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Strong Versions of Kummer-Type Congruences for Genocchi Numbers and Polynomials and Tangent Coefficients

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Abstract. We use the properties of p-adic integrals and measures to obtain general congruences for Genocchi numbers and polynomials and tangent coefficients. These congruences are analogues of the usual Kummer congruences for Bernoulli numbers, generalize known congruences for Genocchi numbers, and provide new congruences systems for Genocchi polynomials and tangent coefficients.

1. Introduction

The Bernoulli polynomials $B_n(x)$ may be defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad (|t|<2\pi)$$

and their values at $x=0$ are called Bernoulli numbers and denoted by B_n. These numbers have been extensively studied and many properties for them are known. One of the most important theorem relating to Bernoulli numbers is the Staudt-Clausen Theorem:

Theorem 1.1. *([2]) For $m\geq 1$,*

$$B_{2m}=A_{2m}-\sum_{(p-1)|2m}\frac{1}{p},$$

where A_{2m} is an integer and the summation is over all primes p such that $(p-1)|2m$.

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Another remarkable result is the Kummer congruences, which in the simplest form state that

$$\frac{B_m}{m} \equiv \frac{B_n}{n} \,(\bmod p\mathbb{Z}_p)$$

for positive integers m and n such that $m \equiv n \,(\bmod\,(p-1))$, where p is an odd prime ([16, Corollary 5.14]). The strong form of the Kummer congruences ([1]) states that if p is an odd prime, $c \equiv 0 \,(\bmod\,(p-1)\,p^a)$ with $a \geq 0$ and $p-1$ does not divide m, then

$$\Delta_c^k \left\{ \left(1-p^{m-1}\right) \frac{B_m}{m} \right\} \equiv 0 \left(\bmod p^{k(a+1)} \mathbb{Z}_p \right),$$

where Δ_c is the forward difference operator with increment c and Δ_c^k denotes the kth compositional iterate of this operator (see preliminary section for definitions in detail).

Recently, several authors obtained generalizations of these congruences by applying either p-adic interpolation or p-adic integration techniques. In [17, 18], Young extended several known congruences, involving Kummer-type congruences for Bernoulli numbers and Euler polynomials of higher order, Stirling and weighted Stirling numbers of the second kind, and the values of Bernoulli and Euler polynomials. Developing the theory of "degenerate number sequences", he showed that degenerate Stirling numbers and degenerate Eulerian polynomials could be expressed as p-adic integrals of generalized factorials, and proved an analogue of Kummer's congruences for expressions involving degenerate Bernoulli numbers and polynomials introduced by Carlitz [3], which extended known congruences for ordinary Bernoulli numbers. He also gave versions of Kummer's congruences modulo powers of a general positive integer n for Bernoulli polynomials with n-adic integer argument ([19, 20]). For the generalized Bernoulli polynomials associated with a primitive Dirichlet character χ having conductor $f_\chi \in \mathbb{N}$, Fox illustrated how a particular expression, involving the generalized Bernoulli polynomials, satisfied the systems of congruence relations if and only if a similar expression, involving the generalized Bernoulli numbers, satisfied the same congruence relations, which particularly included Kummer congruences ([9]). In [12], Jang and Kim gave higher order extensions of generalized Bernoulli polynomials, and proposed a question whether these extensions satisfied Kummer-type congruences. The answer of a part of the question was given in [11].

The Genocchi numbers G_n may be defined by the generating function

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad (|t| < \pi) \tag{1.1}$$

([4, p. 49]), which have several combinatorial interpretations in terms of certain surjective maps on finite sets ([5, 6, 7]). The well known identity

$$G_n = 2\left(1-2^n\right) B_n \tag{1.2}$$

shows the relation between Genocchi and Bernoulli numbers. It follows from (1.2) and the Staudt-Clausen Theorem that the Genocchi numbers are integers.

The Euler polynomials $E_n(x)$ may be defined by the generating function ([13])

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad (|t|<\pi). \tag{1.3}$$

It is easy to see that

$$G_n=2nE_{2n-1}(0), \tag{1.4}$$

and from (1.3), (1.4) we deduce that

$$E_n(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{G_{k+1}}{k+1}x^{n-k}.$$

The generalized Euler numbers $H_n(\lambda)$ attached to an algebraic number $\lambda\neq 1$ have been defined by

$$\frac{1-\lambda}{e^t-\lambda}=\sum_{n=0}^{\infty}H_n(\lambda)\frac{t^n}{n!} \tag{1.5}$$

in [15]. We note from (1.5) and (1.1) that

$$\frac{G_{n+1}}{n+1}=H_n(-1).$$

The Genocchi polynomials $G_n(x)$ can be defined as follows:

$$\frac{2te^{xt}}{e^t+1}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\quad (|t|<\pi).$$

Note that $G_n(0)=G_n$, and

$$G_n(x)=\sum_{k=0}^{n}\binom{n}{k}G_kx^{n-k}.$$

It is well known that the tangent coefficients T_n, defined by

$$\tan t=\sum_{n=1}^{\infty}(-1)^{n-1}T_{2n}\frac{t^{2n-1}}{(2n-1)!},\quad \left(|t|<\frac{\pi}{2}\right),$$

are closely related to the Bernoulli numbers, i.e., ([13, p. 35])

$$T_n=2^n(2^n-1)\frac{B_n}{n}.$$

Ramanujan ([14, p. 5]) observed that $2^n(2^n-1)B_n/n$, therefore tangent coefficients, are integers for $n\geq 1$.

Primary focus of this study is the applications of p-adic integration methods to obtain strong versions of Kummer-type congruences for Genocchi numbers, what remained to prove in [20]. These methods are considered for Genocchi polynomials and tangent coefficients as well. All these congruences, involving the kth compositional iterate of the forward difference operator with increment c and binomial coefficient operator, are deduced from some basic properties of p-adic Γ-transforms which are recorded in Theorem 2.1 below.

2. Preliminaries

Throughout this study p will denote a prime number, $\mathbb{Z}_p$ the ring of p-adic integers, $\mathbb{Z}_p^{\times}$ the multiplicative group of units in $\mathbb{Z}_p$, and $\mathbb{Q}_p$ is the field of p-adic numbers. Define the quantity q by 4 if $p=2$ and p otherwise. We use $\mathbb{Z}_p[T-1]$ and $\mathbb{Z}_p[[T-1]]$ to denote, respectively, the ring of polynomials and of formal power series in the indeterminate $(T-1)$ over $\mathbb{Z}_p$. The p-adic valuation "ord_p" is defined by setting $\mathrm{ord}_p(x)=k$ if $x=p^k y$ with $y\in\mathbb{Z}_p^{\times}$. A congruence $x\equiv y\,(\mathrm{mod} m\mathbb{Z}_p)$ is equivalent to $\mathrm{ord}_p(x-y)\geq \mathrm{ord}_p(m)$, and if x and y are rational numbers, this congruence is equivalent to the definition of congruence $x\equiv y\,(\mathrm{mod} m)$ given by Howard in [10, Section 2] for all primes p.

If c is a non-negative integer, the difference operator Δ_c operates on the sequence $\{a_m\}$ by

$$\Delta_c a_m = a_{m+c} - a_m.$$

The powers Δ_c^k of Δ_c are defined by Δ_c^0 =identity, and $\Delta_c^k=\Delta_c\circ\Delta_c^{k-1}$ for positive integer k, so that

$$\Delta_c^k a_m = \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}a_{m+jc},$$

for all non-negative integers k. To define binomial coefficient operators $\binom{D}{k}$ associated to an operator D, we write the binomial coefficient

$$\binom{X}{k} = \frac{X(X-1)\cdots(X-k+1)}{k!},$$

for $k\geq 0$ as a polynomial in X, and replace X by D.

Define the linear operator φ by

$$(\varphi f)(T) = f(T) - \frac{1}{p}\sum_{\zeta^p=1} f(\zeta T). \tag{2.1}$$

This operator is well defined and stable on rational functions, and also on $\mathbb{Z}_p[[T-1]]$ ([17, Eq. (2.14)]). If $f(e^t)=\sum a_n t^n/n!$, write $(\varphi f)(e^t)=\sum \widehat{a_n}t^n/n!$. The following congruences for numbers $\widehat{a_n}$ were proved in [17].

Theorem 2.1. *Let $f\in\mathbb{Z}_p[[T-1]]$ and write $(\varphi f)(e^t)=\sum_{n=0}^{\infty}\widehat{a_n}t^n/n!$. Then $\widehat{a_n}\in\mathbb{Z}_p$ for all n. Furthermore, if $c\equiv 0\,(mod\phi(q)\,p^a)$, where ϕ is the Euler ϕ-function, with $a\geq 0$ then*

$$\Delta_c^k\widehat{a_m}\equiv 0\left(mod p^{ka'}\mathbb{Z}_p\right)$$

for all $m,k\geq 0$, where $a'=a+1$ if $p>2$ and $a'=a+3$ if $p=2$. Also for $0\leq r\leq a'$ and all $m,k\geq 0$

$$\binom{p^{-r}\Delta_c}{k}\widehat{a_m}\in\mathbb{Z}_p.$$

It can be observed from this theorem that $\left(p^{-a'}\Delta_c\right)^k$ is a polynomial of degree k in Δ_c with leading coefficient $p^{-ka'}$, which sends $\widehat{a_m}$ into $\mathbb{Z}_p$, whereas the binomial coefficient operator $\binom{p^{-a'}\Delta_c}{k}$ is a polynomial of degree k in Δ_c with leading coefficient $p^{-ka'}/k!$, which sends $\widehat{a_m}$ into $\mathbb{Z}_p$.

The proof of this theorem made use of the correspondence

$$\Lambda \leftrightarrow \mathbb{Z}_p\left[\left[T-1\right]\right], \tag{2.2}$$

where Λ denotes the set of all $\mathbb{Z}_p$-valued measures on $\mathbb{Z}_p$, under which each measure $\alpha \in \Lambda$ corresponds to the formal power series $f \in \mathbb{Z}_p\left[\left[T-1\right]\right]$, defined by

$$f\left(T\right) = \int\limits_{\mathbb{Z}_p} T^x d\alpha\left(x\right) = \sum_{m=0}^{\infty}\left(\int\limits_{\mathbb{Z}_p}\binom{x}{m} d\alpha\left(x\right)\right)\left(T-1\right)^m \tag{2.3}$$

([16, Chap. 12]). From this it follows that

$$a_n = \int\limits_{\mathbb{Z}_p} x^n d\alpha\left(x\right).$$

We also observed that ([17, Eq. (2.14)]) that

$$\left(\varphi f\right)\left(T\right) = \int\limits_{\mathbb{Z}_p^\times} T^x d\alpha\left(x\right),$$

which implies

$$\widehat{a_n} = \int\limits_{\mathbb{Z}_p^\times} x^n d\alpha\left(x\right).$$

Since

$$\left(1-\varphi\right) f\left(T\right) = \int\limits_{p\mathbb{Z}_p} T^x d\alpha\left(x\right) = \int\limits_{\mathbb{Z}_p} T^{px} d\alpha\left(px\right),$$

we see that $\left(1-\varphi\right) f\left(T\right) \in \mathbb{Z}_p\left[\left[T-1\right]\right]$ according to correspondences (2.2) and (2.3). Therefore there is a linear operator ψ on $\mathbb{Z}_p\left[\left[T-1\right]\right]$ such that

$$\left(\psi f\right)\left(T\right) = \frac{1}{p}\sum_{Z^p=T} f\left(Z\right),$$

and this operator coincides on $\mathbb{Z}_p\left[\left[T-1\right]\right]$ with Dwork's ψ operator ([8, Chap. 5]). So if we write $\left(\psi f\right)\left(e^t\right) = \sum a_n^* t^n/n!$, then $a_n^* \in \mathbb{Z}_p$ for all n, and

$$\widehat{a_n} = a_n - p^n a_n^*.$$

3. Congruences

From now on we assume that p is a prime number > 2.

Theorem 3.1. *If $c \equiv 0\left(mod\left(p-1\right)p^a\right)$ with $a \geq 0$ and $p-1$ does not divide m, then*

$$\Delta_c^k\left\{\left(1-p^{m-1}\right)\frac{G_m}{m}\right\} \equiv 0\left(mod p^{k(a+1)}\mathbb{Z}_p\right),$$

and

$$\binom{p^{-r}\Delta_c}{k}\left\{\left(1-p^{m-1}\right)\frac{G_m}{m}\right\} \in \mathbb{Z}_p$$

for $0 \leq r \leq a+1$ and all $k \geq 0, m > 0$.

Proof. Let λ be algebraic over $\mathbb{Q}_p$ and $\mathbb{Z}_p(\lambda)[[T-1]]$ denotes the ring of formal power series in the indeterminate $(T-1)$ over $\mathbb{Z}_p(\lambda)$ of the finite extension $\mathbb{Q}_p(\lambda)$ of $\mathbb{Q}_p$. Suppose $1-\lambda \in \mathbb{Z}_p^{\times}(\lambda)$. Then

$$f(T) = \frac{1-\lambda}{T-\lambda} \in \mathbb{Z}_p(\lambda)[[T-1]].$$

We compute

$$(1-\varphi)\left(\frac{1-\lambda}{T-\lambda}\right) = \frac{1}{p}\sum_{\zeta^p=1} \frac{1-\lambda}{\zeta T-\lambda} = \frac{1-\lambda}{T^p-\lambda^p}.$$

Let $\lambda=-1$ and set $T=e^t$. Then we have

$$(1-\varphi)\left(\frac{2}{e^t+1}\right) = \frac{2}{e^{pt}+1},$$

or equivalently

$$\varphi\left(\frac{2}{e^t+1}\right) = \frac{2}{e^t+1} - \frac{2}{e^{pt}+1}.$$

Expanding $(\varphi f)(e^t) = \sum \widehat{a_n} t^n/n!$ as formal power series yields

$$\begin{aligned}\sum_{n=0}^{\infty} \widehat{a_n}\frac{t^n}{n!} &= \frac{1}{t}\sum_{n=0}^{\infty} G_n\frac{t^n}{n!} - \frac{1}{pt}\sum_{n=0}^{\infty} G_n\frac{(pt)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{G_{n+1}}{n+1}\frac{t^n}{n!} - \sum_{n=0}^{\infty} p^n\frac{G_{n+1}}{n+1}\frac{t^n}{n!}.\end{aligned}$$

Equating coefficients of the terms $t^n/n!$ gives

$$\widehat{a_n} = \frac{G_{n+1}}{n+1} - p^n\frac{G_{n+1}}{n+1}.$$

The theorem then follows by taking $m=n+1$ and applying Theorem 2.1. $\square$

Theorem 3.2. *Suppose $p-1$ does not divide m. If $c \equiv 0\,(mod(p-1)\,p^a)$ with $a \geq 0$, then*

$$\Delta_c^k\left\{\left(1-p^{m-1}\right)T_m\right\} \equiv 0\left(mod\frac{1}{2}p^{k(a+1)}\mathbb{Z}_p\right),$$

and

$$\binom{p^{-r}\Delta_c}{k}\left\{\left(1-p^{m-1}\right)T_m\right\} \in \frac{1}{2}\mathbb{Z}_p$$

for $0 \leq r \leq a+1$ and all $m, k \geq 0$.

Proof. Consider the polynomial

$$F(T) = \frac{T^4-1}{T^2-1} = 2+(T-1)(T+1).$$

This polynomial lies in $\mathbb{Z}[T-1]$ and has a constant term $2 \in \mathbb{Z}_p^{\times}$ when viewed as an element of $\mathbb{Z}[T-1]$. Therefore

$$\frac{2}{F(T)} \in 1+\left(T^2-1\right)\mathbb{Z}_p[[T-1]]$$

so that

$$\frac{2}{T^4-1} = \frac{1}{T^2-1} + g(T),$$

where $g(T)$ is a rational function which also lies in $\mathbb{Z}_p[[T-1]]$. Now we substitute $T = e^t$ and expanding as formal power series,

$$\frac{1}{t}\sum_{n=0}^{\infty} 2^n \left(2^n - 1\right) B_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} 2a_n \frac{t^n}{n!},$$

where $a_n \in \mathbb{Z}_p$. For each $n \geq 0$, equating coefficients of $t^n/n!$ gives

$$T_n = 2^{n+1}\left(2^{n+1} - 1\right)\frac{B_{n+1}}{n+1} = 2a_n.$$

Since

$$\varphi\left(\frac{1}{T^k - 1}\right) = \frac{1}{T^k - 1} - \frac{1}{T^{pk} - 1}$$

whenever $(k, p) = 1$, we see that for this function $g(T)$ we have

$$\left(1 - p^{n-1}\right) T_n = 2\widehat{a_n}.$$

Therefore by Theorem 2.1, the result follows. $\square$

Dwork's shift map $x \to x'$ is defined for $x \in \mathbb{Z}_p$ by the relation $px' - x = \mu_x \in \{0, 1, \ldots, p-1\}$ so that μ_x is the representative of $-x \bmod p\mathbb{Z}_p$ which lies in $\{0, 1, \ldots, p-1\}$. The following lemma describes the action of Dwork's ψ operator on certain functions in terms of shift map.

Lemma 3.3. *For $x \in \mathbb{Z}_p$ we have formally*

$$\psi\left(\frac{T^x}{T - c}\right) = c^{p-1-\mu_x}\frac{T^x}{T - c^p},$$

or equivalently

$$\varphi\left(\frac{T^x}{T - c}\right) = \frac{T^x}{T - c} - c^{p-1-\mu_x}\frac{T^{px'}}{T^p - c^p}.$$

Proof. Using (2.1) we compute

$$\begin{aligned}(1-\varphi)\left(\frac{T^x}{T-c}\right) &= \frac{1}{p}\sum_{\zeta^p=1}\frac{\zeta^x T^x}{\zeta T - c} = T^x\left(\frac{1}{p}\sum_{\zeta^p=1}\frac{\zeta^{-\mu_x}}{\zeta T - c}\right)\\ &= c^{p-1-\mu_x}\frac{T^{x+\mu_x}}{T^p - c^p}\end{aligned}$$

by considering the partial fraction decomposition of the latter rational function. Since $x + \mu_x = px'$, the lemma follows. $\square$

Theorem 3.4. *If $c \equiv 0\,(mod\,(p-1)\,p^a)$ with $a \geq 0$, then for all $x \in \mathbb{Z}_p$ we have*

$$\Delta_c^k\left\{\frac{G_m(x)}{m} - (-1)^{p-1-\mu_x} p^{m-1}\frac{G_m(x')}{m}\right\} \equiv 0\left(mod p^{k(a+1)}\mathbb{Z}_p\right),$$

and

$$\binom{p^{-r}\Delta_c}{k}\left\{\frac{G_m(x)}{m} - (-1)^{p-1-\mu_x} p^{m-1}\frac{G_m(x')}{m}\right\} \in \mathbb{Z}_p$$

for $0 \leq r \leq a+1$ and all $k \geq 0$, $m > 0$.

Proof. Suppose $1-\lambda \in \mathbb{Z}_p^{\times}(\lambda)$. Then

$$h(T)=\frac{1-\lambda}{T-\lambda}T^{x}\in \mathbb{Z}_p(\lambda)[[T-1]]$$

for all $x\in\mathbb{Z}_p$. From Lemma 3.3 we have

$$(1-\varphi)h(T)=\lambda^{p-1-\mu_x}(1-\lambda)\frac{T^{px'}}{T^p-\lambda^p}.$$

Let $\lambda=-1$. Setting $T=e^t$ and expanding $(\varphi f)(e^t)=\sum\widehat{a_n}t^n/n!$ as formal power series give

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{a_n}\frac{t^n}{n!} &= \frac{2e^{xt}}{e^t+1}-(-1)^{p-1-\mu_x}\frac{2e^{px't}}{e^{pt}+1}\\ &= \frac{1}{t}\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}-(-1)^{p-1-\mu_x}\frac{1}{pt}\sum_{n=0}^{\infty}G_n(x')\frac{(pt)^n}{n!}.\end{aligned}$$

Comparing the coefficients of the terms $t^n/n!$ yields

$$\widehat{a_n}=\frac{G_{n+1}(x)}{n+1}-(-1)^{p-1-\mu_x}p^n\frac{G_{n+1}(x')}{n+1}.$$

The theorem then follows by taking $m=n+1$ and applying Theorem 2.1. $\square$

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