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ON UNSTABLE NEUTRAL DIFFERENCE EQUATIONS WITH "MAXIMA"

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ABSTRACT. The neutral type difference equation

$$\Delta(x_n - px_{n-\tau}) = q_n \max_{s\in[n-\sigma,n]} x_s\,, \qquad n = 0,1,2,\dots\,,$$

where $p \in \mathbb{R}$, τ is a positive integer, σ is a nonnegative integer, $\{q_n\}_{n=0}^{\infty}$ is a nonnegative real sequence is studied. The existence and asymptotic properties of nonoscillatory solutions are considered. Some oscillation results are also obtained.

1. Introduction

In this paper we study the following neutral type difference equation

$$\Delta(x_n - px_{n-\tau}) = q_n \max_{s\in[n-\sigma,n]} x_s\,, \qquad n = 0,1,2,\dots\,, \tag{1}$$

where $p \in \mathbb{R}$, τ is a positive integer, σ is a nonnegative integer, $\{q_n\}_{n=0}^{\infty}$ is a nonnegative sequence and not identical with the zero sequence. Let $\mu = \max\{\tau, \sigma\}$. By a *solution* of equation (1) we mean a real sequence $\{x_n\}_{n=-\mu}^{\infty}$ which satisfies (1) for all sufficiently large n and is not eventually identically zero. Such a solution is said to be *nonoscillatory* if it is eventually positive or eventually negative. Otherwise it is said to be *oscillatory*.

It is easy to see that $\{x_n\}_{n=-\mu}^{\infty}$ is an eventually positive solution of equation

$$\Delta(x_n - px_{n-\tau}) = q_n x_{n-\sigma}\,, \qquad n = 0,1,2,\dots\,, \tag{2}$$

if and only if $\{-x_n\}_{n=-\mu}^{\infty}$ is its eventually negative solution. However, such a property is not valid for equation (1). Indeed, $\{-x_n\}_{n=-\mu}^{\infty}$ is an eventually negative solution of the equation

$$\Delta(x_n - px_{n-\tau}) = q_n \min_{s\in[n-\sigma,n]} x_s\,, \qquad n = 0,1,2,\dots\,. \tag{3}$$

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Thus, all solutions of (1) are oscillatory if and only if both (1) and (3) do not have any eventually positive solutions.

Nonlinear functional equations involving the maximum function are important since they are often met in the applications, for instance in the theory of automatic control, see e.q. [6]. Some of the qualitative theory of these equations has been developed recently, see for example [2] [4], [8], [9]. In this paper we study the existence and asymptotic behavior of nonoscillatory solutions of equation (1). The difference between the asymptotic properties of eventually positive and eventually negative solutions is illustrated by some examples. Some oscillation results are also obtained.

For the sake of convenience, all inequalities are assumed to hold for all sufficiently large n.

2. Existence of nonoscillatory solutions

In this section, we establish the existence and growth conditions of nonoscillatory solutions of equation (1). We need the following well-known theorem of Stolz, which is a discrete analog of l'Hospital's rule (see [1; Theorem 1.8.9]).

LEMMA 1 (STOLZ'S THEOREM). *Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n-1}^{\infty}$ be two real sequences such that $v_n > 0$ and $\Delta v_n > 0$ for all large n. If $\lim\limits_{n\to\infty} v_n = \infty$ and $\lim\limits_{n\to\infty} \frac{\Delta u_n}{\Delta v_n} = c$, where c may be infinite, then $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = c$.*

THEOREM 1. *Let $p \geq 0$. Then the equation (1) always has an eventually positive solution.*

Proof. For every nonnegative sequence $\{q_n\}_{n=0}^{\infty}$ which is eventually not identical with the zero sequence one can find a positive sequence $\{h_n\}_{n=0}^{\infty}$ such that

$$\sum_{i=0}^{\infty} q_i h_i = \infty$$

and

$$\lim_{n\to\infty} \frac{q_n}{\sum\limits_{i=0}^{n-1} q_i h_i} = 0 .$$

Now, we define a sequence

$$v_n = 2^{\sum\limits_{j=0}^{n-1} 2^{\sum\limits_{i=0}^{j-1} q_i h_i}} . \tag{4}$$

It is easy to check that

$$\lim_{n\to\infty} \frac{v_{n-\tau}}{v_n} = 0\,. \tag{5}$$

By the Stoltz theorem we have

$$\lim_{n\to\infty} \frac{1}{v_n} \sum_{i=0}^{n-1} q_i \max_{s\in[i-\sigma,i]} v_s = 0\,. \tag{6}$$

Let l_∞ be the Banach space of all real bounded sequences $y = \{y_n\}_{n=n_1}^{\infty}$ with the supremum norm and let

$$S = \bigl\{y \in l_\infty :\ (\forall n \ge n_1)(0 \le y_n \le 1)\bigr\}\,.$$

Clearly, S is a bounded, convex and closed subset of l_∞. Now, we define an operator $T\colon S \to l_\infty$ by:

$$(Ty)_n = \begin{cases} \frac{1}{2v_n} + p\frac{v_{n-\tau}y_{n-\tau}}{v_n} + \frac{1}{v_n}\sum\limits_{i=N}^{n-1} q_i \max\limits_{s\in[i-\sigma,i]} v_s y_s\,, & n \ge N\,,\\ \frac{n}{N}(Ty)_N + \bigl(1-\frac{n}{N}\bigr)\,, & n_1 \le n \le N\,, \end{cases} \tag{7}$$

where $n_1 = N - \mu$ and N is chosen so large that

$$p\frac{v_{n-\tau}}{v_n} + \frac{1}{v_n}\sum_{i=N}^{n-1} q_i \max_{s\in[i-\sigma,i]} v_s \le \frac{1}{2} \qquad \text{for} \quad n \ge N\,.$$

We note that, in view of (5) and (6) such an integer N does exist. Thus $TS \subset S$. Let $y^1, y^2 \in S$. Then

$$\begin{aligned} \bigl|(Ty^1)_n - (Ty^2)_n\bigr| &\le p\frac{v_{n-\tau}}{v_n}\bigl|y^1_{n-\tau} - y^2_{n-\tau}\bigr| + \frac{1}{v_n}\sum_{i=N}^{n-1} q_i \max_{s\in[i-\sigma,i]} v_s\,\bigl|y^1_i - y^2_i\bigr| \\ &\le \frac{1}{2}\bigl\|y^1 - y^2\bigr\|\,, \qquad n \ge N\,, \end{aligned}$$

and

$$\begin{aligned} \bigl\|Ty^1 - Ty^2\bigr\| &= \sup_{n\ge n_1}\bigl|Ty^1_n - Ty^2_n\bigr| = \sup_{n\ge N}\bigl|Ty^1_n - Ty^2_n\bigr| \\ &\le \frac{1}{2}\bigl\|y^1 - y^2\bigr\|\,, \end{aligned}$$

which shows that T is a contraction on S. Hence, there exists $y \in S$ such that $Ty = y$. Then, we have

$$y_n = \begin{cases} \frac{1}{2v_n} + p\frac{v_{n-\tau}y_{n-\tau}}{v_n} + \frac{1}{v_n}\sum\limits_{i=N}^{n-1} q_i \max\limits_{s\in[i-\sigma,i]} v_s y_s\,, & n \ge N\,,\\ \frac{n}{N}(Ty)_N + \bigl(1-\frac{n}{N}\bigr)\,, & n_1 \le n \le N\,. \end{cases}$$

Obviously $y_n > 0$ for $n \geq n_1$. Now we set $x_n = v_n y_n$. Then

$$x_n - p x_{n-\tau} = \frac{1}{2} + \sum_{i=N}^{n-1} q_i \max_{s\in[i-\sigma,i]} x_s , \qquad n \geq N . \tag{8}$$

Therefore, $\{x_n\}_{n=n_1}^{\infty}$ is a positive solution of equation (1). This completes the proof. □

To prove the next theorem we need some preparatory results.

Lemma 2. (see [5]) *Let $x, z\colon \mathbb{N} \to \mathbb{R}$ be such that*

$$z_n = x_n - p x_{n+k} , \qquad n \geq \max\{0, -k\} ,$$

where $p \in \mathbb{R}$ and k is an integer. Assume that $\{x_n\}_{n=0}^{\infty}$ is bounded and $\lim_{n\to\infty} z_n = l \in \mathbb{R}$ exists. Then the following statements hold:

(i) *if $p = 1$, then $l = 0$;*
(ii) *if $|p| \neq 1$, then the sequence $\{x_n\}_{n=0}^{\infty}$ is convergent and $\lim_{n\to\infty} x_n = \frac{l}{1-p}$.*

Lemma 3. *Assume that $p \geq 0$. Let $\{x_n\}_{n=-\mu}^{\infty}$ be a positive solution of equation (1) and let $x_n - p x_{n-\tau} \geq 0$. Then $\{x_n\}_{n=-\mu}^{\infty}$ is one of the following types of asymptotic behavior:*

a) $\lim_{n\to\infty} x_n = L \neq 0$;
b) $\lim_{n\to\infty} x_n = \infty$.

Proof. Let $\{x_n\}_{n=-\mu}^{\infty}$ be a positive solution of (1). Set $z_n = x_n - p x_{n-\tau}$. Then $z_n \geq 0$ and $\Delta z_n \geq 0$, and the sequence $\{\Delta z_n\}_{n=0}^{\infty}$ is eventually not identical with the zero sequence since $\{q_n\}_{n=0}^{\infty}$ is eventually not identical with the zero sequence. Hence, since there exists an index n_i such that $\Delta z_{n_i} > 0$, $\{z_n\}_{n=0}^{\infty}$ is eventually a nondecreasing sequence and we have $0 < \lim_{n\to\infty} z_n = l \leq \infty$.

Let $l = \infty$. Since $z_n = x_n - p x_{n-\tau} \leq x_n$, we get $\infty = \lim_{n\to\infty} z_n \leq \lim_{n\to\infty} x_n$, i.e. $\lim_{n\to\infty} x_n = \infty$.

If $l < \infty$ and $\{x_n\}_{n=-\mu}^{\infty}$ is bounded, then Lemma 2 implies that $\lim_{n\to\infty} x_n$ exists when $p \neq 1$. But $x_n \geq z_n$ and $\lim_{n\to\infty} z_n > 0$, so $\lim_{n\to\infty} x_n = 0$ is impossible. When $p = 1$, from Lemma 2 we have $l = 0$, which is a contradiction. If $l < \infty$ and $\{x_n\}_{n=-\mu}^{\infty}$ is unbounded, then there exists a subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ of $\{x_n\}_{n=-\mu}^{\infty}$ such that

$$x_{n_k} = \max_{0\leq n\leq n_k} x_n \qquad \text{and} \qquad \lim_{k\to\infty} x_{n_k} = \infty .$$

For $p \in (0,1)$ we have $z_{n_k} \geq x_{n_k}(1-p) \to \infty$ as $k \to \infty$, which contradicts the boundedness of $\{z_n\}_{n=0}^{\infty}$.
For $p = 1$, we have $x_n \geq x_{n-\tau} + \frac{l}{2} \geq \cdots \geq x_{n-k\tau} + k\frac{l}{2} \to \infty$ as $k \to \infty$.
For $p > 1$, we have $x_n \geq px_{n-\tau} \geq \cdots \geq p^k x_{n-k\tau} \to \infty$ as $k \to \infty$.
This implies that b) holds. The proof is complete. □

From Theorem 1 and the proof of Lemma 3 we obtain following result.

THEOREM 2. *Let $p \geq 0$. Then, based on the range of p we have:*

(i) *if $p = 1$, equation* (1) *has an unbounded positive solution $\{x_n\}_{n=-\mu}^{\infty}$ with $\lim\limits_{n\to\infty} x_n = \infty$;*

(ii) *if $0 \leq p < 1$, and*

$$\sum_{i=0}^{\infty} q_i = \infty\,, \tag{9}$$

then equation (1) *has a positive solution $\{x_n\}_{n=-\mu}^{\infty}$ with $\lim\limits_{n\to\infty} x_n = \infty$;*

(iii) *if $p > 1$, equation* (1) *has an unbounded positive solution $\{x_n\}_{n=-\mu}^{\infty}$ which tends to infinity exponentially.*

Proof. From Theorem 1 there exists a positive solution $\{x_n\}_{n=-\mu}^{\infty}$ of equation (1). By (8), it follows that

$$x_n - px_{n-\tau} > 0 \qquad \text{for} \qquad n \geq N\,.$$

(i) and (iii) follow immediately from the proof of Lemma 3. The assertion (ii) follows from (8) and (9) directly. Indeed, by (8) we have

$$x_n \geq x_n - px_{n-\tau} \geq \sum_{i=N}^{n-1} q_i \max_{s\in[i-\sigma,i]} x_s\,.$$

Let us denote

$$z_n = x_n - px_{n-\tau}\,.$$

Then $x_n \geq z_n$ and by (1), $\Delta z_n \geq 0$. Hence, from the above inequality we get

$$x_n \geq \sum_{i=N}^{n-1} q_i \max_{s\in[i-\sigma,i]} z_s \geq z_N \sum_{i=N}^{n-1} q_i\,, \qquad n \geq N\,.$$

Letting $n \to \infty$, by (9), we get $\lim\limits_{n\to\infty} x_n = \infty$. The proof is complete. □

THEOREM 3. *Assume $p \geq 0$, $p \neq 1$ and $\sum_{n=0}^{\infty} q_i < \infty$. Then, the equation* (1) *has a bounded positive solution.*

Proof. Let l_∞ be the Banach space of all real bounded sequences $x = \{x_n\}_{n=n_1}^{\infty}$ with the supremum norm. We need to consider the following two cases: $0 \leq p < 1$ and $p > 1$.

Case 1: $0 \leq p < 1$.

Let us choose a positive integer N sufficiently large such that $N - \mu \geq n_1$, and

$$\sum_{n=N}^{\infty} q_i \leq \frac{1-p}{2}.$$

We define a subset S of l_∞ as

$$S = \left\{x \in l_\infty : \ (\forall n \geq N)\big(\tfrac{1}{2} \leq x_n \leq 1\big)\right\}.$$

Clearly, S is a bounded, convex and closed subset of l_∞. Now, we define an operator $T\colon S \to l_\infty$ by:

$$(Tx)_n = \begin{cases} 1 - p + px_{n-\tau} - \sum\limits_{i=n}^{\infty} q_i \max\limits_{s\in[i-\sigma,i]} x_s\,, & n \geq N + \tau\,, \\ (Tx)_{N+\tau}\,, & n_1 \leq n \leq N + \tau\,. \end{cases} \tag{10}$$

For every $x \in S$, $n \geq N$ we have

$$(Tx)_n \leq 1 - p + p = 1\,,$$

and

$$(Tx)_n \geq 1 - p + \frac{1}{2}p - \frac{1}{2}(1-p) = \frac{1}{2}\,.$$

Hence, $TS \subset S$. Let $x^1, x^2 \in S$. Then

$$\begin{aligned} \big|(Tx^1)_n - (Tx^2)_n\big| &\leq p\big|x^1_{n-\tau} - x^2_{n-\tau}\big| + \sum_{i=n}^{\infty} q_i \max_{s\in[i-\sigma,i]} \big|x^1_s - x^2_s\big| \\ &\leq p\big\|x^1 - x^2\big\| + \big\|x^1 - x^2\big\| \sum_{i=n}^{\infty} q_i \\ &\leq \big\|x^1 - x^2\big\|\Big(p + \frac{1-p}{2}\Big) = \frac{1+p}{2}\big\|x^1 - x^2\big\|\,, \qquad n \geq n_1\,, \end{aligned}$$

and

$$\big\|Tx^1 - Tx^2\big\| = \sup_{n\geq n_1} \big|(Tx^1)_n - (Tx^2)_n\big| \leq \frac{1+p}{2}\big\|x^1 - x^2\big\|\,,$$

which shows that T is a contraction on S. Hence, there exists $x \in S$ such that $Tx = x$. It is easy to see that $\{x_n\}_{n=n_1}^{\infty}$ is a positive bounded solution of equation (1).

Case 2: $p > 1$.
Let N be so large that $N - \mu \geq n_1$ and

$$\sum_{n=N}^{\infty} q_i \leq \frac{p-1}{2} \,.$$

Define a subset S of l_∞ and a mapping T on S as follows:

$$S = \left\{x \in l_\infty : \ (\forall n \geq N)(p \leq x_n \leq 2p)\right\}$$

and

$$(Tx)_n = \begin{cases} p - 1 + \frac{1}{p} x_{n+\tau} + \frac{1}{p} \sum\limits_{i=n+\tau}^{\infty} q_i \max\limits_{s \in [i-\sigma, i]} x_s \,, & n \geq N + \tau \,, \\ (Tx)_{N+\tau} \,, & n_1 \leq n \leq N + \tau \,. \end{cases} \tag{11}$$

It is easy to show that $TS \subset S$ and

$$\left\|Tx^1 - Tx^2\right\| \leq \frac{1+p}{2p} \left\|x^1 - x^2\right\|$$

for $x^1, x^2 \in S$. Then there exists an element $x \in S$ such that $Tx = x$. Clearly, $\{x_n\}_{n=n_1}^{\infty}$ is a positive bounded solution of (1). □

Remark 1. For eventually negative solutions of equation (3), we can also obtain similar results. They are omitted.

3. Asymptotic behavior of nonoscillatory solutions

In this section, we will obtain asymptotic properties of nonoscillatory solutions of equation (1).

THEOREM 4. *Let $p \geq 0$ and $x_n - px_{n-\tau} > 0$. Assume that* (9) *holds and $\{x_n\}_{n=-\mu}^{\infty}$ is an eventually positive solution of equation* (1). *Then $\lim\limits_{n\to\infty} x_n = \infty$.*

Proof. Let $\{x_n\}_{n=-\mu}^{\infty}$ be an eventually positive solution of (1). Set

$$z_n = x_n - px_{n-\tau} \,. \tag{12}$$

Then $z_n > 0$, $\Delta z_n \geq 0$ and $\{z_n\}_{n=0}^{\infty}$ is eventually not identically zero. Therefore there exists a constant $c > 0$ such that $x_n \geq z_n > c$. Summing (1) from n_0 to $n-1$, with n_0 sufficiently large, we get

$$z_n - z_{n_0} = \sum_{i=n_0}^{n-1} q_i \max_{s \in [i-\sigma, i]} x_s \geq c \sum_{i=n_0}^{n-1} q_i \,.$$

In view of (9) this implies that $\lim\limits_{n\to\infty} z_n = \infty$. Hence, $\lim\limits_{n\to\infty} x_n = \infty$. The proof is complete. □

THEOREM 5. *Assume that* (9) *holds and* $0 \le p < 1$. *If* $\{x_n\}_{n=-\mu}^{\infty}$ *is a nonoscillatory solution of equation* (1), *then either* $\lim_{n\to\infty} |x_n| = \infty$ *or* $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\{x_n\}_{n=-\mu}^{\infty}$ be an eventually positive solution of (1) and let $\{z_n\}_{n=0}^{\infty}$ be defined by (12). If $z_n > 0$, then by Theorem 4, $\lim_{n\to\infty} x_n = \infty$. Let $z_n < 0$. Then $p \ne 0$ and $x_n < p x_{n-\tau}$. Hence, by iteration one can see that

$$x_{n+k\tau} < p^k x_n \to 0 \qquad \text{as} \quad k \to \infty .$$

Therefore. $\lim_{n\to\infty} x_n = 0$. If $\{x_n\}_{n=-\mu}^{\infty}$ is eventually negative, the proof is similar. This completes the proof. □

The following result is an immediate consequence of Theorem 5.

COROLLARY 1. *Suppose that* (9) *holds and* $0 \le p < 1$. *If* $\{x_n\}_{n=-\mu}^{\infty}$ *is a bounded nonoscillatory solution of equation* (1), *then* $\lim_{n\to\infty} x_n = 0$.

THEOREM 6. *Assume that* (9) *holds and* $p > 1$. *If* $\{x_n\}_{n=-\mu}^{\infty}$ *is a bounded nonoscillatory solution of equation* (1), *then* $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\{x_n\}_{n=-\mu}^{\infty}$ be a bounded eventually positive solution of (1) and $\{z_n\}_{n=0}^{\infty}$ is defined by (12). Then $\{z_n\}_{n=0}^{\infty}$ must be negative. Indeed, if $z_n > 0$, then by Theorem 4, $\lim_{n\to\infty} x_n = \infty$, which is a contradiction to the boundedness of $\{x_n\}_{n=0}^{\infty}$. Thus, we have $\lim_{n\to\infty} z_n = l \le 0$ is finite. Summing equation (1) from n to ∞, we get

$$l - z_n = \sum_{i=n}^{\infty} q_i \max_{s\in[i-\sigma,i]} x_s .$$

Therefore, we have $\liminf_{n\to\infty} x_n = 0$. Note that $-p x_{n-\tau} < z_n$, so we have $\lim_{n\to\infty} z_n = 0$. Hence, by Lemma 2, $\lim_{n\to\infty} x_n = \frac{l}{1-p} = 0$. The case when $\{x_n\}_{n=-\mu}^{\infty}$ is eventually negative, is similarly proved. This completes the proof. □

THEOREM 7. *Assume that* $p \ge 1$ *and* $0 < q \le q_n$. *If* $\{x_n\}_{n=-\mu}^{\infty}$ *is an eventually positive solution of equation* (1), *then either* $\lim_{n\to\infty} x_n = \infty$ *or* $\lim_{n\to\infty} x_n = 0$.

Proof. Let $z_n > 0$. Then, as in the proof of Lemma 3, we get $\lim_{n\to\infty} x_n = \infty$.

If $z_n < 0$ holds, then $\lim_{n\to\infty} z_n = l$ is finite. Assume, for the sake of contradiction, that $\{x_n\}_{n=-\mu}^{\infty}$ does not tend to zero as $n \to \infty$. Then, $\limsup_{n\to\infty} x_n = a > 0$. There exists a sequence $\{n_k\}_{k=0}^{\infty}$ such that $n_{k+1} - n_k > \sigma$ and $x_{n_k} > \frac{a}{2}$ for each $k \in \mathbb{N}$. Thus

$$\max_{s\in[n-\sigma,n]} x_s > \frac{a}{2} \qquad \text{for} \qquad n \in [n_k, n_k+\sigma]$$

and

$$\sum_{i=n_k}^{n_k+\sigma-1} q_i \max_{s\in[i-\sigma,i]} x_s \geq \frac{aq\sigma}{2}\,.$$

Summing (1) we get

$$\begin{aligned} l - z_0 &= \sum_{i=0}^{\infty} q_i \max_{s\in[i-\sigma,i]} x_s \\ &\geq \sum_{k=1}^{\infty} \sum_{i=n_k}^{n_k+\sigma-1} q_i \max_{s\in[i-\sigma,i]} x_s \\ &\geq \sum_{k=1}^{\infty} \frac{aq\sigma}{2} = \infty\,, \end{aligned}$$

which is a contradiction. This completes the proof. □

The following corollary is a consequence of Theorem 7.

Corollary 2. *Suppose that $p = 1$ and $0 < q < q_n$. Then a bounded eventually positive solution of equation* (1) *satisfies* $\lim_{n\to\infty} x_n = 0$.

Example 1. Consider the difference equation

$$\Delta\big(x_n - p x_{n-\tau}\big) = \frac{1}{2^{\sigma+1}}\big(2^{\tau}p - 1\big) \max_{s\in[n-\sigma,n]} x_s\,. \tag{13}$$

When $\frac{1}{2^\tau} < p < 1$, equation (13) satisfies all conditions of Theorem 5. If $p > 1$, all conditions of Theorem 6 hold, and if $p = 1$, all conditions of Corollary 2 are satisfied. In fact, (13) has a positive solution $\{x_n\}_{n=-\mu}^{\infty} = \left\{\frac{1}{2^n}\right\}_{n=-\mu}^{\infty}$.

Note that for eventually negative solutions of equation (1), Theorem 7 and Corollary 2 may not be true. It is shown in the following examples.

Example 2. Consider the difference equation

$$\Delta\big(x_n - 4x_{n-2}\big) = \frac{15}{3+(-1)^n} \max_{s\in[n-2,n]} x_s\,, \qquad n = 1, 2\,\dots\,.$$

All assumptions of Theorem 7 are satisfied, yet the above equation has an eventually negative solution $\{x_n\}_{n=-2}^{\infty} = \left\{-2^n\big(1+(-1)^n\big) - 2^{-n}\right\}_{n=-2}^{\infty}$, which satisfies $\limsup_{n\to\infty} x_n = 0$ and $\liminf_{n\to\infty} x_n = -\infty$.

Example 3. Consider the difference equation

$$\Delta\big(x_n - x_{n-2}\big) = \frac{3}{3+(-1)^n} \max_{s\in[n-2,n]} x_s\,, \qquad n = 1, 2, \dots\,.$$

All conditions of Corollary 2 are satisfied, but the above equation has a bounded eventually negative solution $\{x_n\}_{n=-2}^{\infty} = \left\{-\left[1+(-1)^n + 2^{-n}\right]\right\}_{n=-2}^{\infty}$, which is divergent.

4. Oscillation results

Our aim in this section is to establish conditions for the oscillation of all bounded solutions of equation (1). We will need the following lemma, which can be found in [1], [7].

LEMMA 4. *Assume* $\{q_n\}_{n=0}^{\infty}$ *is a positive real sequence and* l *is a positive integer. If*

$$\liminf_{n\to\infty} \sum_{i=n}^{n+l-1} q_i > \left(\frac{l}{l+1}\right)^{l+1},$$

then

(i) *the difference inequality*

$$\Delta x_n - q_n x_{n+l} \geq 0$$

has not any eventually positive solution;

(ii) *the difference inequality*

$$\Delta x_n - q_n x_{n+l} \leq 0$$

has not any eventually negative solution.

Due to Theorem 2, we know that equation (1) has an eventually positive solution when $p \geq 0$. Thus, we will first discuss its oscillation for $p < 0$.

THEOREM 8. *Suppose that* $p < 0$, $p \neq -1$ *and the condition* (9) *holds. Then every bounded solution of equation* (1) *is oscillatory.*

P r o o f. Assume, for the sake of contradiction, that $\{x_n\}_{n=-\mu}^{\infty}$ is bounded eventually positive solution of (1). Let $\{z_n\}_{n=0}^{\infty}$ be defined by (12). Then $z_n > 0$, $\Delta z_n \geq 0$ for sufficiently large n, say $n \geq n_0$, $\lim_{n\to\infty} z_n = l \in \mathbb{R}_+$. For $p \neq -1$, by Lemma 2, there exists $\lim_{n\to\infty} x_n$. Summing (1) from n_0 to ∞ we get

$$l - z_{n_0} = \sum_{i=n_0}^{\infty} q_i \max_{s\in[i-\sigma,i]} x_s < \infty .$$

In view of (9), this implies that $\lim_{n\to\infty} x_n = 0$. Then, by Lemma 2, we obtain $l = (1-p)\cdot 0 = 0$. This contradicts $l > 0$. The proof for eventually negative solution is similar and will be omitted. □

The following result is an immediate consequence of Theorem 5 and Theorem 6.

THEOREM 9. *Assume that* $p \geq 0$, $p \neq 1$ *and condition* (9) *holds. Then every bounded solution of equation* (1) *is either oscillatory or tends to zero as* $n \to \infty$.

The next theorem gives sufficient conditions for the oscillation of all bounded solutions of equation (1) when $p > 0$, $p \neq 1$.

THEOREM 10. *Assume that* $p > 0$, $p \neq 1$ *and* (9) *holds. Suppose further that the inequality*

$$\Delta z_n + \frac{1}{p} q_n z_{n+\tau} \geq 0$$

has not any eventually positive solution and the inequality

$$\Delta z_n + \frac{1}{p} q_n z_{n+\tau} \leq 0$$

has no any eventually negative solution. Then every bounded solution of equation (1) *is oscillatory.*

P r o o f . Assume, for the sake of contradiction, that $\{x_n\}_{n=-\mu}^{\infty}$ is a bounded eventually positive solution of (1). Then, by Theorem 4, $\{z_n\}_{n=0}^{\infty}$ must be negative. Note that by (12), $x_n > -\frac{1}{p} z_{n+\tau}$. Then $\max\limits_{s\in[n-\sigma,n]} x_s \geq -\frac{1}{p} z_{n+\tau}$.

Substituting it to (1) we get

$$\Delta z_n + \frac{1}{p} q_n z_{n+\tau} \geq 0\,,$$

which is a contradiction. Similarly, for a bounded negative solution we have $z_n \geq 0$, $\Delta z_n \leq 0$ and $x_n < -\frac{1}{p} z_{n+\tau}$.

Therefore $\max\limits_{s\in[n-\sigma,n]} x_s \geq -\frac{1}{p} z_{n+\tau}$ and we get

$$\Delta z_n + \frac{1}{p} q_n z_{n+\tau} \leq 0\,.$$

This contradiction completes the proof. □

The following corollary is a consequence of Theorem 10 and Lemma 4.

COROLLARY 3. *Let* $p > 1$, $p \neq 1$ *and* (9) *holds. Furthermore, assume that*

$$\liminf_{n\to\infty} \frac{1}{p} \sum_{i=n}^{n+\tau-1} q_i > \left(\frac{\tau}{\tau+1}\right)^{\tau+1}.$$

Then every bounded solution of equation (1) *is oscillatory.*

THEOREM 11. *Assume that $p = 1$ and (9) holds. Suppose further that the inequality*

$$\Delta^2 y_n \geq \frac{1}{\tau} q_{n+\tau} y_{n+\tau}$$

has not any eventually positive solution and the inequality

$$\Delta^2 y_n \leq \frac{1}{\tau} q_{n+\tau} y_{n+\tau}$$

has not eventually negative solution. Then every bounded solution of equation (1) *is oscillatory.*

Proof. Assume, for the sake of contradiction, that $\{x_n\}_{n=-\mu}^{\infty}$ is a bounded eventually positive solution of (1). Set $w_n = x_{n-\tau} - x_n$. Then, by (1), $\Delta w_n \leq 0$, and by Theorem 4, $w_n > 0$, eventually. Thus we have

$$\begin{aligned} x_n &= x_{n+\tau} + w_{n+\tau} \\ &\geq x_{n+\tau} + \frac{1}{\tau} \sum_{i=n+\tau}^{n+2\tau-1} w_i = x_{n+2\tau} + w_{n+2\tau} + \frac{1}{\tau} \sum_{i=n+\tau}^{n+2\tau-1} w_i \\ &\geq x_{n+2\tau} + \frac{1}{\tau} \sum_{i=n+2\tau}^{n+3\tau-1} w_i + \frac{1}{\tau} \sum_{i=n+\tau}^{n+2\tau-1} w_i \\ &\geq x_{n+2\tau} + \frac{1}{\tau} \sum_{i=n+\tau}^{n+3\tau-1} w_i \\ &\vdots \\ &\geq x_{n+k\tau} + \frac{1}{\tau} \sum_{i=n+\tau}^{n+k\tau-1} w_i \\ &\geq \frac{1}{\tau} \sum_{i=n+\tau}^{\infty} w_i . \end{aligned}$$

Set $y_n = \frac{1}{\tau} \sum_{i=n+\tau}^{\infty} w_i$. Hence, $x_n \geq y_n$ and $\Delta y_n = -\frac{1}{\tau} w_{n+\tau}$. Therefore $\Delta^2 y_n = -\frac{1}{\tau} \Delta w_{n+\tau}$ and by (1) we get

$$\Delta^2 y_n = \frac{1}{\tau} q_{n+\tau} \max_{s \in [n+\tau-\sigma, n+\tau]} x_s \geq \frac{1}{\tau} q_{n+\tau} y_{n+\tau} ,$$

which is a contradiction.

If $\{x_n\}_{n=-\mu}^{\infty}$ is a bounded eventually negative solution of (1), we have $w_n < 0$ and $\Delta w_n \geq 0$ eventually.

Thus, we have

$$x_n = x_{n+\tau} + w_{n+\tau} \leq \frac{1}{\tau} \sum_{i=n+\tau}^{\infty} w_i .$$

Let $\{y_n\}_{n=0}^{\infty}$ be defined as above. Then $x_n \le y_n$ and by (1), we get

$$\Delta^2 y_n \le \frac{1}{\tau} q_{n+\tau} y_{n+\tau} ,$$

which is a contradiction. This completes the proof. □

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