VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-007-0025-4
Math. Slovaca **57** (2007), No. 4, 313–320

STRENGTHENED FIXED POINT PROPERTY AND PRODUCTS IN ORDERED SETS

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(*Communicated by Pavol Zlatoš*)

ABSTRACT. Strengthened fixed point property for ordered sets is formulated. It is weaker than the strong fixed point property due to Duffus and Sauer and stronger than the product property meaning that $A \times Y$ has the fixed point property whenever A has the former and Y has the latter. In particular, doubly chain complete ordered sets with no infinite antichain have the strengthened fixed point property whenever they have the fixed point property, which yields a transparent proof of the well-known theorem saying that doubly chain complete ordered sets with no infinite antichain have the product property whenever they have the fixed point property. The new proof does not require the axiom of choice.

The strong fixed point property formulated by Duffus and Sauer [3] is weakened in such a way that it comprises most known examples of ordered sets with the *product property*, that is whose products with ordered sets with the fixed point property have the fixed point property as well, and that itself is stronger that the latter.

The aim of the paper is to show that the strengthened fixed point property is really stronger than the product property. This substantially improves and generalizes the idea formulated for doubly chain complete ordered sets with no infinite antichain by Schröder in [12, Exercises 25–26, Problem 16 after Chap. 10].

In particular, doubly chain complete ordered sets with no infinite antichain have the strengthened fixed point property. This yields a transparent proof of

2000 Mathematics Subject Classification: Primary 06A06.
Keywords: doubly chain complete ordered set, fixed point property, product.
Presented at the Summer School on General Algebra and Ordered Sets, Malá Morávka, 4 10 September 2005.

the well-known theorem saying that doubly chain complete ordered sets with no infinite antichain have the product property whenever they have the fixed point property. My proof does not require the axiom of choice. The original proof by Li and Milner [5] is based on an analogous result for finite ordered sets due to Roddy [6] and Roddy et al. [8], and an earlier result by the same authors [4], whose proof is rather complicated and apparently dependent on the axiom of choice.

Preliminaries

Let P be an ordered set. For each $M \subseteq P$ we put

$$\mathrm{L}(M) := \{p \in P: \ (\forall m \in M)(p \leq m)\} \quad \text{and}$$
$$\mathrm{U}(M) := \{p \in P: \ (\forall m \in M)(m \leq p)\}.$$

We denote $\mathrm{E}(P)$ the set of all endomorphisms of P, that is isotone mappings of P to P, and $\mathrm{I}(P)$ the set of all retractions of P, that is

$$\mathrm{I}(P) := \{f \in \mathrm{E}(P): \ f \circ f = f\}.$$

Both $\mathrm{E}(P)$ and $\mathrm{I}(P)$ are endowed with the induced pointwise order.

For each $f \in \mathrm{E}(P)$ we define

$$\mathrm{Fix}(f) := \{p \in P: \ f(p) = p\},$$
$$\mathrm{Ext}(f) := \{p \in P: \ p \leq f(p)\} \quad \text{and}$$
$$\mathrm{dExt}(f) := \{p \in P: \ f(p) \leq p\}.$$

The elements of $\mathrm{Fix}(f)$ are called the *fixed points* of f. We say that P *has the fixed point property* if $\mathrm{Fix}(f) \neq \emptyset$ for each $f \in \mathrm{E}(P)$. Recall the *strong fixed point property.*

There exists an isotone mapping $\Phi\colon \mathrm{E}(P) \to P$ such that

$$\Phi(f) \in \mathrm{Fix}(f)$$

for each $f \in \mathrm{E}(P)$.

As shown by Duffus and Sauer in [3], the strong fixed point property is stronger than the product property. We define the *strengthened fixed point property* as follows.

P has the fixed point property, and there exists an isotone mapping $^*\colon \mathrm{E}(P) \to \mathrm{I}(P)$ which satisfies the condition

$$(\forall p \in \mathrm{Ext}(f) \cup \mathrm{dExt}(f))\ f^*(p) \in \mathrm{Fix}(f)$$

for each $f \in \mathrm{E}(P)$.

It is a commonplace that in set theory with the axiom of choice, the following concepts of completeness are equivalent:

Every non-empty well-ordered chain has a supremum.
Every non-empty chain has a supremum.
Every directed subset has a supremum.

We shall choose the weakest one and say that P is *chain complete* if every non-empty well-ordered chain in P has a supremum in P, *dually chain complete* if every non-empty dually well-ordered chain in P has an infimum in P, and *doubly chain complete* if it is both chain complete and dually chain complete.

The main theorem

First we shall investigate the properties of $\mathrm{E}(P \times Q)$. We shall start with some ideas from [6]. For $f \in \mathrm{E}(P \times Q)$ and $p \in P$, $q \in Q$, we shall consider mappings $f_{p.} \in \mathrm{E}(Q)$ and $f_{.q} \in \mathrm{E}(P)$ such that

$$\begin{aligned} f_{p.}(y) &:= \pi_Q f(p, y) \qquad \text{for each} \quad y \in Q \qquad \text{and} \\ f_{.q}(a) &:= \pi_P f(a, q) \qquad \text{for each} \quad a \in P, \end{aligned}$$

where π_P and π_Q are projections. It is obvious that $p_1 \le p_2$ implies that $f_{p_1.} \le f_{p_2.}$ and $q_1 \le q_2$ implies that $f_{.q_1} \le f_{.q_2}$.

The following result is a modification of a more particular result due to Roddy [6].

Lemma 1. *Let the ordered set Q have the fixed point property, let P be an ordered set and let $f \in \mathrm{E}(P \times Q)$ be such that $f_{.q} \in \mathrm{I}(P)$ for each $q \in Q$. Then f has a fixed point.*

Proof. Let us choose $p \in P$. The mapping $y \mapsto f_{f_{.y}(p).}(y)$ where $y \in Q$ is isotone, and therefore it has a fixed point z. Then $f_{f_{.z}(p).}(z) = z$ and hence $f(f_{.z}(p), z) = [f_{.z}f_{.z}(p), f_{f_{.z}(p).}(z)] = [f_{.z}(p), z]$. □

Lemma 2. *Let A be an ordered set, let ${}^*\colon \mathrm{E}(A) \longrightarrow \mathrm{I}(A)$ be an isotone mapping, let Y be an ordered set with the fixed point property and let $f \in \mathrm{E}(A \times Y)$. Then the mapping*

$$f' := ([a,y] \mapsto [(f_{.y})^*(a), f_{a.}(y)]) \tag{1}$$

is isotone and has a fixed point.

Proof. Let $[a_0, y_0] \le [a_1, y_1]$. Then $(f_{.y_0})^*(a_0) \le (f_{.y_1})^*(a_0) \le (f_{.y_1})^*(a_1$ and $f_{a_0.}(y_0) \le f_{a_1.}(y_0) \le f_{a_1.}(y_1)$. Hence $f' \in \mathrm{E}(A \times Y)$. We have $f'_{.y}(a)$ $\pi_A f'(a,y) = (f_{.y})^*(a)$ for each $[a,y] \in A \times Y$, and thus $f'_{.y} - (f_{y})^* \in \mathrm{I}(A)$. The mapping f' has a fixed point in virtue of Lemma 1. □

Recall that a subset Q of a set P is *f-invariant* if $f[\![Q]\!] \subseteq Q$.

We shall prove the following lemma, whose basic idea goes back to Rutkowski [9], [10], but his formulation requires the axiom of choice.

Lemma 3. *Let P be an ordered set with the fixed point property, let $f \in \mathrm{E}(P$, let W be a well-ordered chain in $\mathrm{Ext}(f)$ and let V be a dually well-ordered chain in $\mathrm{U}(W) \cap \mathrm{dExt}(f)$. Then $\mathrm{U}(W) \cap \mathrm{L}(V) \cap \mathrm{Fix}(f) \neq \emptyset$.*

Proof. If W is not topped, then we set $W_1 := W$. Otherwise, let us suppose that t is the greatest element of W. If $t \in \mathrm{Fix}(f)$, then we are done. If not, then $t < f(t) \le f^2(t) \le \dots$. It is obvious that $f^n(t) \in \mathrm{L}(V)$ for each $n \in \mathbb{N}$. If $f^n(t) \in \mathrm{Fix}(f)$ for some $n \in \mathbb{N}$, then we are done. If not, then $W_1 := \{t < f(t) < f^2(t) < \dots\}$ has not the greatest element, and W_1 is a well-ordered chain in $\mathrm{Ext}(f)$ such that $V \subseteq \mathrm{U}(W_1)$. Dually, we obtain V_1. Clearly, $W_1 \subseteq \mathrm{Ext}(f$, $V_1 \subseteq \mathrm{dExt}(f)$, $\mathrm{U}(W_1) \cap \mathrm{L}(V_1) \subseteq \mathrm{U}(W) \cap \mathrm{L}(V)$, and $\mathrm{U}(W_1) \cap \mathrm{L}(V_1)$ is f-invariant. Let us put

$$\begin{aligned} f_1(p) &:= \max\{v \in V_1 :\ p \not\le v\} && \text{for} \quad p \in P \setminus \mathrm{L}(V_1),\\ f_1(p) &:= \min\{w \in W_1 :\ w \not\le p\} && \text{for} \quad p \in \mathrm{L}(V_1) \setminus \mathrm{U}(W_1),\\ f_1(p) &:= f(p) && \text{for} \quad p \in \mathrm{L}(V_1) \cap \mathrm{U}(W_1). \end{aligned}$$

Obviously, $f_1 \in \mathrm{E}(P)$ and therefore $\mathrm{Fix}(f_1) \neq \emptyset$. Hence $\mathrm{U}(W_1) \cap \mathrm{L}(V_1)$ $\mathrm{Fix}(f_1) \neq \emptyset$. We may conclude that $\mathrm{U}(W) \cap \mathrm{L}(V) \cap \mathrm{Fix}(f) \neq \emptyset$. □

Let us notice that either of W and V may be empty.

Theorem 4. *Let A be an ordered set with the strengthened fixed point property, and let Y be an ordered set with the fixed point property. Then $A \times Y$ has the fixed point property.*

Proof. Let $f \in \mathrm{E}(A \times Y)$. By Lemma 2, there exists a fixed point $[a, y]$ of the mapping f' associated with f by rule (1). There also exists a fixed point b of the mapping $f_{.y}$. Since A has the fixed point property, it is connected. There exists a fence

$$a = a_0 \gtrless a_1 \gtrless a_2 \gtrless \cdots \gtrless a_n = b.$$

Without loss of generality we may assume that a, y, b, $a_0, \ldots, a_n$ were chosen in such a way that n is the least possible. Applying $(f_{.y})^*$ to $a_0, \ldots, a_n$, we also obtain a fence with required properties. Indeed, $(f_{.y})^*(a) = a$, and $b \in \mathrm{Fix}(f_{.y})$ implies that $(f_{.y})^*(b) \in \mathrm{Fix}(f_{.y})$. Thus we may assume that

$$a_0 = (f_{.y})^*(a_0), \ \ldots, \ a_n = (f_{.y})^*(a_n).$$

Suppose for contradiction that $a \neq b$, that is $n \neq 0$. Then $a_0 \neq a_1$. Without loss of generality we may assume that $a_0 < a_1$. Let us denote $c := a_1$. Clearly $y = f_{a.}(y) \leq f_{c.}(y)$. Define $g\colon Y \to Y$ by the rule

$$g(z) := f_{(f_{.z})^*(c).}(z).$$

Then $g(y) = f_{(f_{.y})^*(c).}(y) = f_{c.}(y)$. Clearly $y \leq g(y) \leq g^2(y) \leq \ldots$, and thus g has a fixed point y' in $\mathrm{U}(\{y, g(y), g^2(y), \ldots\})$ by Lemma 3. Obviously, $[(f_{.y'})^*(c), y']$ is a fixed point of f'. Denote $a'_i := (f_{.y'})^*(a_i)$. Now, $c' = a'_1 \gtrless a'_2 \gtrless \cdots \gtrless a'_n = b'$ is a fence which is shorter than the original one, $[c', y']$ is a fixed point of f' and b' is a fixed point of $f_{.y'}$. Indeed, $b = f_{.y}(b) \leq f_{.y'}(b)$, and thus $(f_{.y'})^*(b) \in \mathrm{Fix}(f_{.y'})$ by assumption. This is a contradiction. We may conclude that $b = a$ and $[a, y]$ is a fixed point of f. □

Schröder in [12, Exercises 25-26, Problem 16 after Chap. 10], observed that in the particular case of doubly chain complete ordered sets with no infinite antichain there is an isotone mapping $^*\colon \mathrm{E}(A) \to \mathrm{I}(A)$ which satisfies the conditions

$$(\forall f \in \mathrm{E}(A))(f^*)^* = f^*, \text{ that is } ^* \text{ is a retraction, and}$$
$$(\forall f \in \mathrm{E}(A))f^*[\![A]\!] = \{p \in A : \ (\exists n \in \mathbb{N}) f^n(p) = p\}.$$

He did not realize that the condition involved in the strengthened fixed point property rather than that the mapping * should be a retraction is important, see [12, Problem 16 after Chap. 10].

Applications to some classes of ordered sets

[3, Theorem 1, assertion (a)] is the first corollary of Theorem 4.

COROLLARY 5. *Let A be an ordered set with the strong fixed point property and let Y be an ordered set with the fixed point property. Then $A \quad Y$ has the fixed point property.*

P r o o f. If A has the strong fixed point property, that is $\Phi\colon \mathrm{E}(A) \to A$ is isotone and $\Phi(f) \in \mathrm{Fix}(f)$, then for each $f \in \mathrm{E}(A)$ we define f^* to be the constant mapping sending all elements of A to $\Phi(f)$. Hence A has the strengthened fixed point property. □

[12, Theorem 10.2.11] is a further corollary of Theorem 4.

COROLLARY 6. *Let A be a doubly chain complete ordered set with no infinite antichain and let Y be an ordered set, both with the fixed point property. Then $A \times Y$ has the fixed point property.*

In the proof of Corollary 6 we shall use the following CPO fixpoint theorem III from [2], also called the Abian-Brown theorem.

ABIAN-BROWN THEOREM. *Let P be a chain complete ordered set, let $f \in \mathrm{E}(P)$ and $p \in \mathrm{Ext}(f)$. Then $\mathrm{U}(p) \cap \mathrm{Fix}(f)$ has the least element.*

A b i a n in [1] observed that the preceding theorem can be proved without the axiom of choice. Some caution is necessary because in set theory without the axiom of choice different notions of chain completeness exist. Nevertheless, it can be easily checked that the Abian-Brown theorem holds in our setting.

P r o o f o f C o r o l l a r y 6. We shall verify that A has the strengthened fixed point property. In [12, Exercise 25 after Chap. 10], it is claimed that there exists an isotone mapping $^*\colon \mathrm{E}(A) \to \mathrm{I}(A)$. The technical details of the proof are therefore omitted. It only remains to check that the mapping * satisfies the condition

$$(\forall p \in \mathrm{Ext}(f) \cup \mathrm{dExt}(f))\; f^*(p) \in \mathrm{Fix}(f)$$

for each $f \in \mathrm{E}(A)$. The crucial idea how to obtain f^* in doubly chain complete ordered sets with no infinite antichain is using factorial powers of the mapping f.

Let $f \in \mathrm{E}(A)$ and $p \in A$. There obviously exists $\mu(f,p) \in \mathbb{N}_1$ such that either

$$(\forall w \in \mathbb{N}_1)\big(f^{w!}(p) \in \mathrm{dExt}(f^{w!}) \iff \mu(f,p) \le w\big)$$

or

$$(\forall w \in \mathbb{N}_1)\big(f^{w!}(p) \in \mathrm{Ext}(f^{w!}) \iff \mu(f,p) \le w\big).$$

Here $\mathbb{N}_1$ denotes the set of all positive integers. Thus we can denote

$$f^*(p) := \max(\mathrm{L}(f^{\mu(f,p)!}(p)) \cap \mathrm{Fix}(f^{\mu(f,p)!}))$$

in the former case and

$$f^*(p) := \min(\mathrm{U}(f^{\mu(f,p)!}(p)) \cap \mathrm{Fix}(f^{\mu(f,p)!}))$$

in the latter case. If $p \in \mathrm{Ext}(f) \cup \mathrm{dExt}(f)$, then $\mu(f,p) = 1$, and thus $f^*(p) \in \mathrm{Fix}(f)$. $\square$

Concluding remark

In his more recent paper [7], Roddy proved by different means that $A \times Y$ has the fixed point property whenever both A and Y have it and A is of width at most three.

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Received 22. 9. 2005
Revised 6. 4. 2006

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