{Z}\,, \;\; ad - bc = 1\,. \tag{4}$$

Another way of saying this is that τ, τ' define the same torus if and only if τ, τ' lie in the same orbit under the modular group $\mathrm{PSL}(2, \mathbb{Z})$ which consists of all Möbius transformations of the form (4). The modular group has the well-known fundamental region $\mathcal{F}$ shown in Figure 5. Thus every torus can be represented by a point in $\mathcal{F}$. This representation is unique except that the transformation $\tau \mapsto \tau + 1$ identifies points on the vertical boundaries, and $\tau \mapsto -1/\tau$ identifies points on the boundary, which lie on the unit circle $|\tau| = 1$. These identified points correspond to the same torus. Particularly important for us will be the points i and $\rho = \left(1 + \sqrt{-3}\right)/2$, for as we shall see, these are the only tori to carry regular maps.

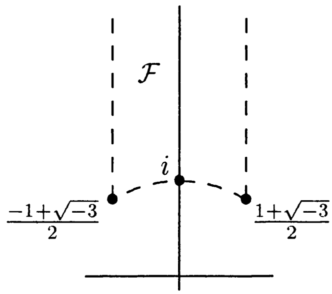

FIGURE 5.

4. Uniform maps on the torus

The regular maps of genus 1 have either square, triangular or hexagonal faces and are denoted by their Schläfli symbols $\{4,4\}_{p,q}$, $\{3,6\}_{p,q}$ or $\{6,3\}_{p,q}$ respectively (see [CMo]). We have proved that the uniform maps on the torus are precisely the toroidal maps with a Euclidean structure and so, coming from the universal Euclidean maps $\hat{\mathcal{M}}(4,4)$, $\hat{\mathcal{M}}(6,3)$ and $\hat{\mathcal{M}}(3,6)$, must have type $(4,4)$, $(6,3)$ or $(3,6)$. The following lemma, proved in [SSy], defines a correspondence between toroidal maps with a Euclidean structure and elliptic curves with moduli $\tau \in \mathbb{Q}(\mathrm{i})$ or $\tau \in \mathbb{Q}(\rho)$.

LEMMA 4. *The values of $\tau \in \mathbb{H}$ that correspond to maps on the torus with a Euclidean structure are precisely those for which $\tau = (p+\mathrm{i}q)/r$ or $\tau = (p+\rho q)/r$, where $p,q,r \in \mathbb{Z}$ and $q,r > 0$.*

Elliptic curves with moduli $\tau \in \mathbb{Q}(\mathrm{i})$ correspond to sublattices of the Gaussian integer lattice

$$\Lambda(1,\mathrm{i}) = \{m + n\,\mathrm{i} \mid m,n \in \mathbb{Z}\}.$$

Given a sublattice $\Lambda \leq \Lambda(1,\mathrm{i})$ the corresponding uniform map of type $(4,4)$ is obtained by taking the natural projection $\Lambda(1,\mathrm{i}) \to \mathbb{C}/\Lambda$. For example if we take $\Lambda(1-\mathrm{i}, 2+3\,\mathrm{i}) \leq \Lambda(1,\mathrm{i})$, then the corresponding map is shown in Figure 6.

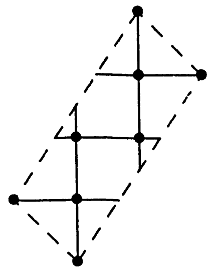

FIGURE 6.

Elliptic curves with moduli $\tau \in \mathbb{Q}(\rho)$ correspond to sublattices of

$$\Lambda(1,\rho) = \{m + n\rho \mid m, n \in \mathbb{Z}\},$$

where $\rho = (1+\sqrt{-3}\,)/2$. If we consider a sublattice $\Lambda \leq \Lambda(1,\rho)$, the corresponding uniform map of type $(6,3)$ is obtained by taking the natural projection $\Lambda(1,\rho) \to \mathbb{C}/\Lambda$. For example if we take the lattice $\Lambda(2,2\rho) \leq \Lambda(1,\rho)$, then the corresponding map of type $(6,3)$ is shown in Figure 7(a). Every uniform map of type $(3,6)$ occurs as the dual of some uniform map of type $(6,3)$, and the map of type $(3,6)$ corresponding to the lattice $\Lambda(2,2\rho)$ is shown in Figure 7(b). The maps in Figure 7 form a dual pair.

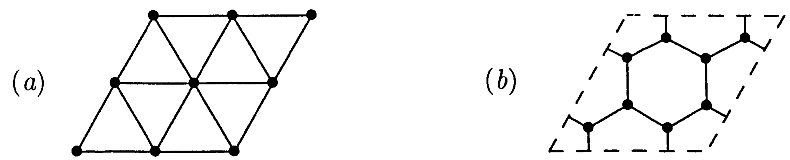

FIGURE 7.

We recall that elliptic curves are parametrized by the set of moduli in the modular fundamental region $\tau \in \mathcal{F}$ (we assume the equivalence relation on the boundary points of $\mathcal{F}$ induced by the side-pairing transformations); using this classification of elliptic curves we can derive a classification of the toroidal uniform maps of type $(4,4)$ and $(6,3)$ extending the notation of Coxeter and Moser (for precise details see [Sy]).

Given a sublattice $\Lambda \leq \Lambda(1,\mathrm{i})$ we choose a basis $\Lambda = \Lambda(\alpha+\beta\,\mathrm{i}, \gamma+\delta\,\mathrm{i})$ such that $\tau = (\gamma+\delta\,\mathrm{i})/(\alpha+\beta\,\mathrm{i}) \in \mathcal{F}$. Writing

$$\begin{aligned}\alpha+\beta\,\mathrm{i} &= (p+q\,\mathrm{i})(a+b\,\mathrm{i}),\\ \gamma+\delta\,\mathrm{i} &= (p+q\,\mathrm{i})(c+d\,\mathrm{i}),\end{aligned}$$

where $(a+b\mathrm{i})$ and $(c+d\mathrm{i})$ are coprime Gaussian integers, we use the notation

$$\left\{\frac{c+d\mathrm{i}}{a+b\mathrm{i}}\right\}_{p+q\mathrm{i}}$$

to describe the corresponding uniform map of type $(4,4)$. Similarly, every genus 1 uniform map of type $(6,3)$ can be expressed in the form

$$\left\{\frac{c+d\rho}{a+b\rho}\right\}_{p+q\rho},$$

where $\tau=(c+d\rho)/(a+b\rho)\in\mathcal{F}$ and $a+b\rho$, $c+d\rho$ are coprime in the ring $\mathbb{Z}[\rho]$. The map in Figure 6 can be expressed as $\{\frac{2+3\mathrm{i}}{1-i}\}_1$ while the map in Figure 7(a) has the form $\{\rho\}_2$. Hence every toroidal uniform map of type $(4,4)$ or $(6,3)$ has the form $\{\tau\}_{p+q\mathrm{i}}$ for $\tau\in\mathbb{Q}(\mathrm{i})$ or $\{\tau\}_{p+q\rho}$ for $\tau\in\mathbb{Q}(\rho)$. The following lemma is proved in [Sy]:

Lemma 5.

(i) *Let $\tau,\tau'\in\mathbb{Q}(\mathrm{i})$ with $\tau,\tau'\in\mathcal{F}$. Then the uniform maps $\{\tau\}_{p+q\mathrm{i}}$ and $\{\tau'\}_{p'+q'\mathrm{i}}$ are isomorphic if and only if $\tau=\tau'$ (or are equivalent boundary points) and $p+q\mathrm{i}$, $p'+q'\mathrm{i}$ are associates in $\mathbb{Z}[\mathrm{i}]$.*

(ii) *Let $\tau,\tau'\in\mathbb{Q}(\rho)$ with $\tau,\tau'\in\mathcal{F}$. Then the uniform maps $\{\tau\}_{p+q\rho}$ and $\{\tau'\}_{p'+q'\rho}$ are isomorphic if and only if $\tau=\tau'$ (or are equivalent boundary points) and $p+q\rho$, $p'+q'\rho$ are associates in $\mathbb{Z}[\rho]$.*

In particular, if $\tau=\mathrm{i}$ or $\tau=\rho$, we obtain the regular maps

$$\{\mathrm{i}\}_{p+q\mathrm{i}}\longleftrightarrow\{4,4\}_{p,q}\qquad\textit{or}\qquad\{\rho\}_{p+q\rho}\longleftrightarrow\{3,6\}_{p,q}$$

in the notation of [CMo].

5. Toroidal embeddings of some complete graphs

We now look at some examples involving both Euclidean and hyperbolic maps on the torus. It is well-known that the complete graph K_n can be embedded as a toroidal map for $n=5,6,7$. Figure 8 shows the complete graphs K_5 and K_7 embedded into the torus as the regular maps $\{\mathrm{i}\}_{2+\mathrm{i}}$ and $\{\rho\}_{2+\rho}$ respectively; in particular we note that they are maps with Euclidean structures.

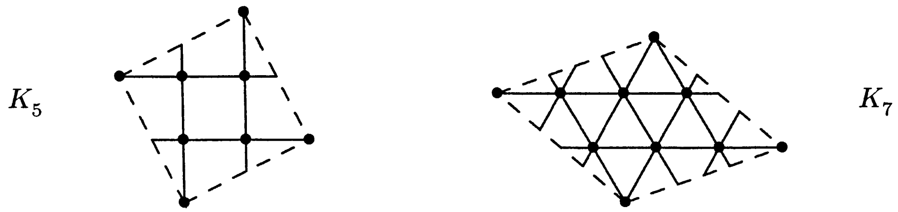

FIGURE 8.

One might ask about the embedding of K_6 into the torus. Of course, the obvious way of obtaining K_6 on the torus is to take the genus 1 embedding of K_7 (shown in Figure 9(a) with a slightly shifted fundamental parallelogram) and to remove one vertex and all incident edges, as shown in Figure 9(b). We note, however, that the Euclidean structure of the map has now been lost, since the vertices of the map have valency 5 and not all the angles are equal to $2\pi/5$ (see the note before Definition 3).

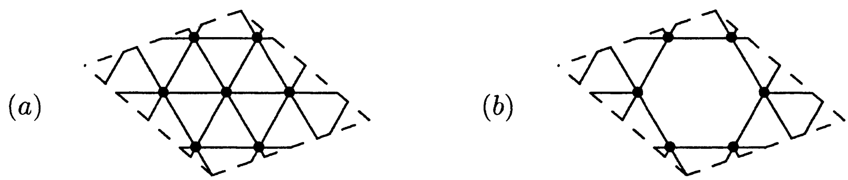

FIGURE 9.

In Figure 9(b) all regions are hexagons or triangles, and all vertices have valency 5. Therefore, the genus 1 embedding of K_6 must have a hyperbolic structure, and come from the universal map $\hat{\mathcal{M}}(5,6)$. In Figure 10 we show such a hyperbolic embedding of K_6 into the torus. The map has 30 darts, so its associated map subgroup $\Lambda \leq \Gamma(5,2,6)$ must have index 30. The map is found by gluing together 30 fundamental regions of the $\Gamma(5,2,6)$ triangle group. Observe that the torus is now obtained by folding up a hyperbolic 18-gon, and contains a cone point in the centre of each of the triangular faces. However, as can be seen from Figure 10, the polygon is "essentially" still a hexagon with opposite sides identified. (For those interested in Fuchsian groups, the map subgroup has signature $(1;2,2,2,2,2,2,2,2) < \Gamma(5,2,6)$, where the eight periods 2 correspond to the eight cone points.)

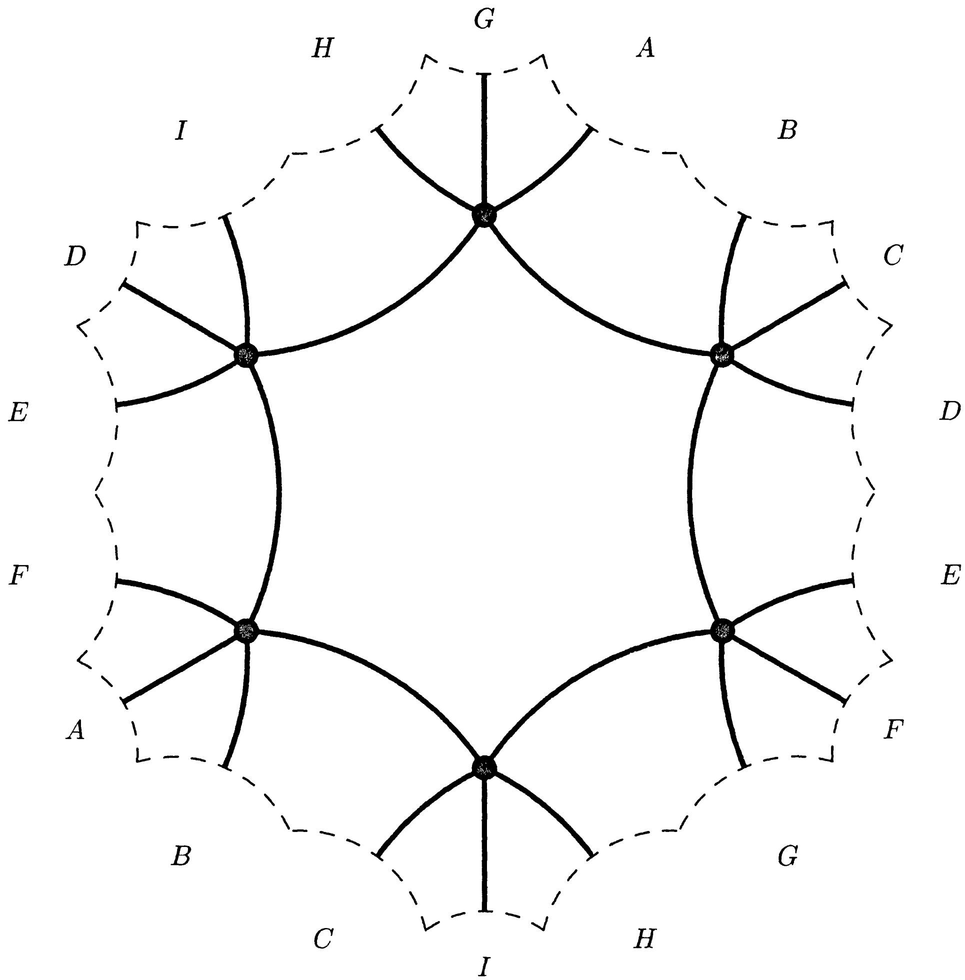

FIGURE 10. A hyperbolic embedding of K_6 into the torus.

6. Elliptic curves defined over $\overline{\mathbb{Q}}$

We return to elliptic curves. We would really like to know when a torus corresponds to an elliptic curve defined over the rationals $\mathbb{Q}$. This question is too difficult, but there is a beautiful answer to the question as to when an elliptic curve is defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers. This is a special case of Theorem 1:

THEOREM 6. *A torus defines an elliptic curve defined over $\overline{\mathbb{Q}}$ if and only if it carries a map.*

Graph theorists might object to this, since in their context every torus carries a map! What we mean here is that the torus carries a map with a geometric structure as described in §2. This geometric structure then defines a map subgroup, which in turn defines a torus. Belyĭ's Theorem then implies that this torus corresponds to an elliptic curve defined over $\overline{\mathbb{Q}}$. Our aim is to give some examples so that we can relate the value of τ defining the torus to the equation of the elliptic curve. In order to do this, we introduce the remarkable function $j(\tau)$, called the *elliptic modular function*. This is defined as a function on $\mathbb{H}$ as follows:

$$j(\tau) = \frac{1728 g_2^3}{g_2^3 - 27 g_3^2}\,.$$

We note that the functions g_2, g_3 are functions of a lattice, and not of a complex number. However, if we pass from a lattice to a similar lattice by multiplication by a non-zero constant c, then this constant cancels in the above formula. Hence $j(\tau)$ is really a function of the similarity class of the lattice and so depends only on one complex number $\tau \in \mathbb{H}$.

Properties of the j-function:

(a) $j(T(\tau)) = j(\tau)$ for all $T \in \mathrm{PSL}(2,\mathbb{Z})$. Thus j is invariant with respect to the modular group. Also $j\colon \mathbb{H} \to \mathbb{C}$ is an analytic function.

(b) $j(\tau_1) = j(\tau_2)$ if and only if there exists $T \in \mathrm{PSL}(2,\mathbb{Z})$ such that $T(\tau_1) = \tau_2$. (Thus j is one-to-one on the fundamental region $\mathcal{F}$, except that j will take the same value on equivalent boundary points.) Given an elliptic curve, we can determine a value of τ and now we see that the values of $j(\tau)$ determine this curve.

(c) If τ is a quadratic imaginary (i.e. $a\tau^2 + b\tau + c = 0$ for $a, b, c \in \mathbb{Z}$, where $a > 0$, $(a, b, c) = 1$ and $b^2 - 4ac < 0$), then $j(\tau)$ is an algebraic integer of degree $h(d)$, where $d = b^2 - 4ac < 0$ is the *discriminant* of τ, and $h(d)$ is the *class number* of d as defined below.

The class number.

A primitive (binary) quadratic form is a function of x and y of the form

$$ax^2 + bxy + cy^2 ,$$

where $a, b, c \in \mathbb{Z}$ and $(a, b, c) = 1$. A fundamental problem in number theory is to determine which integers are represented by such a form (for example, which integers are sums of two squares). The *discriminant* of the above form is defined by $d = b^2 - 4ac$. If we perform a change of variable by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} ,$$

where $p, q, r, s \in \mathbb{Z}$ and $ps - qr = 1$, then we get a new quadratic form $AX^2 + BXY + CY^2$, which we call equivalent to the above form; they clearly represent the same set of integers. Under such a transformation it can also be shown that the discriminant d is unchanged. We define the class number $h(d)$ to be the number of equivalence classes of primitive quadratic forms of discriminant d. A classical problem going back to the time of Gauss is to find the values of $d < 0$ for which $h(d) = 1$. This was finally solved around 1960 by Baker, Heegner and Stark who found that the values of $d < 0$ for which $h(d) = 1$ are $d = -3, -4, -7, -8, -11, -12, -16, -19, -27, -28, -43, -67, -163$. ([St]).

We now use these results to find the values of τ representing Euclidean toroidal maps that give rational elliptic curves. Lemma 4 tells us in this case that $\tau = (p+iq)/r$ or $\tau = (p+\rho q)/r$, where $p, q, r \in \mathbb{Z}$ and $q, r > 0$ (we will also assume that $(p, q, r) = 1$). Thus τ is a quadratic imaginary with discriminant d. If the elliptic curve corresponding to τ is rational then by property (c) of the j-function above, $h(d) = 1$. Now, if $\tau = (p + iq)/r$, then

$$(r\tau - p)^2 + q^2 = 0$$

and thus

$$r^2\tau^2 - 2pr\tau + (p^2 + q^2) = 0 .$$

Therefore, the discriminant

$$d = 4p^2r^2 - 4(p^2 + q^2)r^2 = -4q^2r^2$$

and thus the discriminant is -4 times a square. Similarly, if $\tau = (p + \rho q)/r$, then we find that the discriminant is -3 times a square. By comparing this with the solution of the class number 1 problem above, we obtain

$$d = -3, -4, -12, -16, -27 .$$

Thus we have proved the following theorem:

THEOREM 7. *There are 5 rational elliptic curves that correspond to Euclidean toroidal maps. All other rational elliptic curves (by Belyĭ's Theorem) correspond to hyperbolic toroidal maps.*

Given d, it is easy using the above computations to find the value of τ in the fundamental region $\mathcal{F}$ with discriminant d. The values of $j(\tau)$ can then be found from the classical literature, and knowing $j(\tau)$ we can find the equation of the elliptic curve. We include the details in Table 1. In [SSy] we also found those elliptic curves defined over quadratic and cubic extensions of $\mathbb{Q}$ which correspond to Euclidean toroidal maps.

TABLE 1. The five rational elliptic curves which correspond to Euclidean toroidal maps.

d	τ	$j(\tau)$	Elliptic curve E_τ
-3	ρ	0	$y^2 = 4x^3 - 1$
-4	i	1728	$y^2 = 4x^3 - x$
-12	$-1 + 2\rho$	54000	$y^2 = 4x^3 - 15x - 11$
-16	$2\mathrm{i}$	287496	$y^2 = 4x^3 - 11x - 7$
-27	$-1 + 3\rho$	-12288000	$y^2 = 4x^3 - 120x - 253$

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