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Abstract. We prove explicit upper bounds for the density of universality for Dirichlet series. This complements previous results [15]. Further, we discuss the same topic in the context of discrete universality. As an application we sharpen and generalize an estimate of Reich concerning small values of Dirichlet series on arithmetic progressions in the particular case of the Riemann zeta-function.

1. Introduction: the effectivity problem in universality theorems

The Riemann zeta-function is one of the most-studied functions in mathematics; however, it still does not reveal its mysteries and *simplest* questions concerning the value-distribution are still open. For $\operatorname{Re} s > 1$, the zeta-function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

and by analytic continuation elsewhere, except for a simple pole at $s = 1$. In particular, the distribution of zeros of $\zeta(s)$ is of interest in number theory: the error term in the prime number theorem is as small as possible if and only if the Riemann hypothesis is true; i.e., $\zeta(s)$ does not vanish in the half-plane $\operatorname{Re} s > \frac{1}{2}$. Here we are concerned with an analytic property of the zeta-function which, however, is related to zero-free regions as well.

In 1975, Voronin [19] proved his remarkable universality theorem which roughly states that any non-vanishing analytic function can be approximated uniformly by certain shifts of the Riemann zeta-function. In a more precise form: *Let $f(s)$ be a non-vanishing continuous function defined on a disk $\{s \in \mathbb{C} : |s| \leq r\}$ with some $r \in (0, \frac{1}{4})$, and analytic in the interior. Then, for any $\epsilon > 0$, there exists $\tau > 0$ such that*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{\tau \in [0, T] : \max_{|s|\leq r} |\zeta(s + \tfrac{3}{4} + i\tau) - f(s)| < \epsilon \right\} > 0. \tag{1}$$

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Thus, the set of τ for which shifts of the zeta-function approximate $f(s)$ with a given accuracy has positive lower density (with respect to the Lebesgue measure). Bagchi [1] improved Voronin's universality theorem significantly in replacing the disk by an arbitrary compact subset of the right half of the critical strip with connected complement. It is easily seen that $\zeta(s)$ cannot approximate functions having zeros since otherwise the Riemann hypothesis would be violated; moreover, approximation of a function with a zero on a set of positive lower density contradicts classic density estimates. Bagchi [1] proved that the Riemann hypothesis is true if and only if Voronin's theorem holds with $f(s) = \zeta(s + \frac{3}{4} + i\tau_0)$ for any real τ_0.

In the meantime, similar universality results were obtained for many Dirichlet series. For instance, Voronin [20] proved joint universality for Dirichlet L-functions; i.e., simultaneous uniform approximation by a family of L-functions associated with non-equivalent Dirichlet characters. Reich [13] proved discrete universality for Dedekind zeta-functions $\zeta_{\mathbb{K}}(s)$; here *discrete* means that the shifts τ are taken from an arithmetic progression and the statement of universality takes the form: *Let* $\mathbb{K}$ *be a an algebraic number field of degree* d *and let* $\mathcal{K}$ *be a closed disk lying inside the strip* $\{s \in \mathbb{C} \,:\, \max\{\frac{1}{2}, 1 - \frac{1}{d}\} < \operatorname{Re} s < 1\}$. *Further, let* $f(s)$ *be a non-vanishing continuous function on* $\mathcal{K}$ *which is analytic in the interior of* $\mathcal{K}$. *Then, for any* $\Delta \neq 0$ *and any* $\epsilon > 0$,

$$\limsup_{N \to \infty} \frac{1}{N} \sharp \left\{ 1 \leq n \leq N \,:\, \max_{s \in \mathcal{K}} |\zeta_{\mathbb{K}}(s + in\Delta) - f(s)| < \epsilon \right\} > 0. \tag{2}$$

Notice that the strip of universality depends on the degree of the number field. In the case of normal extensions $\mathbb{K}/\mathbb{Q}$ one can deduce from the joint universality of Dirichlet L-functions that $\zeta_{\mathbb{K}}(s)$ is universal in $\frac{1}{2} < \sigma < 1$; this was first observed by Gonek [4]. Laurinčikas [6] obtained universality for Lerch zeta-functions (which in general even allow to approximate functions having zeros). For further examples we refer to [7, 9, 16] (see also the remark below Theorem 1). It is conjectured that any *reasonable* Dirichlet series is universal.

The known proofs of universality theorems are ineffective, giving neither an estimate for the first τ which yields an approximation to a given $f(s)$ of the desired quality nor bounds for the positive lower density of such τ. Recently, Garunkštis [3] obtained for a *small* class of functions a remarkable effective version of Voronin's theorem; however, it seems hopeless to extend his approach significantly. In fact, he proved that *if* $g(s)$ *is analytic in* $|s| \leq 0.05$ *with* $\max_{|s| \leq 0.06} |g(s)| \leq 1$, *then, for any* $0 < \epsilon < \frac{1}{2}$, *there exists a*

$$0 \leq \tau \leq \exp\left(\exp\left(10\epsilon^{-13}\right)\right) \tag{3}$$

such that

$$\max_{|s| \leq 0.0001} \left|\log \zeta\left(s + \tfrac{3}{4} + i\tau\right) - g(s)\right| < \epsilon.$$

Moreover,

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] \,:\, \max_{|s| \leq 0.0001} \left|\log \zeta\left(s + \tfrac{3}{4} + i\tau\right) - g(s)\right| < \epsilon \right\} \geq \exp\left(-\epsilon^{-13}\right).$$

Note that any non-vanishing analytic function possesses an analytic logarithm; thus, taking the exponential and writing $f(s) = \exp(g(s))$ the latter estimate gives an explicit lower bound for the lower density of universality.

In this paper we are concerned with the complementary question. We shall prove upper bounds for the upper density of universality. This generalizes the results from [15] where we were restricted to approximations of analytic isomorphisms $g(s)$.

2. Statement of the main result

We define for a meromorphic function $\mathcal{D}(s)$, an analytic function $f : \{s \in \mathbb{C} : |s| \leq r\} \to \mathbb{C}$ with fixed $r \in \left(0, \frac{1}{4}\right)$, and positive ϵ the densities

$$\begin{aligned} \underline{\mathrm{d}}(\epsilon, r, f; \mathcal{D}) &= \liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] \,:\, \max_{|s|\leq r}\left|\mathcal{D}\left(s + \tfrac{3}{4} + i\tau\right) - f(s)\right| < \epsilon\right\}, \\ \overline{\mathrm{d}}(\epsilon, r, f; \mathcal{D}) &= \limsup_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] \,:\, \max_{|s|\leq r}\left|\mathcal{D}\left(s + \tfrac{3}{4} + i\tau\right) - f(s)\right| < \epsilon\right\}. \end{aligned}$$

We will prove for a large class of functions upper bounds for the upper density $\overline{\mathrm{d}}(\epsilon, r, f; \mathcal{D})$ which tend with ϵ to zero. For this purpose, we assume that $\mathcal{D}(s)$ satisfies the following axioms:

- **Dirichlet series.** For $\operatorname{Re} s > 1$,

$$\mathcal{D}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s};$$

- **Analytic continuation.** $\mathcal{D}(s)$ has an analytic continuation to the half-plane $\operatorname{Re} s > \frac{1}{2}$ except for at most finitely many poles;
- **Polynomial growth.** There exists a positive constant c such that, for $\sigma > \frac{1}{2}$,

$$\mathcal{D}(\sigma + it) \ll t^c \qquad \text{as} \quad t \to \infty; \tag{4}$$

- **Mean-square.** $\mathcal{D}(s)$ has a bounded mean-square on vertical lines:

$$\limsup \frac{1}{T} \int_0^T |\mathcal{D}(\sigma + it)|^2 \,\mathrm{d}t < \infty \qquad \text{for} \quad \tfrac{1}{2} < \sigma < 1; \tag{5}$$

- **Universality.** $\mathcal{D}(s)$ is universal: for any continuous non-vanishing function $f(s)$ on $|s| \leq r$, $r \in (0, \frac{1}{4})$, which is analytic for $|s| < r$,

$$\underline{\mathrm{d}}(\epsilon, r, f; \mathcal{D}) > 0 \qquad \text{for any} \quad \epsilon > 0. \tag{6}$$

In this case we say that $\mathcal{D}(s)$ belongs to the class D. Of course, $\mathsf{D} \neq \emptyset$; e.g., the zeta-function and Dirichlet L-functions fulfill these axioms.

Theorem 1. *Suppose that $\mathcal{D} \in \mathsf{D}$ and f is a non-constant, non-vanishing, analytic function defined on $|s| \leq r$, where $r \in (0, \frac{1}{4})$. Then, for any sufficiently small $\epsilon > 0$,*

$$\overline{\mathrm{d}}(\epsilon, r, f; \mathcal{D}) \ll \epsilon.$$

Thus, in this rather general setting $\lim_{\epsilon\to 0} \overline{\mathrm{d}}(\epsilon, r, f; \mathcal{D}) = 0$ which answers part of a question raised in [15]. Hence, for functions in D the decay of the upper density of universality as $\epsilon \to 0$ is at least linear. In the particular case of the Riemann zeta-function we can further obtain a slight improvement:

$$\overline{\mathrm{d}}(\epsilon, r, f; \zeta) = o(\epsilon), \tag{7}$$

valid under the same assumption on f and r as in the theorem.

In [16], Steuding proved universality for certain polynomial Euler products from the Selberg class. Roughly speaking, these functions are Dirichlet series with analytic continuation, functional equation of Riemann-type, a polynomial Euler product representation,

$$\mathcal{L}(s) = \prod_p \prod_{j=1}^{m} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1},$$

where the $\alpha_j(p)$ are complex numbers with $|\alpha_j(p)| \le 1$, and which satisfy a certain reasonable normality conjecture (a very weak form of Selberg's orthogonality conjecture, known to be true in many instances). For these functions the region of universality is restricted by an analytic quantity, namely, the degree d for elements $\mathcal{L}(s)$ in the Selberg class, a quantity defined by the data of the functional equation; in the case of Dedekind zeta-functions this *analytic* degree coincides with the degree of the corresponding field extension, and the strip of universality with $\{s \in \mathbb{C} : \max\{\frac{1}{2}, 1 - \frac{1}{d}\} < \operatorname{Re} s < 1\}$. Under assumption of the analogue of the Lindelöf hypothesis for such universal Dirichlet series, this restriction is not necessary and $\mathcal{L}(s)$ is universal in the right half of the critical strip. This restriction relies on the mean-square estimate (5) which is not known to hold in the range $\frac{1}{2} < \sigma \le 1 - \frac{1}{d}$ if the degree of $\mathcal{L}(s)$ is greater than 2 is the degree of $\mathcal{L}(s)$. (For more information we refer to [16].) It should be mentioned that the statement of Theorem 1 can also be applied to Dirichlet series with a restricted strip of universality, e.g., Dedekind zeta-functions. For this one simply has to apply a suitable linear mapping from the restricted strip of universality to the right half of the critical strip. Furthermore, it can also be applied to the situation when the function $f(s)$ is defined on an arbitrary compact subset $\mathcal{K}$ of the strip of universality with connected complement provided that $\mathcal{K}$ has a non-empty interior. In this case one can find a closed disk with sufficiently small radius inside and the estimate for the upper density with respect to this disk is an upper bound for the upper density with respect to $\mathcal{K}$ too. For the sake of simplicity, here we have chosen the form of Voronin's original universality theorem.

We shall prove the theorem and (7) in the following section. In the final section, we discuss the case of discrete universality. As an application we improve and generalize an estimate of Reich concerning small values of Dirichlet series on arithmetic progressions in the particular case of the Riemann zeta-function.

3. Proof of Theorem 1 and Estimate (7)

The idea of proof is more or less the same as in [15]. We shall apply Rouché's theorem to obtain information about the value-distribution of $\mathcal{D}(s)$ in the strip

$\frac{3}{4} - r \leq \operatorname{Re} s \leq \frac{3}{4} + r$ from the distribution of values of $f(s)$ on the disk $|s| \leq r$. However, the fact that we do not have strong restrictions on $f(s)$ requires some further ideas.

Proof of Theorem 1. First, assume that there exists a complex number c in the interior of $f(\{s \in \mathbb{C} : |s| \leq r\})$ (which is not empty since $f(s)$ is not constant) such that

$$f(s) = c + \gamma(s - \lambda_c) + O\left(|s - \lambda_c|^2\right) \tag{8}$$

for some λ_c of modulus less than r and some $\gamma \neq 0$; this means that λ_c is a c-value of $f(s)$ of multiplicity one.

Now suppose that

$$\max_{|s|=r} \left|\{\mathcal{D}\left(s + \tfrac{3}{4} + i\tau\right) - c\} - \{f(s) - c\}\right| < \min_{|s|=r} |f(s) - c|.$$

Then, by Rouché's theorem, $\mathcal{D}(z)$ has at least one c-value ρ_c in $\{z = s + \frac{3}{4} + i\tau : |s| < r\}$. Moreover, the inequality in question holds for sufficiently small ϵ whenever

$$\max_{|s|\leq r} \left|\mathcal{D}\left(s + \tfrac{3}{4} + i\tau\right) - f(s)\right| < \epsilon \leq \min_{|s|=r} |f(s) - c|. \tag{9}$$

By (6), the first inequality is assumed to hold for a set of τ with positive lower density. The second one follows for sufficiently small ϵ from the fact that $c = f(\lambda_c)$ has positive distance to the boundary of $f(\{s \in \mathbb{C} : |s| \leq r\})$. Thus, a c-value of $f(s)$ *generates* many c-values of $\mathcal{D}(z)$.

However, it can happen that for different τ the generated c-values ρ_c of $\mathcal{D}(z)$ are the same (universality is a phenomenon that occurs in intervals). First, we shall show that then the corresponding shifts differ only by a small quantity.

Assume that $\rho_c = s_j + \frac{3}{4} + i\tau_j$ with $|s_j| < r$ for $j = 1, 2$. It follows from (9) that

$$|f(\lambda_c) - f(s_j)| = |c - f(s_j)| < \epsilon. \tag{10}$$

Since $f'(\lambda_c) = \gamma \neq 0$, there exists a neighborhood of c where the inverse f^{-1} exists and is a one-valued continuous function. In view of the continuity, (10) implies

$$|s_j - \lambda_c| < \varepsilon = \varepsilon(\epsilon), \tag{11}$$

where $\varepsilon(\epsilon)$ tends with ϵ to zero; since $f(s)$ behaves locally as a linear function by (8), we have $\varepsilon(\epsilon) \asymp \epsilon$. Now (11) implies

$$|\tau_2 - \tau_1| = |s_1 - s_2| \leq |s_1 - \lambda_c| + |s_2 - \lambda_c| < 2\varepsilon. \tag{12}$$

Denote by $\mathcal{I}_j(T)$ the disjoint intervals in $[0, T]$ such that (9) is valid exactly for

$$\tau \in \bigcup_j \mathcal{I}_j(T) =: \mathcal{I}(T).$$

Inequality (12) implies that in every interval $\mathcal{I}_j(T)$ lie at least

$$1 + \left[\frac{1}{2\varepsilon} \operatorname{meas} \mathcal{I}_j(T)\right] \geq \frac{1}{2\varepsilon} \operatorname{meas} \mathcal{I}_j(T)$$

c-values ρ_c of $\mathcal{D}(s)$ in the strip $\frac{1}{2} < \operatorname{Re} s < 1$; here $[x]$ stands for the greatest integer $\leq x$. Thus, the number $\mathcal{N}_c(T)$ of these c-values ρ_c (counting multiplicities) satisfies the estimate

$$2\varepsilon \mathcal{N}_c(T) \geq \operatorname{meas} \mathcal{I}(T). \tag{13}$$

The next step is to locate the real parts of these c-values a bit more precisely. Obviously, by (12),

$$\mathrm{Re}\,\lambda_c + \tfrac{3}{4} - \varepsilon < \mathrm{Re}\,\rho_c = \mathrm{Re}\,s_j + \tfrac{3}{4} < \mathrm{Re}\,\lambda_c + \tfrac{3}{4} + \varepsilon.$$

Clearly, for sufficiently small ε this range for the c-values lies in the interior of the strip of universality. Hence, if we let $N_c(\sigma_1, \sigma_2, T; \mathcal{D})$ count all c-values of $\mathcal{D}(s)$ in the region $\sigma_1 < \mathrm{Re}\,s < \sigma_2$, $0 < \mathrm{Im}\,s \le T$ (again, according multiplicities), then we can rewrite (13) as

$$\mathrm{meas}\,\mathcal{I}(T) \le 2\varepsilon N_c\left(\mathrm{Re}\,\lambda_c + \tfrac{3}{4} - \varepsilon, \mathrm{Re}\,\lambda_c + \tfrac{3}{4} + \varepsilon, T; \mathcal{D}\right). \tag{14}$$

In view of (6) there exists an increasing sequence (T_k) with $\lim_{k\to\infty} T_k = \infty$ such that for any $\delta > 0$

$$\mathrm{meas}\,\mathcal{I}(T_k) \ge (\overline{\mathrm{d}}(\epsilon, r, f; \mathcal{D}) - \delta) T_k.$$

Consequently, this together with (14) leads to

$$(\overline{\mathrm{d}}(\epsilon, r, f; \mathcal{D}) - \delta) T_k \le 2\varepsilon N_c\left(\mathrm{Re}\,\lambda_c + \tfrac{3}{4} - \varepsilon, \mathrm{Re}\,\lambda_c + \tfrac{3}{4} + \varepsilon, T; \mathcal{D}\right).$$

Sending $\delta \to 0$, yields

$$\overline{\mathrm{d}}(\epsilon, r, f; \mathcal{D}) \le \limsup_{T\to\infty} \frac{2\varepsilon}{T} N_c\left(\mathrm{Re}\,\lambda_c + \tfrac{3}{4} - \varepsilon, \mathrm{Re}\,\lambda_c + \tfrac{3}{4} + \varepsilon, T; \mathcal{D}\right). \tag{15}$$

Since the set of poles of $\mathcal{D}(s)$ in $\sigma > \frac{1}{2}$ has zero density but $\underline{\mathrm{d}}(\epsilon, r, f; \mathcal{D}) > 0$, these poles do not affect the above observations. Next we want to replace the right-hand side of (15) by a more suitable expression. For this purpose we define

$$\ell(s) = \begin{cases} \dfrac{1}{a(1) - c}(\mathcal{D}(s) - c) & \text{if} \quad c \ne a(1), \\ \dfrac{m^s}{a(m)}(\mathcal{D}(s) - c) & \text{if} \quad c = a(1), \end{cases}$$

where m is the minimum over all positive integers $n > 1$ for which $a(n) \ne 0$. Then the c-values of $\mathcal{D}(s)$ correspond one-to-one to the zeros of $\ell(s)$ (having the same multiplicity) and

$$\ell(\sigma + it) = 1 + \lambda^{-\sigma - it} + O(\Lambda^{-\sigma}) \tag{16}$$

with some constants λ, Λ satisfying $1 < \lambda < \Lambda$, as $\sigma \to \infty$. Now let $N(\sigma, T)$ count the number of zeros of $\ell(s)$ (resp. the number of c-values of $\mathcal{D}(s)$) in the region $\mathrm{Re}\,s > \sigma, 0 < \mathrm{Im}\,s \le T$ (according multiplicities). Then Littlewood's lemma (see [17], §9.9) yields

$$\int_{\sigma_1}^{\sigma_2} N(\sigma, T)\,\mathrm{d}\sigma = \frac{1}{2\pi i}\int_{\mathcal{R}} \log \ell(s)\,\mathrm{d}s + O(1), \tag{17}$$

where $\mathcal{R}$ is the rectangular contour with vertices $\sigma_1, \sigma_2, \sigma_1 + iT, \sigma_2 + iT$ with $\frac{1}{2} < \sigma_1 < 1 < \sigma_2$, and where the error term arises from the possible poles of $\mathcal{D}(s)$ (to define here $\log \ell(s)$ we choose the principal branch of the logarithm on the real axis whereas for other points s the value of the logarithm is obtained by continuous variation). By (16) we may choose σ_2 such that $\ell(s)$ has no zeros in the half plane

$\operatorname{Re} s \geq \sigma_2$. A standard application of Jensen's formula shows that the right-hand side of (17) can be replaced by

$$\begin{aligned} & \frac{1}{2\pi}\int_0^T \log|\mathcal{D}(\sigma_1+it)|\,\mathrm{d}t + O(T) \\ \leq\ & \frac{T}{4\pi}\log\left(\frac{1}{T}\int_0^T |\mathcal{D}(\sigma_1+it)|^2\,\mathrm{d}t\right) + O(T); \end{aligned}$$

here we have to use (4) and (16) (the reader can find the details, for example, in [8]). The right-hand side can be bounded by the mean-square estimate (5). This gives in (17)

$$\sum_{\substack{\operatorname{Re}\rho_c>\sigma_1 \\ 0<\operatorname{Im}\rho_c\leq T}} (\operatorname{Re}\rho_c - \sigma_1) \ll T, \tag{18}$$

as $T \to \infty$; here the sum on the left-hand side is taken over all c-values ρ_c of $\mathcal{D}(s)$, not necessarily generated by λ_c. Since, for $\frac{1}{2} < \sigma_1 < \sigma_3$,

$$N(\sigma_3, T) \leq \frac{1}{\sigma_3 - \sigma_1} \sum_{\substack{\operatorname{Re}\rho_c>\sigma_1 \\ 0<\operatorname{Im}\rho_c\leq T}} (\operatorname{Re}\rho_c - \sigma_1),$$

we may estimate

$$\begin{aligned} & N_c\left(\operatorname{Re}\lambda_c + \tfrac{3}{4} - \varepsilon, \operatorname{Re}\lambda_c + \tfrac{3}{4} + \varepsilon, T; \mathcal{D}\right) \\ \leq\ & N(\tfrac{1}{2}(\tfrac{1}{2} + \operatorname{Re}\lambda_c + \tfrac{3}{4} - \varepsilon), T) \ll T. \end{aligned}$$

Thus, the estimate of the theorem follows from (15) provided we can find a c for which (8) holds.

Finally, suppose that for all c in the interior of $f(\{s \in \mathbb{C} : |s| \leq r\})$ the local expansion is different than (8), i.e., $f'(s)$ vanishes identically in the interior. Then f is a constant function, a contradiction to the assumption of the theorem.

It remains to give the

Proof of (7). Here we shall use an old result of Bohr & Jessen: by *Hilfssatz 6* from [2], for any complex $c \neq 0$,

$$\lim_{T\to\infty} \frac{1}{T} N_c\left(\operatorname{Re}\lambda_c + \tfrac{3}{4} - \varepsilon, \operatorname{Re}\lambda_c + \tfrac{3}{4} + \varepsilon, T; \zeta\right) = o(1). \tag{19}$$

Substituting this in (15) implies (7).

4. Value-distribution on arithmetic progressions

We conclude with the special case of discrete universality, introduced by Reich [13] for Dedekind zeta-functions $\zeta_{\mathbb{K}}(s)$ (see (2)). The argument in the proof of Theorem 1 which gave us a factor ϵ for the upper bound does not apply if we consider discrete shifts and so, in general, we do not get an upper bound which tends with ϵ to zero. Anyway, for the zeta-function we obtain via (19)

$$\limsup_{N\to\infty} \frac{1}{N}\sharp\left\{1 \leq n \leq N : \max_{|s|\leq r}\left|\zeta(s + \tfrac{3}{4} + in\Delta) - f(s)\right| < \epsilon\right\} = o(1) \tag{20}$$

as $\epsilon \to 0$. This is of interest with respect to an estimate of Reich concerning small values of Dirichlet series on arithmetic progressions. In [14], he proved: *Let $f(s)$ be a Dirichlet series, not identically zero, which has a half-plane of absolute convergence $\sigma > \sigma_a$, an analytic continuation to $\sigma > \sigma_m$ ($\sigma_m < \sigma_a$) except for at most a finite number of poles on the line $\sigma = \sigma_a$, such that its mean square exists and $f(s)$ is of finite order of growth in any closed strip in $\sigma_m < \sigma < \sigma_a$. Then, for any $\sigma > \sigma_m, \sigma \neq \sigma_a$, any sufficiently small $\epsilon > 0$, and any Δ, neither being equal to zero nor of the form $2\pi\ell\cos(q/r)$ with positive integers ℓ, q, r and $q \neq r$,*

$$\limsup_{N\to\infty} \frac{1}{N} \sharp\{1 \leq n \leq N : |f(\sigma + in\Delta)| < \epsilon\} < 1.$$

In particular, it follows that $f(\sigma + i\Delta n)$ cannot converge to zero as $n \to \infty$, and hence $s_n = \sigma + i\Delta n$ cannot be a sequence of zeros of $f(s)$. It should be noticed that Reich's class of Dirichlet series shares many axioms with D.

Now we shall consider the special case of the Riemann zeta-function. Reich's theorem also includes estimates for c-values on arithmetic progressions (since with $f(s)$ also $f(s)-c$ satisfies the conditions). We shall note an improvement of Reich's theorem:

Theorem 2. *Let c be any constant, $\sigma \in (\frac{1}{2}, 1)$, and $0 \neq \Delta \in \mathbb{R}$. Then*

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \sharp\{1 \leq n \leq N : |\zeta(\sigma + in\Delta) - c| < \epsilon\} = 0.$$

In particular, there does not exist an arithmetic progression $s_n = \sigma + i\Delta n$ (with σ and Δ as in the theorem) on which $\zeta(s)$ converges to any complex number c.

Proof. Let $f(s)$ be a non-constant, non-vanishing, analytic function defined on a small disk centered at $\sigma \in (\frac{1}{2}, 1)$ such that its closure lies inside the strip of universality for the zeta-function. Further assume that

$$|f(s) - c| < \epsilon;$$

this choice for $f(s)$ is certainly possible for any complex number c. By the triangle inequality,

$$|\zeta(\sigma + in\Delta) - c| \leq |\zeta(\sigma + in\Delta) - f(s)| + |f(s) - c|$$

for any s. Hence, applying (20) yields

$$\limsup_{N\to\infty} \frac{1}{N} \sharp\{1 \leq n \leq N : |\zeta(\sigma + in\Delta) - c| < 2\epsilon\} \ll \epsilon.$$

This is the assertion of the theorem.

An alternative proof can be given by using the deep *Hauptsatz I* of Bohr & Jessen [2]; this approach does not depend on the universality property of $\zeta(s)$. As a matter of fact, this theorem may also be used to prove the estimate $\overline{\mathrm{d}}(\epsilon, r, f; \zeta) \ll \epsilon^2$ for constant $f \neq 0$, a case not included in Theorem 1.

There are remarkable results for a related problem. Putnam [10], [11] showed that $\zeta(s)$ does not have an infinite vertical arithmetic progressions of zeros (or even approximate zeros). Lapidus & van Frankenhuijsen [5], Chapter 9, gave a different proof of Putnam's theorem. Watkins (cf. [18]) was the first to give upper bounds for the length of such arithmetic progressions (valid for any Dirichlet L-functions). Recently, van Frankenhuijsen [18] improved these bounds by showing that

$$\zeta(\sigma + in\Delta) = 0 \qquad \text{for} \quad 0 < |n| < N$$

with $\sigma, \Delta > 0$ and $N \geq 2$ cannot hold for

$$N \geq 60 \left(\frac{\Delta}{2\pi}\right)^{\frac{1}{\sigma}-1} \log \Delta$$

(his method also applies to Dirichlet L-functions). It is conjectured that there are no arithmetic progressions at all; there are even no zeros known of the form $\frac{1}{2} + i\gamma$ and $\frac{1}{2} + i2\gamma$. The methods of Putnam, Lapidus and van Frankenhuijsen do not apply to c-values.

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